INSTITUTE OF PHYSICS PUBLISHING REPORTS ON PROGRESS IN PHYSICS

Rep. Prog. Phys. **68** (2005) 2337–2408 doi:10.1088/0034-4885/68/10/R02

Disorder-driven non-Fermi liquid behaviour of correlated electrons

E Miranda[1] **and V Dobrosavljević**[2]

[1] Instituto de Física Gleb Wataghin, Unicamp, Caixa Postal 6165, 13083-970 Campinas, SP, Brazil
[2] Department of Physics and National High Magnetic Field Laboratory, Florida State University, Tallahassee, FL 32306, USA

Received 1 April 2005
Published 22 August 2005
Online at stacks.iop.org/RoPP/68/2337

Abstract

Systematic deviations from standard Fermi-liquid behaviour have been widely observed and documented in several classes of strongly correlated metals. For many of these systems, mounting evidence is emerging that the anomalous behaviour is most likely triggered by the interplay of quenched disorder and strong electronic correlations. In this review, we present a broad overview of such disorder-driven non-Fermi liquid behaviour, and discuss various examples where the anomalies have been studied in detail. We describe both their phenomenological aspects as observed in experiment, and the current theoretical scenarios that attempt to unravel their microscopic origin.

 © 2005 IOP Publishing Ltd Printed in the UK

Contents

	Page
1. Introduction	2340
2. General predictions of Fermi liquid theory and its limitations	2341
2.1. Fermi liquid theory for clean systems	2341
2.2. Fermi liquid theory for disordered systems	2343
2.2.1. Drude theory	2343
2.2.2. Perturbative quantum corrections	2344
2.2.3. Scaling theories of disordered Fermi liquids	2345
2.2.4. Fermi liquid near localization transitions	2346
2.2.5. Critical scaling	2347
2.3. Typical departures from Fermi liquid theory in real systems	2348
3. Phenomenology of NFL behaviour and experimental realizations	2350
3.1. NFL behaviour in correlated systems with weak or no disorder	2350
3.1.1. Heavy fermion materials near magnetic quantum critical points	2350
3.1.2. Marginal Fermi liquid behaviour of high T_c superconductors	2352
3.2. Disorder-driven NFL behaviour in correlated systems: doped semiconductors	2355
3.2.1. Metal–insulator transition	2355
3.2.2. Thermodynamic anomalies	2356
3.2.3. Two-fluid model	2357
3.2.4. Asymptotic critical behaviour or mean-field scaling?	2359
3.3. Disorder-driven NFL behaviour in correlated systems: disordered heavy fermion systems	2359
3.4. Disorder-driven NFL behaviour in correlated systems: metallic glass phases	2362
3.4.1. Metallic glass phases	2362
3.4.2. Glassiness in the charge sector	2362
4. Theoretical approaches	2364
4.1. Disordered Hubbard and Kondo lattice models	2364
4.1.1. Disordered-induced local moment formation	2364
4.1.2. Random singlet phases	2365
4.1.3. Phenomenological Kondo disorder model	2366
4.1.4. Dynamical mean-field theory	2367
4.1.5. Statistical DMFT: localization effects and results of numerical calculations	2369
4.1.6. Electronic Griffiths phase in disordered Kondo lattice models of dirty heavy-fermion materials	2371
4.1.7. Electronic Griffiths phase near the Mott–Anderson MIT	2373
4.1.8. Incoherent metallic phase and the anomalous resistivity drop near the two-dimensional MIT	2374

4.2. Magnetic Griffiths phases 2375
4.2.1. Quantum Griffiths phases in insulating magnets with disorder: random singlet formation and the IRFP 2375
4.2.2. General properties of quantum Griffiths phases 2379
4.2.3. Possibility of quantum Griffiths phases in itinerant random magnets 2381
4.2.4. On the applicability of the magnetic Griffiths phase phenomenology to metallic disordered systems 2388
4.3. Itinerant quantum glass phases and their precursors 2390
4.3.1. Inherent instability of the electronic Griffiths phases to spin-glass ordering 2390
4.3.2. Quantum critical behaviour in insulating and metallic spin glasses 2391
4.3.3. Spin-liquid behaviour, destruction of the Kondo effect by bosonic dissipation, and fractionalization 2394
4.3.4. Electron glasses, freezing in the charge sector and the quantum AT line 2398
5. Conclusions and open problems 2401
Acknowledgments 2402
References 2402

1. Introduction

Soon after the discovery of high temperature superconductivity in the cuprates it was realized that the optimally doped compounds showed clear deviations from Fermi liquid behaviour. This was not the first time that a real system seemed to violate Landau's paradigm. The heavy fermion superconductor UBe_{13} had previously been found to show anomalous normal state behaviour (Cox 1987). Since then, the quest for such systems has been vigorous and numerous puzzling compounds have been discovered. However, a coherent theoretical framework with which to understand this behaviour has not been obtained. One of the few situations where there seems to be a general agreement as to the inadequacy of the Fermi liquid theory is the critical region governed by a zero temperature quantum phase transition. Here, deviations from Landau's predictions seem to be the norm, although a general theory of this behaviour is still lacking. However, the quantum critical non-Fermi liquid (NFL) requires fine tuning to a special location in the phase diagram. In many of the known compounds, NFL behaviour seems to be better characterized as the property of a whole phase instead of just a phase boundary.

Interestingly, NFL behaviour in disordered systems rests on a much firmer theoretical basis. Even an exactly solvable model is known where diverging thermodynamical properties can be calculated (McCoy 1969, McCoy and Wu 1968). In addition, this anomalous behaviour persists over an entire region of the phase diagram and is not confined to a specific point. These so-called Griffiths–McCoy phases had been originally proposed in disordered classical systems where the accompanying singularities are rather weak (Griffiths 1969). The quantum version of this phenomenon is, however, much stronger and has now been established theoretically in many models. Furthermore, their properties are not believed to be a peculiarity of the one-dimensional exactly solvable model. Their relevance to real compounds has also been advocated in many systems, from doped semi-conductors to disordered heavy fermion compounds. The emergence of NFL behaviour in a disordered context is thus seen to be fairly natural.

The question of NFL behaviour in various clean situations has been extensively discussed in the literature. It is the purpose of this review to attempt to bring together, in a single paper, a number of theoretical models and analyses in which disorder-induced NFL behaviour has arisen. Although reference will be made to the most important experimental findings, the focus will be on theory rather than on experiments. For the latter, we will refer the reader to other reviews.

The issue of disorder-induced NFL behaviour was discussed for the first time in connection with the anomalous thermodynamic responses near the metal–insulator transition (MIT) in doped semiconductors (Paalanen and Bhatt 1991, Sarachik 1995). In this context, the random singlet phase of Bhatt and Lee was an important pioneering work (Bhatt and Lee 1981, 1982). Since then, anomalous behaviour has been observed in various disordered heavy fermion compounds, where local moments are well formed even in the clean limit (Stewart 2001). More recently, two-dimensional electron systems in semiconductor heterostructures have also shown intriguing behaviour (Abrahams *et al* 2001). Finally, intrinsic and extrinsic heterogeneities in HTS have led to the consideration of these disorder effects in that context as well. These systems, therefore, have become the main focus of the theoretical efforts and will take centre stage in this review.

We will attempt to emphasize the general aspects of the various contexts in which NFL behaviour arises. As mentioned above, Griffiths phases play a central role in many of these models. Although they come in different guises, all Griffiths phases seem to share a few common generic features, which we will describe. Once these features are present, most physical quantities can be immediately obtained. The question of distinguishing among the

different microscopic mechanisms is, therefore, rather subtle. However, a few criteria can be posed and perhaps used to make this distinction. We hope these will serve as a guide to experimentalists.

This review is organized as follows. In section 2, we briefly summarize the main ideas of Fermi liquid theory as applied to both clean and disordered systems. Section 3 is devoted to a general discussion of the phenomenology of the main NFL systems. Section 3.1 focuses on NFL behaviour of clean systems, whereas sections 3.2–3.4 describe the anomalous behaviour of doped semiconductors, disordered heavy fermion systems and the glassy regime close to the MIT. The main theoretical approaches are reviewed in section 4. Section 4.1 is devoted to results on the disordered Hubbard and Kondo/Anderson lattice models. We highlight the physics of local moment formation in a disordered environment (4.1.1), the random singlet phase (4.1.2), the Kondo disorder model (4.1.3 and 4.1.4), the electronic Griffiths phase close to the disorder-induced MIT (4.1.5–4.1.7), and the incoherent transport near the two-dimensional MIT (4.1.8). The general topic of magnetic Griffiths phases is discussed in section 4.2. We first review their well-established properties in the case of insulating magnets, where we emphasize the important role played by the infinite randomness fixed point (IRFP) (4.2.1). The generic mechanism behind magnetic Griffiths phases is described in section 4.2.2, the important effect of dissipation on the dynamics of the Griffiths droplets is considered in section 4.2.3 and further remarks on the applicability of this scenario to real metallic systems are made in section 4.2.4. Section 4.3 is devoted to the physics of glassy dynamics and ordering in metals. We wrap up with conclusions and open questions in section 5.

2. General predictions of Fermi liquid theory and its limitations

2.1. Fermi liquid theory for clean systems

Interacting homogeneous systems of fermions in three dimensions are believed to be described at low energies and long wave lengths by Landau's Fermi liquid theory (Landau 1957a, 1957b). Although there is no general rigorous justification for it, Landau was able to use diagrammatic arguments to show that his general framework is at least internally consistent (Landau 1959). More recently, these early demonstrations were put on a firmer basis in the modern framework of the renormalization group (RG), in which the asymptotic correctness of the theory can be shown (Benfatto and Gallavotti 1990, Shankar 1994). In addition, our confidence in its correctness is boosted by simple model systems where controlled calculations can be carried out, such as a dilute Fermi gas with short-range interactions (Abrikosov and Khalatnikov 1958, Galitskii 1958), the electron–phonon system (Migdal 1958) and the very dense degenerate electron plasma (Abrikosov 1962). More importantly, its direct experimental verification in the prototypical Fermi liquid system, the normal state of liquid ^{3}He between the onset of superfluidity at $T_c \approx 3\,\mathrm{mK}$ and about 100 mK, can be taken as one of the most striking examples of Landau's theory at work (see, e.g. Leggett (1975)). Finally, Fermi liquid theory, properly generalized to charged systems (Silin 1958a, 1958b), forms the foundation of our understanding of the behaviour of electrons in metals.

There are excellent reviews of Fermi liquid theory to which the reader is referred to (Abrikosov *et al* 1975, Baym and Pethick 1991, Leggett 1975, Pines and Nozières 1965). We will content ourselves with a brief outline of its main assumptions and predictions. Fermi liquid theory starts by recognizing that the low energy excitations of a Fermi sea, say an added electron at a wave vector $\boldsymbol{k}$ with $|\boldsymbol{k}| = k > k_F$ and $k - k_F \ll k_F$, where k_F is the Fermi wave vector (we assume a rotationally invariant system, for simplicity), have a very long lifetime. This is due to restrictions imposed by energy and momentum conservation and the blocking of

further occupation of the Fermi sea by the Pauli exclusion principle. In this case, the lifetime can be shown to be $\tau_k \sim (k - k_F)^2$. The stability of these quasi-particle excitations allows us to ignore their decay at first and treat them (perturbatively) only at a later stage. The second ingredient of the theory is sometimes called 'adiabatic continuity': the excitations of the interacting fermions are in one-to-one correspondence to those of a non-interacting Fermi gas, in other words, the slow 'switching-on' of the interactions in a Fermi gas does not alter the nature of the excitation spectrum. These two assumptions led Landau to write the total energy of weakly excited states of the system as a functional of the various occupation numbers of states labelled by momentum $\boldsymbol{k}$ and spin projection $\sigma = \pm$, the quasi-particle states. More precisely, he used the deviations $\delta n_{\boldsymbol{k},\sigma}$ from their occupations in the ground state (in this review, we use units such that $\hbar = 1$ and $k_B = 1$)

$$E = E_0 + \sum_{\boldsymbol{k},\sigma} \epsilon(\boldsymbol{k}) \delta n_{\boldsymbol{k},\sigma} + \sum_{\boldsymbol{k},\sigma;\boldsymbol{k}',\sigma'} f(\boldsymbol{k},\sigma;\boldsymbol{k}',\sigma') \delta n_{\boldsymbol{k},\sigma} \delta n_{\boldsymbol{k}',\sigma'}. \tag{1}$$

Here, E_0 is the (usually unknown) ground state energy, $\epsilon(\boldsymbol{k}) \approx v_F(k - k_F)$ is the quasi-particle dispersion, parametrized by $v_F = k_F/m^*$, where m^* is the effective mass of the quasi-particle, usually distinct from its bare, free-electron value. Note that the Fermi wave vector k_F is not modified by interactions, a fact known as Luttinger's theorem (Luttinger 1960). The last term, quadratic in the occupations, incorporates the self-consistent interactions among the quasi-particles and is easily seen to be of the same order as the second term for weakly excited states.

Since only momenta close to the Fermi surface are needed for low-energy excitations, we can take, to leading order, $|\boldsymbol{k}| = |\boldsymbol{k}'| = k_F$ in the last term of equation (1) and then f only depends on the angle θ between $\boldsymbol{k}$ and $\boldsymbol{k}'$. Moreover, in the absence of an applied magnetic field, spin rotation invariance imposes additional constraints

$$f(\boldsymbol{k},\sigma;\boldsymbol{k}',\sigma') = f^s(\theta) + \sigma\sigma' f^a(\theta). \tag{2}$$

Finally, it is convenient to decompose the angular dependence in Legendre polynomials

$$v_F f^{s,a}(\theta) = \sum_l (2l+1) F_l^{s,a} P_l(\cos\theta), \tag{3}$$

where $v_F = V m^* k_F/\pi^2$ is the total density of states (DOS) at the Fermi level and V is the volume of the system. The dimensionless constants $F_l^{s,a}$ are known as Landau coefficients.

We thus see that the low-energy sector of the interacting system is completely parametrized by the effective mass m^* and the Landau coefficients, in terms of which important physical properties can be written. For example, the low temperature specific heat is linear in temperature with a coefficient determined by the effective mass

$$\frac{C_V(T)}{T} = \frac{1}{3} m^* k_F. \tag{4}$$

The magnetic susceptibility is a constant at low temperatures and related to the Landau parameter F_0^a

$$\chi(T) = \frac{m^* k_F \mu_B^2}{\pi^2 (1 + F_0^a)}, \tag{5}$$

where μ_B is the Bohr magneton. The compressibility $\kappa = -V(\partial P/\partial V)$, which can be accessed through the sound velocity, is related to F_0^s

$$\kappa = \frac{9\pi^2 m^*}{k_F^5 (1 + F_0^s)}. \tag{6}$$

There are several other predictions which come out of the basic framework outlined above, especially with regard to collective excitations (zero sound, plasmons, spin waves) and transport

properties, but we will not dwell on them, referring the interested reader to the available reviews. We will only note that the quadratic dependence on its excitation energy of the quasi-particle lifetime leads to a quadratic temperature dependence of the resistivity at the lowest temperatures. Moreover, in the $T \to 0$ limit the resistivity tends to a constant value determined by lattice imperfections and extrinsic impurities, such that the Fermi liquid prediction is

$$\rho(T) = \rho_0 + AT^2. \tag{7}$$

The thermal conductivity of a Fermi liquid is proportional $T\sigma(T)$ (the Wiedemann–Franz law), where $\sigma(T) = 1/\rho(T)$ is the electric conductivity.

The application of Fermi liquid theory to metals helps explain why a strongly interacting electron gas in a clean crystal can behave almost like a free electron gas and conduct heat and electricity so well at low temperatures. Interestingly, very strong electronic correlations can put the theory to test in the most extreme circumstances. In particular, in a certain class of compounds containing rare-earth or actinide elements such as Ce or U (to be reviewed later in section 3.1.1), the effective mass is observed to be 2 to 3 orders of magnitude greater than the electron mass. These compounds were therefore named 'heavy fermions'. Nevertheless, even in these cases, Fermi liquid theory provides a valid description in a number of cases, although a growing number of exceptions have been discovered and investigated in recent years. Furthermore, the idea of adiabatic continuation (the one-to-one correspondence between the excitations of a simpler reference system and another one of interest) has proven fruitful in contexts which go far beyond Landau's original proposal. In particular, this philosophy has been extended to the superfluid phases of ^{3}He (Leggett 1975) and to nuclear physics (Migdal 1967). More importantly for the subject of this review, Fermi liquid ideas form the basis of much of our understanding of interacting electrons in disordered metals, as we will expand upon in the next section.

2.2. *Fermi liquid theory for disordered systems*

While the original formulation and much of the later work on Fermi liquid theory concentrated on clean metals, the relevant physical principles have a much more general validity. This framework is flexible enough to be also applicable not only in presence of arbitrary forms of randomness, but even for finite size systems such as quantum dots, molecules, atoms or atomic nuclei. In electronic systems, the first systematic studies of the interplay of interactions and disorder (Lee and Ramakrishnan 1985) emerged only in the last 25 years or so. Most progress was achieved in the regime of weak disorder, where controlled many-body calculations are possible using the disorder strength as a small parameter.

2.2.1. Drude theory. To lowest order in the disorder strength, one obtains the semi-classical predictions of the Drude theory (Lee and Ramakrishnan 1985), where the conductivity takes the form

$$\sigma \approx \sigma_0 = \frac{n\mathrm{e}^2\tau_{\mathrm{tr}}}{m}, \tag{8}$$

where n is the carrier concentration, e the electron charge and m its band mass. According to Matthiessen's rule, the transport scattering rate takes additive contributions from different scattering channels, namely,

$$\tau_{\mathrm{tr}}^{-1} = \tau_{\mathrm{el}}^{-1} + \tau_{\mathrm{ee}}^{-1}(T) + \tau_{\mathrm{ep}}^{-1}(T) + \cdots. \tag{9}$$

Here, τ_{el}^{-1} is the elastic scattering rate (describing impurity scattering), and $\tau_{\mathrm{ee}}^{-1}(T)$, $\tau_{\mathrm{ep}}^{-1}(T)$, ..., describe inelastic scattering processes from electrons, phonons, etc. It is important

to note that in this picture the resistivity $\rho = \sigma^{-1}$ is a monotonically increasing function of temperature

$$\rho(T) \approx \rho_0 + AT^n, \tag{10}$$

since inelastic scattering is assumed to only increase at higher temperatures. The residual resistivity $\rho_0 = \sigma(T = 0)^{-1}$ is thus viewed as a measure of impurity (elastic) scattering. Within this formulation, thermodynamic quantities are generally not expected to acquire any singular or non-analytic corrections due to impurity scattering.

The Drude theory encapsulates a very simple physical picture. It implicitly assumes that the itinerant carriers undergo many collisions with unspecified scattering centres, but that these scattering events remain independent and uncorrelated, justifying Matthiessen's rule. At this level, inelastic scattering processes are therefore assumed to be independent of impurity scattering, and thus assume the same form as in standard Fermi liquid theory, e.g. $\tau_{\rm el}^{-1} \sim T^2$, etc. This simplifying assumption is better justified at higher temperatures, where inelastic scattering processes erase the phase memory of the electrons, and suppress quantum interference processes arising from multiple scattering events.

2.2.2. Perturbative quantum corrections. At weak disorder, systematic corrections to the Drude theory were found (Lee and Ramakrishnan 1985) to consist of several additive terms,

$$\sigma = \sigma_0 + \delta\sigma_{\rm wl} + \delta\sigma_{\rm int}, \tag{11}$$

corresponding to the so-called 'weak localization' and 'interaction' corrections. These 'hydrodynamic' corrections are dominated by infrared singularities, i.e. they acquire non-analytic contributions from small momenta or equivalently large distances, and which generally lead to an instability of the paramagnetic Fermi liquid in two dimensions. Specifically, the weak localization corrections take the form

$$\delta\sigma_{\rm wl} = \frac{e^2}{\pi^d}[l^{-(d-2)} - L_{\rm Th}^{-(d-2)}], \tag{12}$$

where $l = v_F\tau$ is the mean free path, d is the dimension of the system and $L_{\rm Th}$ is the length scale over which the wave functions are coherent. This effective system size is generally assumed to be a function of temperature of the form $L_{\rm Th} \sim T^{p/2}$, where the exponent p depends on the dominant source of decoherence through inelastic scattering. The situation is simpler in the presence of a weak magnetic field where the weak localization corrections are suppressed and the leading dependence comes from the interaction corrections first discovered by Altshuler and Aronov (1979)

$$\delta\sigma_{\rm int} = \frac{e^2}{\hbar}(c_1 - c_2\tilde{F}_\sigma)(T\tau)^{(d-2)/2}. \tag{13}$$

Here, c_1 and c_2 are constants and $\tilde{F}_\sigma$ is an interaction amplitude. In $d = 3$, this leads to a square-root singularity $\delta\sigma_{\rm int} \sim \sqrt{T}$ and in $d = 2$ to a logarithmic divergence $\delta\sigma_{\rm int} \sim \ln(T\tau)$. These corrections are generally more singular than the temperature dependence of the Drude term, and thus they are easily identified experimentally at the lowest temperatures. Indeed, the $T^{1/2}$ law is commonly observed (Lee and Ramakrishnan 1985) in transport experiments in many disordered metals at the lowest temperatures, typically below 500 mK.

Similar corrections have been predicted for other physical quantities, such as the tunnelling DOS and, more importantly, for thermodynamic response functions. As in the Drude theory, these quantities are not expected to be appreciably affected by non-interacting localization processes, but singular contributions are predicted from interaction corrections. In particular,

corrections to both the spin susceptibility χ and the specific heat coefficient $\gamma = C_V/T$ were expected to take the general forms $\delta\chi \sim \delta\gamma \sim T^{(d-2)/2}$ and thus in three dimensions

$$\chi = \chi_0 + m_\chi \sqrt{T}, \tag{14}$$

$$\gamma = \gamma_0 + m_\gamma \sqrt{T}, \tag{15}$$

where m_χ and m_γ are constants.

As in conventional Fermi liquid theories, these corrections emerged already when the interactions were treated at the lowest Hartree–Fock level, as was done in the approach of Altshuler and Aronov (1979). Higher order corrections in the interaction amplitude were first incorporated by Finkel'stein (1983, 1984), demonstrating that the predictions remained essentially unaltered, at least within the regime of weak disorder. In this sense, the Fermi liquid theory has been generalized to weakly disordered metals, where its predictions have been confirmed in numerous materials (Lee and Ramakrishnan 1985).

2.2.3. Scaling theories of disordered Fermi liquids. In several systems, experimental tests were extended beyond the regime of weak disorder, where at least at face value, the perturbative predictions seem of questionable relevance. Interestingly, a number of transport experiments (Lee and Ramakrishnan 1985, Paalanen and Bhatt 1991, Sarachik 1995) seemed to indicate that some predictions, such as the $T^{1/2}$ conductivity law, appear to persist beyond the regime of weak disorder. At stronger disorder, the system approaches a disorder-driven MIT. Since the ground-breaking experiment of Paalanen, Rosenbaum and Thomas in 1980 (Rosenbaum *et al* 1980), it became clear that this is a continuous (second order) phase transition (Paalanen *et al* 1982), which bears many similarities to conventional critical phenomena. This important observation has sparked a veritable avalanche of experimental (Lee and Ramakrishnan 1985, Paalanen and Bhatt 1991, Sarachik 1995) and theoretical (Abrahams *et al* 1979, Schaffer and Wegner 1980, Wegner 1976, 1979) works, most of which have borrowed ideas from studies of second order phase transitions. Indeed, many experimental results were interpreted using scaling concepts (Lee and Ramakrishnan 1985), culminating with the famous scaling theory of localization (Abrahams *et al* 1979).

The essential idea of these approaches focuses on the fact that a weak, logarithmic instability of the clean Fermi liquid arises in two dimensions, suggesting that $d = 2$ corresponds to the lower critical dimension of the problem. In conventional critical phenomena, such logarithmic corrections at the lower critical dimension typically emerge due to long wavelength fluctuations associated with spontaneously broken continuous symmetry. Indeed, early work of Wegner (Schaffer and Wegner 1980, Wegner 1976, 1979) emphasized the analogy between the localization transition and the critical behaviour of Heisenberg magnets. It mapped the problem onto a field theoretical non-linear σ-model and identified the hydrodynamic modes leading to singular corrections in $d = 2$. Since the ordered (metallic) phase is only marginally unstable in two dimensions, the critical behaviour in $d > 2$ can be investigated by expanding around two dimensions. Technically, this is facilitated by the fact that in dimension $d = 2 + \varepsilon$ the critical value of disorder W for the MIT is very small ($W_c \sim \varepsilon$), and thus can be accessed using perturbative RG approaches in direct analogy to the procedures developed for Heisenberg magnets.

In this approach (Abrahams *et al* 1979, Schaffer and Wegner 1980, Wegner 1976, 1979), conductance is identified as the fundamental scaling variable associated with the critical point, which is an unstable fixed point of the RG flows. In this picture, temperature scaling is obtained from examining the system at increasingly longer length scales $L_{\mathrm{Th}} \sim T^{p/2}$, which follows from the precise form of the RG flows. In the metallic phase, under rescaling the conductance

$g \to \infty$ (corresponding to the reduction of effective disorder) and the long distance behaviour of all correlation functions is controlled by the approach to the stable fixed point at $W = 0$. In other words, the leading low temperature behaviour of all quantities should be *identical* to that calculated at infinitesimal disorder. This scaling argument therefore provides strong justification for using the weak disorder predictions throughout the entire metallic phase, *provided that the temperature is low enough.*

2.2.4. Fermi liquid near localization transitions. In the following years, these ideas were extended with a great deal of effort in the formulation of a scaling theory of interacting disordered electrons by Finkel'stein (1983) and many followers (Belitz and Kirkpatrick 1994, Castellani *et al* 1984). While initially clad in an apparent veil of quantum field theory jargon, these theories were later given a simple physical interpretation in terms of Fermi liquid ideas (Castellani *et al* 1987, Kotliar 1987) for disordered electrons. Technical details of these theories are of considerable complexity and the interested reader is referred to the original literature (Belitz and Kirkpatrick 1994, Castellani *et al* 1984, Finkel'stein 1983, 1984). Here we just summarize the principal results, in order to clarify the constraints imposed by these Fermi liquid approaches.

Within the Fermi liquid theory for disordered systems (Castellani *et al* 1987, Kotliar 1987), the low energy (low temperature) behaviour of the system is characterized by a small number of effective parameters, which include the diffusion constant D, the frequency renormalization factor Z and the interaction amplitudes γ_s and γ_t. These quantities can also be related to the corresponding quasi-particle parameters which include the quasi-particle DOS

$$\rho_Q = Z\rho_0 \tag{16}$$

and the quasi-particle diffusion constant

$$D_Q = \frac{D}{Z} \sim \frac{D}{\rho_Q}. \tag{17}$$

Here, ρ_0 is the 'bare' DOS which describes the non-interacting electrons. In the absence of interactions, the single-particle DOS is only weakly modified by disorder and remains non-critical (finite) at the transition (Wegner 1981).

Using these parameters, we can now express the thermodynamic response functions as follows. We can write the compressibility

$$\chi_c = \frac{\mathrm{d}n}{\mathrm{d}\mu} = \rho_Q[1 - 2\gamma_s], \tag{18}$$

the spin susceptibility

$$\chi_s = \mu_B^2 \rho_Q[1 - 2\gamma_t] \tag{19}$$

and the specific heat

$$C_V = \frac{2\pi^2 \rho_Q T}{3}. \tag{20}$$

In addition, we can use the same parameters to express transport properties such as the conductivity

$$\sigma = \frac{\mathrm{d}n}{\mathrm{d}\mu} D_c = \rho_Q D_Q, \tag{21}$$

as well as the density–density and spin–spin correlation functions

$$\pi(q,\omega) = \frac{\mathrm{d}n}{\mathrm{d}\mu}\frac{D_c q^2}{D_c q^2 - \mathrm{i}\omega}, \qquad \chi_s(q,\omega) = \chi_s \frac{D_s q^2}{D_s q^2 - \mathrm{i}\omega}. \tag{22}$$

Here, we have expressed these properties in terms of the spin and charge diffusion constants, which are defined as

$$D_c = \frac{D}{Z(1-2\gamma_s)}, \qquad D_s = \frac{D}{Z(1-2\gamma_t)}. \tag{23}$$

Note that the the quantity D is *not* the charge diffusion constant D_c that enters the Einstein relation (equation (21)). As we can write $\sigma = \rho_0 D$, and since ρ_0 is not critical at any type of transition, the quantity D (also called the 'renormalized diffusion constant') has a critical behaviour identical to that of the conductivity σ. We also mention that the quasi-particle diffusion constant $D_Q = D/Z$ has been physically interpreted as the heat diffusion constant.

2.2.5. Critical scaling. Finally, we discuss the scaling behaviour of observables in the critical region. In particular, if the scaling description (Belitz and Kirkpatrick 1994, Castellani *et al* 1984, Finkel'stein 1983, 1984) is valid, the conductivity can be written as

$$\sigma(t,T) = b^{-(d-2)} f_\sigma(b^{1/\nu} t, b^z T). \tag{24}$$

Here, b is the length rescaling factor and $t = (n - n_c)/n_c$ is the dimensionless distance from the transition. We have also introduced the correlation length exponent ν and the 'dynamical exponent' z. This expression, first proposed by Wegner (Abrahams *et al* 1979, Schaffer and Wegner 1980, Wegner 1976, 1979), is expected to hold for 'regular' types of transitions, where the critical values of the interaction amplitudes remain finite. The conductivity exponent μ in $\sigma(T=0) \sim t^\mu$ can be obtained by working at low temperatures and choosing $b = t^{-\nu}$. We immediately see that

$$\sigma(T) \sim t^\mu \phi_\sigma\left(\frac{T}{t^{\nu z}}\right), \tag{25}$$

where $\phi_\sigma(x) = f_\sigma(1,x)$ and

$$\mu = (d-2)\nu, \tag{26}$$

a relation known as 'Wegner scaling'. Finite temperature corrections in the metallic phase are obtained by expanding

$$\phi_\sigma(x) \approx 1 + a x^\alpha, \tag{27}$$

giving the low temperature conductivity of the form

$$\sigma(t,T) \approx \sigma_0(t) + m_\sigma(t) T^\alpha. \tag{28}$$

Here, $\sigma_0(t) \sim t^\mu$ and $m_\sigma(t) \sim t^{\mu - \alpha\nu z}$. Since the scaling function $\phi_\sigma(x)$ is independent of t, the exponent α must take a universal value in the entire metallic phase, and thus it can be calculated at weak disorder, giving $\alpha = 1/2$. This scaling argument provides a formal justification for using the predictions from perturbative quantum corrections as giving the leading low temperature dependence in the entire metallic phase. Note, however, that according to this result the prefactor $m_\sigma(t)$ is not correctly predicted by perturbative calculations, since it undergoes Fermi liquid renormalizations which can acquire a singular form in the critical region near the MIT.

The temperature dependence at the critical point (in the critical region) is obtained if we put $t = 0$ ($n = n_c$), and choose $b = T^{-1/z}$. We get

$$\sigma(t=0,T) \sim T^{(d-2)/z}. \tag{29}$$

An analogous argument can be used (Castellani and Castro 1986) for the specific heat coefficient $\gamma = C_V/T$ (it is important not to confuse this quantity with the interaction amplitude γ_t).

$$\gamma(t, T) = b^{\kappa/\nu} f_\gamma(b^{1/\nu} t, b^z T). \tag{30}$$

Choosing $b = t^{-\nu}$ and expanding in T we find

$$\gamma(t, T) \approx \gamma_0(t) + m_\gamma(t) T^{1/2}, \tag{31}$$

with $\gamma_0(t) \sim t^{-\kappa}$, $m_\gamma(t) \sim t^{-(\kappa+\nu z/2)}$. Even though $\gamma_0 = \gamma(t, T = 0)$ can become singular at the transition, it is expected to remain finite away from the transition ($t \neq 0$). We stress that the specific heat exponent κ and the dynamical exponent z are *not* independent quantities. In fact, Castellani and Castro (1986) have proved that within the Fermi liquid theory, the following relation is obeyed

$$z = d + \frac{\kappa}{\nu}. \tag{32}$$

Similar conclusions apply also to other quantities, such as the spin susceptibility χ_s, which should also remain bounded away from the critical point and acquire universal $T^{1/2}$ corrections within the metallic phase. Finally, the compressibility $\chi_c = \mathrm{d}n/\mathrm{d}\mu$ is generically expected to remain non-singular (finite) at the transition, since $n(\mu)$ is expected to be a smooth function, except at special filling fractions. For example, if one approaches (Mott 1990) a band or a Mott insulator, the compressibility vanishes as a precursor of the gap opening at the Fermi surface.

All the above expressions are quite general, and can be considered to be a phenomenological description (Castellani *et al* 1987) of disordered Fermi liquids. On the other hand, these relations tell us nothing about the specific *values* of the Landau parameters, or how they behave in the vicinity of the MIT. Perturbative RG calculations (Belitz and Kirkpatrick 1994, Castellani *et al* 1984, Finkel'stein 1983, 1984) based on the $2 + \varepsilon$ expansion have been used to make explicit predictions for the values of the critical exponents and scaling functions for different universality classes. Despite a great deal of effort invested in such calculations, the predictions of these perturbative RG approaches have not met almost any success in explaining the experimental data in the critical region of the MIT. We should emphasize, though, that limitations associated with these weak-coupling theories do not invalidate the potential applicability of Fermi liquid ideas *per se*. On the other hand, even in their most general form, these Fermi liquid considerations predict that thermodynamic response functions such as χ and γ remain finite at $T = 0$ even within the disordered metallic phase. All experiments that find a more singular behaviour of these quantities should therefore be regarded as violating the Fermi liquid theory—even when it is properly generalized to disordered systems.

2.3. Typical departures from Fermi liquid theory in real systems

The Fermi liquid theory describes the leading low energy excitations in a system of fermions. These, as we have seen, are weakly interacting quasi-particles characterized by several Landau parameters. Most remarkably, its validity is by no means limited to systems with weak interactions. In several materials, most notably heavy fermion systems, these many body renormalizations are surprisingly large (e.g. $m^*/m \sim 1000$), yet the Fermi liquid theory is known to apply at the lowest temperatures.

On the other hand, we should make it clear that precisely in such strongly correlated systems, the *temperature range* where Fermi liquid theory applies is often quite limited. The 'coherence temperature' T^* below which it applies is typically inversely proportional to the effective mass enhancement, and thus can be much smaller than the Fermi temperature. Above

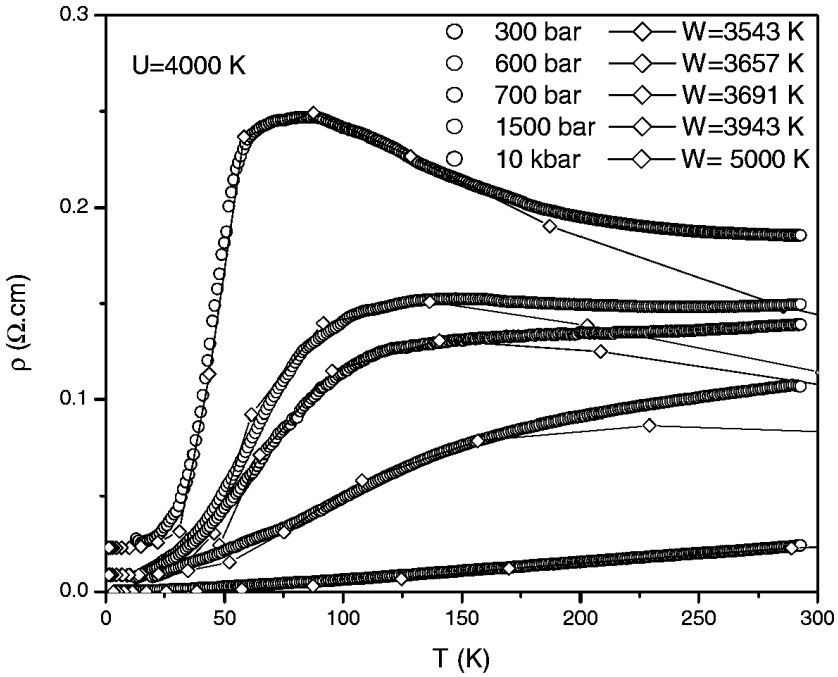

Figure 1. Temperature dependence of the resistivity at different pressures for a two-dimensional organic charge transfer κ–(BEDT–TTF)$_2$Cu[N(CN)$_2$]Cl, following Limelette *et al* (2003). The data (○) are compared to the DMFT predictions (◇), with a pressure dependence of the bandwidth as indicated.

T^* all physical quantities are dominated by large incoherent electron–electron scattering, and the Fermi liquid theory simply ceases to be valid. In many heavy fermion systems T^* is found to be comparable to the single-ion Kondo temperature, thus to typically be of order 10–10^2 K.

Another interesting class of materials, where clear departures from the Fermi liquid theory have been observed at sufficiently high temperatures, are systems close to the Mott transition. These include the transition metal oxides such as $(V_{1-x}Cr_x)_2O_3$ and chalcogenides such as $NiS_{2-x}Se_x$ (for a review see Imada *et al* (1998)). In a very recent work, similar behaviour was observed in another Mott system, a two-dimensional organic charge transfer salt κ–(BEDT–TTF)$_2$Cu[N(CN)$_2$]Cl (Limelette *et al* 2003). In this material the system can be pressure-tuned across the Mott transition, and on the metallic side the resistivity follows the conventional T^2 law below a crossover temperature $T^* \sim 50$ K. Above this temperature transport crosses over to an insulating-like form, reflecting the destruction of heavy quasi-particles by strong inelastic scattering (see figure 1). Clearly, the Fermi liquid theory does not apply above this coherence scale, but alternative dynamical mean-field theory (DMFT) approaches (Limelette *et al* 2003) proved very successful in quantitatively fitting the data over the entire temperature range.

The incoherent metallic behaviour could be even more important in disordered systems, where the coherence temperature T^* can be viewed as a random, position-dependent quantity $T^*(\boldsymbol{x})$. If this random quantity is characterized by a sufficiently broad probability distribution function $P(T^*)$, then the Fermi liquid theory may not be applicable in any temperature regime. In the following, we examine several physical systems where these phenomena are of potential importance, and then describe theoretical efforts to address these issues and produce a consistent physical picture of a disorder-driven NFL metallic state.

3. Phenomenology of NFL behaviour and experimental realizations

3.1. NFL behaviour in correlated systems with weak or no disorder

3.1.1. Heavy fermion materials near magnetic quantum critical points. Heavy fermion systems have been the focus of intense study over the last three decades as prime examples of strongly correlated electronic behaviour. These are metallic systems with an array of localized moments formed in the incompletely-filled f-shells of rare-earth or actinide elements. An important requirement for heavy fermion behaviour is that the f-shell electrons should be sufficiently close to a valence instability and therefore should have a fairly low ionization energy in the metallic host. This restricts the interesting behaviour to a few elements, most often Ce and U, but also Yb, Pr, Sm and some other less common cases. The proximity to a valence instability promotes the enhancement of the hybridization between the f-shell electrons and the conduction bands. This, in turn, leads to an antiferromagnetic (AFM) interaction between the f-shell moment and the local conduction electron spin density through essentially a super-exchange mechanism. This interaction is otherwise much too weak in the lanthanide or actinide series. The tendency towards local singlet formation through the Kondo effect serves to strongly suppress the ubiquitous magnetic order seen in most other intermetallics with localized f-moments. The hybridization of the metallic carriers with the virtual bound states in the strongly correlated f-shell leads to large effective mass renormalization factors (of order 10^2–10^3), hence the name 'heavy fermions'. There are many excellent reviews of heavy fermion physics to which the reader is referred to (Grewe and Steglich 1991, Hewson 1993, Lee *et al* 1986, Stewart 1984).

A great part of the interesting physics of these compounds is a result of competing tendencies: localization versus delocalization of the f-electrons and magnetic order versus Kondo singlet formation. This complex interplay provides the background where a wide variety of low temperature phases arise: there are metals, insulators (Aeppli and Fisk 1992) and superconductors (Heffner and Norman 1996). Furthermore, many exotic phenomena are also found, such as small moment antiferromagnetism (Buyers 1996), coexistence of superconductivity and magnetism and unconventional superconductivity with more than one phase (Heffner and Norman 1996).

Despite a bewildering zoo of correlated behaviour, in recent years a common theme has been the focus of attention. In many systems, the Néel temperature is low enough to be tuned to zero by an external parameter, such as pressure, chemical pressure (by alloying with an element with a different ionic radius) or applied magnetic field. The zero temperature phase transition between an antiferromagnet and a paramagnet as a function of the external parameter is called a quantum phase transition. If this happens to be a second order phase transition, the system will exhibit a diverging correlation length and we expect the critical behaviour to be classified in 'universality classes', much like thermal second order phase transitions (Continentino 1994, 2001, Sachdev 1999). Indeed, many heavy fermion systems can be tuned in just this way and a general theory of this so-called quantum critical point (QCP) has been sought vigorously. The possibility of such a quantum phase transition in heavy fermion systems is generally attributed to Doniach, who discussed it early on in the context of a one-dimensional effective model (Doniach (1977) see also Continentino *et al* (1989)). The associated phase diagram is thus usually referred to as the Doniach phase diagram. It is a natural consequence of the above mentioned competition between the Kondo singlet formation and the AFM order.

Besides the natural interest in a catalogue of the possible universality classes of such quantum phase transitions, the low temperature region in the vicinity of the zero temperature QCP is observed to be characterized by strong deviations from Landau's Fermi liquid theory.

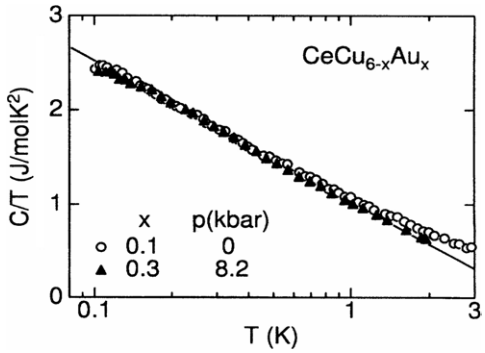

Figure 2. Specific heat of $CeCu_{6-x}Au_x$ showing the logarithmic divergence at criticality. Data are from Bogenberger and Löhneysen (1995).

A recent exhaustive review of the experimental data on many heavy fermion systems showing NFL behaviour, including but not restricted to those where QCP physics seems relevant, can be found in (Stewart 2001). In many cases, the following properties are observed:

- A diverging specific heat coefficient, often logarithmically, $C(T)/T \sim \gamma_0 \log(T_0/T)$ (see figure 2).
- An anomalous temperature dependence of the resistivity, $\rho(T) \sim \rho_0 + AT^{\alpha}$, where $\alpha < 2$.
- An anomalous Curie–Weiss law, $\chi^{-1}(T) \sim \chi_0^{-1} + CT^{\beta}$, where $\beta < 1$.

Despite an intensive theoretical onslaught, a coherent theoretical picture is still lacking (Coleman *et al* 2001, Millis 1999). The natural theoretical description of the metallic QCP would seem to be one where the critical modes are subject to the interactions generated in the presence of a Fermi sea, as in the old paramagnon theory (Berk and Schrieffer 1966, Doniach and Engelsberg 1966, Izuyama *et al* 1963). In particular, the low-frequency dependence of the RPA susceptibility leads to the overdamping of the critical modes, which can decay into particle–hole pairs (Landau damping). As emphasized by Hertz, this increases the effective dimensionality of the critical theory by the dynamical exponent z: $d_{\text{eff}} = d + z$ (Hertz (1976) see also Béal-Monod and Maki (1975)). The dynamical exponent z determines the relation between energy and length scales $E \sim L^{-z}$. Landau damping leads to $\omega \sim -\mathrm{i}k^3 (z = 3)$ in clean itinerant ferromagnets and $\omega \sim -\mathrm{i}q^2$ $(z = 2)$ in clean itinerant antiferromagnets (q being the deviation from the ordering vector), the latter being the case of interest in heavy fermion materials. As a result, three-dimensional systems are generically above their upper critical dimension, the dimension above which mean field exponents become exact (equal to 4 in the usual thermal case for Ising and Heisenberg symmetries). In the effective critical theory, the critical modes are the more weakly interacting the longer the length scales, rendering the theory asymptotically tractable (Continentino 1994, Hertz 1976, Millis 1993, Moryia 1985). This is the weak coupling spin density wave approach, sometimes named the Hertz–Millis scenario. In particular, at the AFM QCP the specific heat is non-singular, $C(T)/T \sim \gamma_0 - a\sqrt{T}$. Furthermore, scattering off the incipient AFM order is singular only along *lines* on the Fermi surface which are connected by the AFM ordering wave vector $\boldsymbol{Q}$ ('hot spots'). These are effectively short-circuited by the remainder of the Fermi surface ('cold spots'), which are hardly affected by criticality, leading in clean samples to a Fermi liquid response $\rho(T) \sim \rho_0 + AT^2$ (Hlubina and Rice 1995). These results cannot explain the observed properties of heavy fermion systems close to quantum criticality. Another consequence of the higher effective dimensionality is the fact that the theory does not obey hyperscaling.

Hyperscaling is usually associated with the Josephson scaling law: $2 - \alpha = \nu d$, where α is the specific heat critical exponent and ν is the correlation length exponent (Goldenfeld 1992). In a quantum phase transition, d should be replaced by the effective dimensionality d_{eff}. However, the most important consequence of the violation of hyperscaling is the absence of ω/T scaling in dynamical responses which, in the AFM case, should involve the combination $\omega/T^{3/2}$ instead (Coleman *et al* 2001). However, ω/T scaling is observed in a quantum critical heavy fermion system $CeCu_{5.9}Au_{0.1}$ (Schröder *et al* 1998, 2000). These results present major difficulties for the description of the NFL behaviour observed near a QCP in heavy fermion systems.

There have been proposals for amending the Hertz–Millis scenario in an effort to account for the experimental results. In particular, we mention the effect of disorder close to a QCP (Rosch 1999, 2000) and the possible two-dimensional character of the spin fluctuation spectrum (Mathur *et al* 1998, Rosch *et al* 1997, Schröder *et al* 2000, Stockert *et al* 1998). Disorder acts by relaxing the momentum conservation which restricts strong scattering to the 'hot lines', effectively enhancing their influence on transport. We stress that in this case disorder plays only an ancillary role and is not the driving mechanism for NFL behaviour. On the other hand, a two-dimensional spin fluctuation spectrum places the system at the upper critical dimension and is able to account for some of the observed critical exponents. None of these approaches, however, are able to account for all the available data. Alternatively, radical departures from the spin density wave critical theory have been proposed. In one of these, the long time dynamics of the localized moment acquires a non-trivial power-law dependence, whereas the spatial fluctuations retain the usual mean-field form (Si *et al* 2001). This has been dubbed 'local quantum criticality' and is a natural scenario to incorporate the ω/T scaling with non-trivial exponents observed over much of the Brillouin zone. Its current form, however, still relies on a two-dimensional spin fluctuation spectrum. In another approach, there is a phase transition from the usual paramagnetic heavy Fermi liquid state to a non-trivial state where the localized spins form a 'spin liquid' weakly coupled to the conduction sea (Senthil *et al* 2003, 2004). An interesting consequence of the latter proposals would be a drastic reduction of the Fermi surface volume across the transition, presumably measurable through the Hall constant (Coleman *et al* 2001). This is still an open arena where perhaps new ideas and new experiments will be necessary before further progress can be achieved.

3.1.2. Marginal Fermi liquid behaviour of high T_c superconductors HTS have taken the centre stage of modern condensed matter theory ever since their discovery in 1987. Most of their features continue to defy theoretical understanding, and even the origin of superconducting pairing in these materials remains controversial. Most remarkably, many features of the superconducting state seem to be less exotic than the extremely unusual behaviour observed in the normal phase. In fact, HTS materials presented one of the first well-documented examples featuring deviations from Fermi liquid phenomenology, initiating much theoretical activity and debate. On a phenomenological level, behaviour close to 'optimal doping' (the regime where the superconducting transition temperature T_c is the highest) seems to display 'marginal Fermi liquid' (MFL) behaviour. The most prominent feature observed in experiments is the linear resistivity (see figure 3), which at optimal doping persists in an enormous temperature range from a few kelvin all the way to much above room temperature (Takagi *et al* 1992). A phenomenological model describing the MFL behaviour of cuprates has been put forward a long time ago (Varma *et al* 1989), but its microscopic origin remains highly controversial. To our knowledge no microscopic theory has so far been able to provide a microscopic explanation for this puzzling behaviour, despite years of effort and hundreds of papers published on the subject.

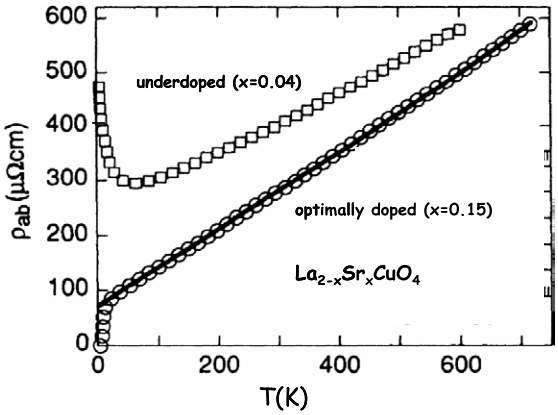

Figure 3. Normal state in-plane resistivity of optimally doped cuprate $La_{2-x}Sr_xCuO_4$ (○). The striking linear temperature dependence is observed around optimal doping, persisting far above room temperature. Deviations from this behaviour are seen both below (□) and above optimal doping. Data are from Takagi *et al* (1992).

Much theoretical and experimental effort over the years concentrated on unravelling the microscopic origin of the superconducting pairing, a mechanism that, one hopes, would also explain the puzzling features of the normal state. Most proposed scenarios concentrated on exotic many-body mechanisms in an assumed homogeneous conductor, thus ignoring the role of disorder or the possible emergence of some form of random ordering. Many excellent reviews exist detailing the current status of these research efforts and, given our focus on the effects of disorder, will not be elaborated here. In the following, we follow Panagopoulos and Dobrosavljević (2005) and briefly review those experimental and theoretical efforts suggesting that the homogeneous picture may be incomplete.

The first evidence of random ordering in HTS was obtained from the observation of low temperature spin-glass ordering in the pseudo-gap phase. At doping concentration x larger than a few per cent, AFM ordering is suppressed, but the short range magnetic order persists. The low-field susceptibility displays a cusp at temperature $T = T_g$ and a thermal hysteresis below it, characteristic of a spin glass transition. At $T < T_g$ the material displays memory effects like 'traditional' spin glasses and is described by an Edwards–Anderson order parameter (Chou *et al* 1995). Such magnetic measurements are not possible in the superconducting regime ($x > x_{sc}$), but muon spin relaxation (μSR) has been successful in identifying the freezing of electronic moments under the superconducting dome of various HTS systems (Kanigel *et al* 2002, Panagopoulos *et al* 2002, 2003, Sanna *et al* 2004). From these studies one may conclude that in this regime glassiness coexists with superconductivity on a microscopic scale throughout the bulk of the material. Most remarkably, the observed spin-glass phase seems to end at a QCP precisely at optimal doping ($x = x_{opt}$), suggesting that NFL behaviour in the normal phase may be related to the emergence of glassy ordering in the ground state (figure 4).

The correlation between spin order and charge transport is further emphasized by experiments in high magnetic fields (Boebinger *et al* 1996) where bulk superconductivity was suppressed, revealing information about low-T charge transport in the normal phase. The resistivity data show a crossover in $\rho_{ab}(T)$ from metallic to insulating-like behaviour (resistivity minimum) at a characteristic temperature T^* which, similarly to T_g, decreases upon doping, and seems to vanish at the putative QCP. In addition, a crossover temperature T_m at $x > x_{opt}$ separating MFL transport at $T > T_m$ from more conventional metallic behaviour at $T < T_m$ also seems to drop (Naqib *et al* 2003) to very small values around optimal doping (see figure 4).

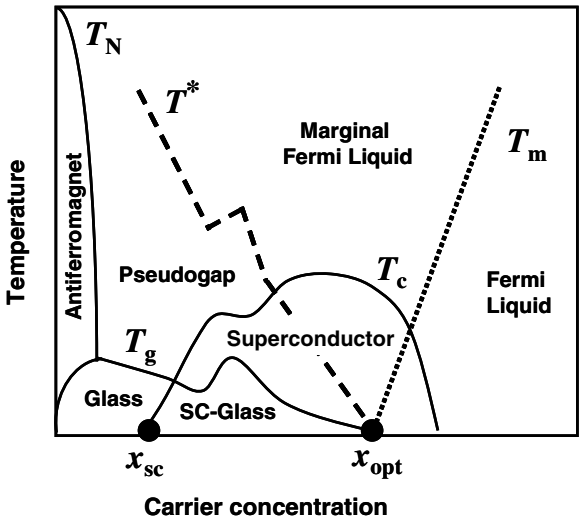

Figure 4. Phase diagram of the archetypal HTS, following Panagopoulos and Dobrosavljević (2005). T_N is the Néel temperature, T_F and T_g the onset of short range freezing to an electronic glass and T_c the superconducting transition temperature. At $x < x_{sc}$ the material is a glassy insulator. At $x_{sc} < x < x_{opt}$ a microscopically inhomogeneous conducting glassy state emerges, with intercalated superconducting and magnetic regions. At $x = x_{opt}$ the system experiences a quantum glass transition and at $x > x_{opt}$ the material transforms into a homogeneous metal with BCS-like superconducting properties. The superfluid density is maximum at $x = x_{opt}$. The crossover scales T^* and T_m characterizing normal-state transport (see text for details) vanish at the quantum glass transition.

At $x > x_{opt}$ the ground state becomes metallic and homogeneous, with no evidence for glassiness or other form of nano-scale heterogeneity (Balakirev *et al* 2003, Boebinger *et al* 1996, Panagopoulos *et al* 2002, 2003). Remarkable independent evidence that a QCP is found precisely at $x = x_{opt}$ is provided by the observation of a sharp change in the superfluid density $n_s(0) \sim 1/\lambda^2_{ab}(0)$ (where $\lambda_{ab}(0)$ is the absolute value of the in-plane penetration depth). At $x > x_{opt}$, $n_s(0)$ is mainly doping independent (figure 4), while the T-dependence is in good agreement with the BCS weak-coupling d-wave prediction (Panagopoulos *et al* 2003). At dopings below the quantum glass transition $n_s(0)$ is rapidly suppressed (note the enhanced depletion near $x = 1/8$ precisely where T_g and T^* are enhanced) and there is a marked departure of $n_s(T)$ from the canonical weak coupling curve (Panagopoulos *et al* 2003). All these results provide strong evidence for a sharp change in ground state properties at $x = x_{opt}$, and the emergence of vanishing temperature scales as this point is approached—just as one expects at a QCP.

The formation of an inhomogeneous state below optimal doping is consistent with those theoretical scenarios that predict phase separation (Dagotto 2002, Gor'kov and Sokol 1987, Kivelson *et al* 2003) at low doping. Coulomb interactions, however, enforce charge neutrality and prevent (Kivelson *et al* 2003) global phase separation; instead, the carriers are expected (Schmalian and Wolynes 2000) to segregate into nano-scale domains—to form a stripe/cluster glass (Schmalian and Wolynes 2000). As quantum fluctuations increase upon doping (Dobrosavljević *et al* 2003, Pastor and Dobrosavljević 1999), such a glassy phase should be eventually suppressed at a QCP, around $x = x_{opt}$. These ideas find striking support in very interesting STM studies revealing nano-scale domains forming in the underdoped phase, as recently reported by several research groups.

This scenario is consistent with another mysterious aspect of normal state transport in the weakly underdoped regime ($x \lesssim x_{\rm opt}$). Here, DC transport has a much weaker (Boebinger *et al* 1996) (although still insulating-like) temperature dependence. However, the observed $\log T$ resistivity upturn in this region has been shown (Boebinger *et al* 1996) to be inconsistent with conventional localization/interaction corrections which could indicate an insulating ground state. Instead, estimates (Beloborodov *et al* 2003) reveal this behaviour to be consistent with that expected for metallic droplet charging/tunnelling processes, as seen in quantum dots and granular metals (Beloborodov *et al* 2003). These results suggest that in this regime HTS are inhomogeneous metals, where conducting droplets connect throughout the sample, and a MIT in the normal phase happens *exactly* at $x = x_{\rm sc}$. At lower densities the conducting droplets remain isolated, and the material may be viewed as an insulating cluster or stripe glass. As carrier concentration increases they connect and the carriers are free to move throughout the sample, forming filaments or 'rivers'. This is, in fact, the point where free carriers emerge in the Hall effect data (Balakirev *et al* 2003) and phase coherent bulk superconductivity arises at $x > x_{\rm sc}$. This observation suggests that it is the *inhomogeneous* nature of the underdoped glassy region which controls and limits the extent of the superconducting phase at low doping.

At this time we still do not know if the formation of such inhomogeneous states is indeed a fundamental property of HTS materials, or merely a by-product of strong frustration and extrinsic disorder. Nevertheless, the emerging evidence seems compelling enough by itself, as it opens the possibility that inhomogeneities and glassy ordering may not be disregarded as the possible origin of NFL behaviour in the cuprates.

3.2. Disorder-driven NFL behaviour in correlated systems: doped semiconductors

Doped semiconductors (Shklovskii and Efros 1984) have been studied for a long time, not the least because of their enormous technological applications, but also because of their relatively simple chemical composition allowing the possibility for simple theoretical modelling (Shklovskii and Efros 1984). Typically, they are used to fabricate transistors and other devices, which essentially can be used as electrical switches in logical integrated circuits. For this reason, much attention has been devoted to understanding their behaviour close to the MIT (Mott 1990), which typically takes place when the Bohr radius a_0 of the donor atoms becomes comparable to the typical carrier separation. Most applications are based on doped silicon devices, where the critical concentration $n = n_c \sim 10^{18}\,{\rm cm}^{-3}$. Since the Fermi energy of the carriers in this regime is fairly low ($T_F \approx 100\,{\rm K}$) (Paalanen and Bhatt 1991, Sarachik 1995), the behaviour at room temperatures is easy to understand using conventional solid-states theories (Paalanen and Bhatt 1991, Sarachik 1995), and as such it has been qualitatively and even quantitatively understood for many years. The evolution of the physical properties as a function of n, however, is smooth at elevated temperatures due to thermal activation.

3.2.1. Metal–insulator transition. A sharp MIT is seen only at the lowest accessible temperatures, and it is in this regime where most of the interesting physics associated with quantum effects comes into play.

In the last 30 years, a great deal of effort has been devoted to the study of detailed properties (Paalanen and Bhatt 1991, Sarachik 1995) of doped semiconductors in the critical region near the MIT. Early experiments (Paalanen and Bhatt 1991, Sarachik 1995) concentrated on transport measurements, and reported a temperature dependence of the conductivity of the form

$$\sigma(T) = \sigma_0 + m\sqrt{T}, \tag{33}$$

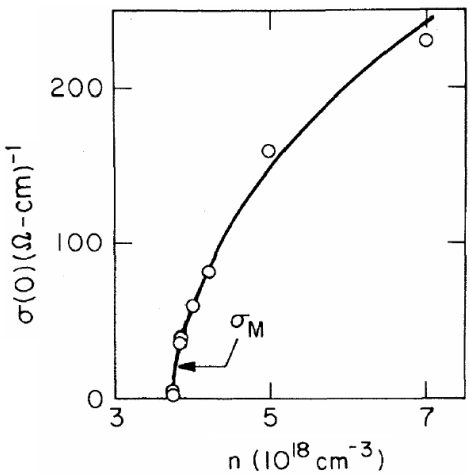

Figure 5. Critical behaviour of the conductivity extrapolated to $T \to 0$ for uncompensated Si : P. Note a sharp power-law behaviour with exponent $\mu \approx 1/2$, extending over a surprisingly large concentration range. The results reported are taken from the classic stress-tuned experiment of Paalanen, Rosenbaum and Thomas (Rosenbaum *et al* 1980).

where σ_0 and m are parameters that generally depended on the carrier concentration n and the magnetic field B. Using these observations, one typically extrapolated these results to $T = 0$, and examined the critical behaviour as the transition is approached. In agreement with scaling predictions (Abrahams *et al* 1979, Belitz and Kirkpatrick 1994, Finkel'stein 1983, 1984), power-law behaviour was reported (Paalanen and Bhatt 1991, Rosenbaum *et al* 1980, Sarachik 1995) of the form (see figure 5)

$$\sigma(T \to 0) \sim (n - n_c)^{\mu}. \tag{34}$$

The conductivity exponent was generally reported to take the value $\mu \approx 0.5$ for uncompensated materials (e.g. only donor dopants present as in Si : P or only acceptors as in Si : B) in zero magnetic field. In the presence of a magnetic field experiments (Paalanen and Bhatt 1991, Sarachik 1995) showed $\mu \approx 1$, and the same behaviour was reported in compensated materials (both donors and acceptors present, such as Si : P, B). These findings had the general form expected from Fermi liquid considerations and scaling approaches, although the specific values for the conductivity exponent μ were not easy to understand from the available microscopic theories (Belitz and Kirkpatrick 1994). It was, however, generally felt (Lee and Ramakrishnan 1985) that the observed compensation dependence reflects the enhanced role of the electronic correlations in the uncompensated case, where the dopant impurity band is half-filled, and one would expect a Mott MIT in the absence of disorder.

3.2.2. Thermodynamic anomalies. If the proximity to the Mott transition (Mott 1990) is an important consideration, then magnetism may be expected to have unusual features, since the electrons turn into local magnetic moments (Mott 1990) in the Mott insulator. Indeed, many strongly correlated metals near the Mott transition (e.g. transition metal oxides (Mott 1990) such as V_2O_3) show large spin susceptibility and specific heat enhancements, which are known to be dominated by local magnetic moment (Quirt and Marko 1971, Ue and Maekawa 1971) physics.

To test these ideas and also to explore the validity of generalized Fermi liquid considerations (Castellani *et al* 1987), subsequent experiments turned to exploring the

thermodynamic behaviour of the system. On the insulating side, one expects (Shklovskii and Efros 1984) the donor electrons to be tightly bound to ionic centres, thus behaving as spin 1/2 local magnetic moments. Because of the overlap of the donor electron wave-functions, AFM exchange is generated between pairs of such magnetic moments, which takes the form (Shklovskii and Efros 1984)

$$J(R) = J_0 \exp\left\{-\frac{R}{a}\right\}. \tag{35}$$

The parameters J_0 and a (the Bohr radius) can be calculated with precision for all the known shallow impurity centres. Since the donor atoms are randomly distributed throughout the host matrix, the insulator can therefore be described as a random quantum antiferromagnet, which at low temperature can be expected to exhibit random spin freezing, i.e. spin glass behaviour. Despite considerable effort, the search for such spin glass ordering in the insulating phase has proven unsuccessful (Quirt and Marko 1971, Ue and Maekawa 1971). Instead, a very unusual thermodynamic response has been observed, with a diverging spin susceptibility and sub-linear specific heat as a function of temperature, namely,

$$\chi(T) \sim T^{\alpha-1}, \tag{36}$$

$$\gamma(T) = \frac{C}{T} \sim T^{\alpha-1}. \tag{37}$$

The exponent α was found (Paalanen and Bhatt 1991, Sarachik 1995) to weakly depend on the details of a specific system, generally taking the value $\alpha \approx 0.4$ for uncompensated and $\alpha \approx 0.3$ for compensated systems. Soon after the initial observations, this behaviour on the insulating side was qualitatively and even quantitatively explained by the 'random singlet' theory of Bhatt and Lee (1981, 1982) (see section 4.1.2).

3.2.3. Two-fluid model. The real surprise, however, came from the observation that essentially the same low temperature anomalies persisted (Paalanen *et al* 1988, Schlager and Lohneysen 1997) well into the metallic side of the transition. The data were fitted by a phenomenological 'two-fluid' model (Paalanen *et al* 1988, Quirt and Marko 1971), which assumed coexistence of local magnetic moments and itinerant electrons, so that (see figure 6)

$$\frac{\gamma}{\gamma_0} = \frac{m^*}{m_0^*} + \left(\frac{T_0}{T}\right)^{\alpha-1}, \tag{38}$$

$$\frac{\chi}{\chi_0} = \frac{m^*}{m_0^*} + \beta\left(\frac{T_0}{T}\right)^{\alpha-1}. \tag{39}$$

In these expressions, the first term is ascribed to itinerant electrons which are assumed to form a Fermi liquid with band mass m_0^* and effective mass m^*. The second term describes the contribution from localized magnetic moments, where the deviation from the Curie law ($\alpha > 1$) is viewed as a result of spin–spin interactions, similarly as in the Bhatt–Lee theory (Bhatt and Lee 1981, 1982). The exponent α was found (Paalanen *et al* 1988, Schlager and Lohneysen 1997) to have a weak density dependence, and essentially takes the same value as on the insulating side. The fitted values of the parameter T_0 can be used to estimate the relative fraction of the electrons that form the local moments. According to these estimates, between 10% and 25% of the electrons contribute to the formation of these localized moments in the vicinity of the transition.

These experimental findings represented drastic violations of the Fermi liquid predictions. Within the Fermi liquid picture (Castellani *et al* 1987), in any metal the local magnetic moments should be screened by conduction electrons through the Kondo effect or spin–spin interactions,

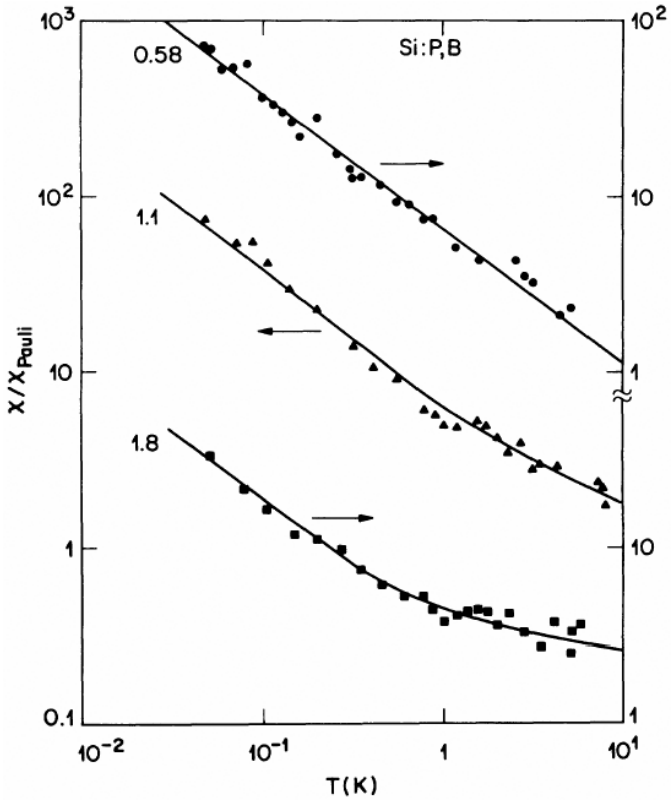

Figure 6. Temperature dependence of the normalized susceptibility for three different Si : P, B samples, with electron densities $n/n_c = 0.58, 1.1, 1.8$, as reported by Hirsch *et al* (1992). The low temperature behaviour looks qualitatively the same in a very broad concentration interval, persisting well into the metallic side of the transition, in contrast to Fermi liquid predictions.

so that γ and χ should remain finite as $T \to 0$. Their precise value should be a function of the corresponding Fermi liquid parameters m^* and F_0^a (see equations (4) and (5)), and as such should have a singular behaviour only as a phase transition is approached, not within the metallic phase. In contrast, the singularities observed in these materials display *no observable anomaly* (Hirsch *et al* 1992) as one crosses from the metallic to the insulating side. In fact, the thermodynamic data on their own have such a weak density dependence, that just on their basis one could not even determine the critical concentration n_c. This behaviour is in sharp contrast to that observed in conventional Fermi liquids (e.g. clean heavy fermion compounds), where sharp anomalies in both the transport and the thermodynamic properties are clearly seen and well documented.

Since the physics associated with precursors of local moment magnetism is typically associated with strong correlation effects, it seems very likely (Lee and Ramakrishnan 1985) that the surprising behaviour of doped semiconductors has the same origin. If this is true, then the conventional weak-coupling approaches (Belitz and Kirkpatrick 1994) to electronic correlations seem ill-suited to describe this puzzling behaviour. These ideas seem in agreement with the original physical picture proposed by Mott (1990), where the Coulomb repulsion is viewed as a major driving force behind the MIT. Indeed, even early attempts to address the role of correlation effects based on Hubbard models by Milovanović *et al* (1989) were sufficient

to indicate the emergence of disorder-induced local moment formation around the transition region. This theory was too simple to address more subtle questions such as the interaction of such local moments and the conduction electrons (Bhatt and Fisher 1992, Dobrosavljević *et al* 1992, Lakner *et al* 1994), but later work (Dobrosavljević and Kotliar 1993, 1994, 1997, 1998) based on extended DMFT approaches provided further support to the strong correlation picture.

3.2.4. Asymptotic critical behaviour or mean-field scaling? One more important aspect of the experimental data is worth emphasizing. Several well defined features of these materials have been clearly identified, both with respect to the thermodynamic and the transport properties. These include a sharply defined power-law behaviour of the conductivity, with an exponent μ being a strong function of the magnetic field or the deviation from half-filling and a smooth but singular thermodynamic response. All of these features are clearly defined over a *very broad* range of parameters covering, for example, in excess of 100% of the reduced concentration $\delta n = (n - n_c)/n_c$. Such behaviour provides a strong indication that the puzzling observations are not associated with complications arising within the asymptotic critical region of a second-order phase transition. In contrast, the behaviour in the immediate vicinity of the critical point is still a subject of some controversy (Stupp *et al* 1993, Waffenschmidt *et al* 1999), and may be dominated by extrinsic (e.g. self-compensation) effects (Itoh *et al* 2004).

Therefore, theories designed to capture only long wavelength and low energy excitations do not seem likely to be sufficient in describing the most puzzling features of the experiments. To our knowledge, none of the microscopic predictions of such 'field theoretical' approaches (Belitz and Kirkpatrick 1994) have been experimentally verified; most observations seem drastically different from what one expects based on these theories. Such robust and systematic behaviour observed in a broad parameter range is more reminiscent of mean-field behaviour, which typically describes most of the experimental data around phase transitions, except in extremely narrow critical regions. Typically, one has to approach the critical point within a reduced temperature $t = (T - T_c)/T_c$ of the order of 10^{-3} or less in order to observe the true asymptotic critical behaviour. Unfortunately, an appropriate mean-field description is not available yet for MIT, despite years of effort. In our opinion, what is needed to correctly describe the observed NFL features are approaches that can explicitly address the interplay of strong correlations and disorder.

3.3. Disorder-driven NFL behaviour in correlated systems: disordered heavy fermion systems

Anomalous NFL behaviour has been observed in a large number of non-stoichiometric heavy fermion compounds. An ample review of the experimental situation until 2001 can be found in Stewart's review (Stewart 2001). Since the main focus of this review is on the theoretical approaches to disordered NFL systems, we will not dwell much on the available data on these systems. We will, however, give a bird's eye view of the subject.

We would first like to distinguish between the main systems of interest here and those whose detailed description requires the consideration of disorder effects but in which disorder is most likely *not* the driving mechanism of NFL behaviour. This is especially important in those cases where the proximity to a QCP seems to be the origin of the anomalous behaviour. Indeed, the external tuning parameter in these systems is often chemical pressure, namely, the substitution of small amounts of an isoelectronic element with a slightly larger ionic radius with the aim of expanding the lattice. Non-stoichiometry is then unavoidable but its effects are considered secondary when compared with, say, the variation of the couplings between neighbouring atoms. Some studies of these effects, especially on transport have been

undertaken (Rosch 1999, 2000). A caveat is in order, however. A well-known criterion due to Harris establishes that if $\nu d < 2$, where ν is the correlation length critical exponent of the clean system, then disorder is a *relevant* perturbation, i.e. the actual critical behaviour is modified in the dirty system (Harris 1974). The clean Hertz–Millis theory discussed in section 3.1.1 has a mean-field exponent $\nu = 1/2$ and the Harris criterion indicates that disorder is relevant. Therefore, even in this case it is not clear that disorder does not play a more crucial role. We will discuss the interplay of disorder and quantum critical behaviour in both insulating and metallic systems in section 4.2.

There is of course no clear-cut way to separate systems whose behaviour is governed by the proximity to a QCP and those in which disorder is the driving mechanism of NFL behaviour. A practical rule of thumb is to check whether the anomalous behaviour exists in regions of the phase diagram distant from a clean quantum phase transition, typically antiferromagnetism. This rule has been adopted, for instance, by Stewart in his review (Stewart 2001). Another indication of the importance of disorder in heavy fermion alloys is a large value of the residual resistivity $\rho_0 = \rho(T \to 0)$. Clean heavy fermion systems are characterized by large amounts of scattering of the metallic carriers by the localized magnetic moments at high temperatures, the so-called Kondo scattering (Kondo 1964). As the temperature is lowered, however, lattice translation invariance sets in and a precipitous drop of the resistivity, by orders of magnitude, occurs below the so-called coherence temperature $T_{\rm coh}$, which is typically a few tens of kelvin. This delicate state is easily destroyed by disorder and large incoherent Kondo scattering is then able to survive to the lowest temperatures leading to values of ρ_0 on the order of hundreds of micro-ohms-centimetres. NFL behaviour is observed in a number of heavy fermion alloys where no trace of coherence remains. In these cases, besides a large value of ρ_0, the leading low-temperature behaviour is often *linear* in temperature with a negative coefficient: $\rho(T) \approx \rho_0 - AT$, in sharp contrast to the Fermi liquid behaviour of dilute Kondo impurities $\rho(T) \approx \rho_0 - BT^2$ (Nozières 1974, Wilson 1975). Prominent examples of systems where this linear in-temperature behaviour is observed are $U_xY_{1-x}Pd_3$ (Andraka and Tsvelik 1991, Ott *et al* 1993, Seaman *et al* 1991), $UCu_{5-x}Pd_x$ ($x = 1$ and 1.5, see figure 7) (Andraka and Stewart 1993, Chau and Maple 1996, Weber *et al* 2001), $UCu_{5-x}Pt_x$ ($x = 0.5$, 0.75 and 1) (Chau and Maple 1996, Stewart 2001), $U_{1-x}Th_xPd_2Al_3$ (Maple *et al* 1995), $Ce_{0.1}La_{0.9}Cu_{2.2}Si_2$ (Andraka 1994), UCu_4Ni (de la Torre *et al* 1998), URh_2Ge_2 (as grown) (Süllow *et al* 2000) and U_2PdSi_2 (Li *et al* 1998). The last two systems and those of the form UCu_4M, although nominally stoichiometric are actually crystallographically disordered, as evidenced by wide NMR lines (Bernal *et al* 1995), μSR (MacLaughlin *et al* 1998) and EXAFS measurements (Bauer *et al* 2002, Booth *et al* 1998, 2002).

Besides the anomalous transport, heavy fermion alloys also exhibit NFL thermodynamic properties. Singular behaviour is observed in both the specific heat coefficient $\gamma(T) = C(T)/T$ and the magnetic susceptibility $\chi(T)$ (see figure 8). Ambiguities in the determination of the exact singularity are common since the available temperature range is often not very broad. Therefore, there are conflicting claims of logarithmic [$\sim \log(T_0/T)$] or power law [$\sim T^{\lambda-1}$] divergences, a small value of λ in the latter being very difficult to discern from the former dependence. Once again, we refer the reader to Stewart's review (Stewart 2001) for a summary of the published results. In that paper, the author also presents a great deal of data previously published as being well-described by a logarithmic dependence, replotted and fitted to a weak power law with a small λ. These discrepancies highlight the difficulty of extracting the nature of the singular behaviour from a limited range of temperatures. The only reliable way to resolve these ambiguities is through the determination of error bars for the exponents (as in classical critical phenomena). Unfortunately, this has not, to our knowledge, been attempted.

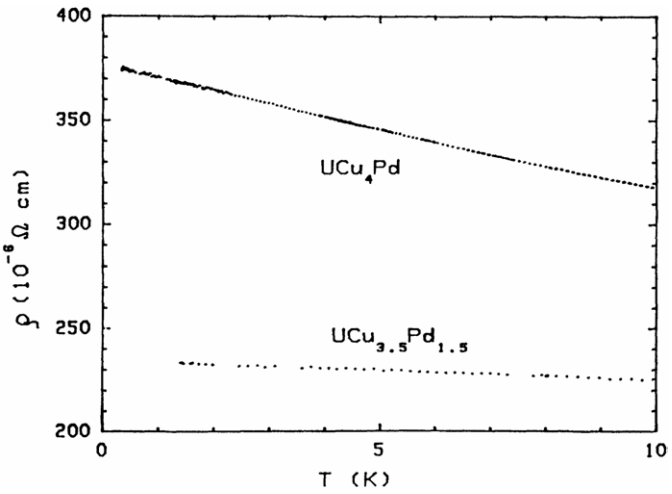

Figure 7. Anomalous NFL resistivity of $UCu_{5-x}Pd_x$. Data are from Andraka and Stewart (1993).

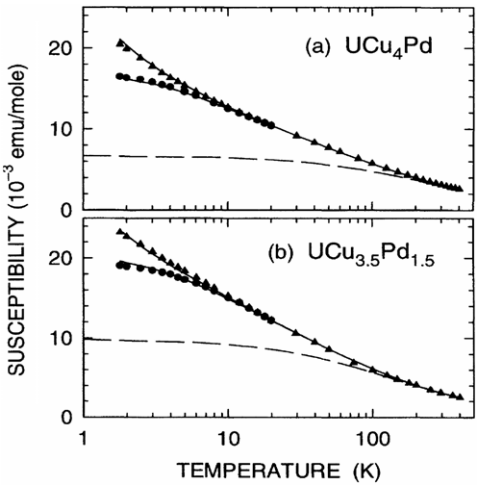

Figure 8. Anomalous NFL magnetic susceptibility of $UCu_{5-x}Pd_x$ for different applied fields: $H = 5$ kOe (▲) and $H = 50$ kOe (●). Data are from Bernal *et al* (1995).

Other experimental probes have also revealed NFL behaviour in heavy fermion alloys. Optical conductivity studies have shown that the frequency dependence of the scattering rate at low temperatures is linear as opposed to the quadratic dependence predicted by the Fermi liquid theory. This behaviour has been seen in $UCu_{5-x}Pd_x$ (Degiorgi and Ott 1996), $U_xY_{1-x}Pd_3$ (Degiorgi *et al* 1995) and $U_{1-x}Th_xPd_2Al_3$ (Degiorgi *et al* 1996 see also the review Degiorgi (1999) for a more detailed discussion). Furthermore, neutron scattering data on $UCu_{5-x}Pd_x$ have revealed ω/T scaling in the dynamical spin susceptibility over a broad temperature and frequency range without any significant wave vector dependence (other than the trivial Uranium atom form factor) (Aronson *et al* 1995, 2001). Finally, we should mention extensive NMR and μSR work on these systems (Büttgen *et al* 2000, Liu *et al* 2000, MacLaughlin *et al* 1996, 2000, 2001, 2002a, 2002b, 2003, 2004). These studies, together with the already cited EXAFS

technique (Bauer *et al* 2002, Booth *et al* 1998, 2002), are invaluable tools to access spatial fluctuations of local quantities, which are inevitably introduced by disorder.

3.4. Disorder-driven NFL behaviour in correlated systems: metallic glass phases

3.4.1. Metallic glass phases. Although usually not close to an ordered magnetic phase, disordered heavy fermion systems very often show spin-glass freezing. This is, perhaps, not too surprising given the interplay of disorder, local magnetic moments and Ruderman–Kittel–Kasuya–Yosida (RKKY)-induced frustration in these systems. However, freezing temperatures are surprisingly low given the large concentrations of magnetic moments. A possible explanation is the Kondo compensation by the conduction electrons, which tends to favour a paramagnetic heavy-fermion state. Related to this is the proposal that the QCP separating the heavy-fermion paramagnet and the spin glass phase is at the origin of the NFL behaviour of these alloys (see section 4.3.2). Spin-glass behaviour is usually detected by the difference between zero-field-cooled and field-cooled magnetic susceptibility curves. Direct measurements of the spin dynamics in μSR experiments have also proven to be a useful tool. Examples of systems where spin-glass phases have been detected are $Y_{1-x}U_xPd_3$ ($x \gtrsim 0.2$) (Gajewski *et al* 1996), $UCu_{5-x}Pd_x$ ($x \gtrsim 1$) (Andraka and Stewart 1993, MacLaughlin *et al* 2001, Vollmer *et al* 2000), URh_2Ge_2 (Süllow *et al* 2000), U_2PdSi_2 (Li *et al* 1998), $Ce_{0.15}La_{0.85}Cu_2Si_2$ (Andraka 1994), $U_{0.07}Th_{0.93}Ru_2Si_2$ (Stewart 2001), $U_{1-x}Y_xAl_2$ ($0.3 \leqslant x \leqslant 0.7$) (Mayr *et al* 1997) and $U_2Cu_{17-x}Al_x$ ($x = 8$) (Pietri *et al* 1997). From these numerous examples, it becomes clear that spin-glass ordering is a common low-temperature fate of heavy fermion alloys. It is likely that a complete theoretical picture of the NFL behaviour of these systems will have to encompass the spin-glass state.

In many of these systems one can tune through a $T = 0$ spin glass transition by varying parameters such as doping pressure. In such cases one expects new physics associated with a spin-glass–paramagnet QCP, but very few experiments exist where systematic studies of this regime have been reported in heavy fermion systems. On general grounds, one may expect such disorder-driven QCPs to have a more complicated form than in the clean cases. In recent works, theoretical scenarios have been proposed (Dobrosavljević and Miranda 2005, Vojta 2003, Vojta and Schmalian 2004), suggesting that rare events and dissipation may 'smear' such phase transitions, possibly making it difficult even to detect the precise location of the critical point. More experimental work is urgently needed to settle these interesting issues, especially using the local probes (STM, NMR, μSR) which are better suited to characterize such strongly inhomogeneous systems.

3.4.2. Glassiness in the charge sector. An interesting alternative to spin glass ordering is offered by the possibility of glassy freezing of charge degrees of freedom. In many disordered electronic systems (Lee and Ramakrishnan 1985), electron–electron interactions and disorder are equally important, and lead to a rich variety of behaviours which remain difficult to understand. Their competition often leads to the emergence of many metastable states and the resulting history-dependent glassy dynamics of electrons. Theoretically, the possibility of glassy behaviour in the charge sector was anticipated a long time ago (Efros and Shklovskii 1975) in situations where the electrons are strongly localized due to disorder. In the opposite limit, for well-delocalized electronic wave functions, one expects a single well-defined ground state and absence of glassiness. The behaviour in the intermediate region has proved more difficult to understand, and at present little is known as to the precise role and stability of the glassy phase close to the MIT (Mott 1990).

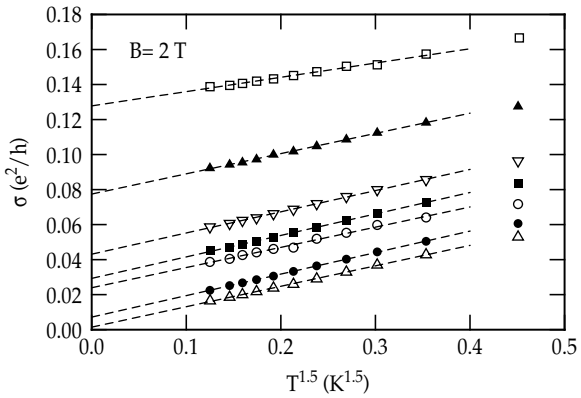

Figure 9. Temperature dependence of the conductivity of two-dimensional electrons in silicon in the metallic glass phase, in the presence of a parallel magnetic field ($B = 2$ T). The data taken from Jaroszynski *et al* (2004) show results for several electron densities ($n_s(10^{10}\,\mathrm{cm}^{-2}) =$ 11.9, 11.6, 11.3, 11.2, 11.0, 10.9, 10.7), as the MIT is approached.

From the experimental point of view, glassy behaviour has often been observed in sufficiently insulating materials (Ovadyahu and Pollak 1997, Vaknin *et al* 1998, 2000), but more recent experiments (Bogdanovich and Popović 2002, Jaroszyński *et al* 2002) have provided striking and precise information on the regime closer to the MIT. These experiments on low density electrons in silicon MOSFETs have revealed the existence of an intermediate metallic glass phase in low mobility (highly disordered) samples (Bogdanovich and Popović 2002). Precise experimental studies of low temperature transport in this regime have revealed an unusual low temperature behaviour of the resistivity of the form

$$\rho(T) = \rho(0) + AT^{3/2}.$$

This behaviour is observed throughout the metallic glass phase, allowing one to systematically extrapolate the resistivity to $T = 0$, and thus characterize the approach to the MIT. Such NFL behaviour is consistent with some theoretical predictions for metallic glasses (Arrachea *et al* 2004, Dalidovich and Dobrosavljević 2002, Sachdev and Read 1996), but the same exponent $3/2$ is expected for glassiness both in the spin and the charge sector. To determine the origin of glassiness, the experiments were carried out (Jaroszynski *et al* 2004) in the presence of large magnetic fields parallel to the two-dimensional channel (such that the field couples only to the electron spin), as shown in figure 9. Since the Fermi energy is relatively small in this density regime ($T_F \sim 10$ K), it is possible to completely spin-polarize the electrons with accessible fields, and thus completely eliminate any spin fluctuations. Remarkably, all the signatures of the glassy behaviour, as well as the $T^{3/2}$ resistivity persisted in this regime, demonstrating that the anomalies originate from the charge degrees of freedom.

These experimental results are certainly very compelling, as they indicate that glassy ordering in the charge sector may very well play a crucial role in most systems close to inhomogeneous insulating states. This possibility seems even more reasonable keeping in mind that screening effects can be expected to weaken as one approaches an insulator—thus strengthening the effects of the Coulomb interactions. More experimental work is necessary to test these ideas, especially using those techniques that directly couple to the charge degrees of freedom. In this respect, global and local compressibility measurements, as well as frequency-dependent dielectric constant experiments seem particularly promising. Such experiments have recently been reported on some transition-metal oxides, revealing glassy ordering, but similar

work on other materials are called for in order to determine how general these phenomena may be.

4. Theoretical approaches

4.1. Disordered Hubbard and Kondo lattice models

4.1.1. Disordered-induced local moment formation. The experiments of Paalanen *et al* were very successfully described within a two-fluid model of itinerant carriers and localized magnetic moments (Paalanen *et al* 1988, Quirt and Marko 1971), as explained in section 3.2. A first attempt to directly test this phenomenology within a well-defined model calculation was due to Milovanović *et al* (1989). These authors investigated a disordered Hubbard model with both diagonal (site energies) and off-diagonal (hopping) disorder. The model parameters were chosen so as to faithfully describe the situation of Si : P. In contrast to the weak-disorder approach of the scaling theory, their work treats disorder exactly by numerical calculations while relying on the mean-field Hartree–Fock treatment of interactions, as was done for the single impurity Anderson model (Anderson 1961). The latter is known to describe well the phenomenon of local moment formation, here taken to signify a broad temperature crossover with a Curie law magnetic susceptibility. However, it is not capable of correctly accounting for the low-temperature quenching of these moments by the conduction electrons, the Kondo effect. The results of Milovanović *et al* show that a finite fraction of the electrons do indeed exhibit a local moment Curie response even if the wave functions at the Fermi level are extended, i.e. the system is still metallic. The fraction of localized moments obtained is of order 10% at 25% above the critical density, in reasonable order of magnitude agreement with the experiments (Paalanen *et al* 1988). In other words, the results are compatible with the observation of local moment responses on the metallic side of the disorder-induced MIT. A rather complete exploration of the phase diagram of this model with the same method has also been carried out (Tusch and Logan 1993), confirming this picture.

Extremal statistics arguments were used by Bhatt and Fisher (1992) to analyse local moment formation and also to determine the effects of residual interactions between these moments. They considered a disordered Hubbard model very similar to the one studied by Milovanović *et al*. In such a system, rare disorder fluctuations can give rise to sites weakly coupled to the rest of the lattice which, for sufficiently strong interactions, can form localized moments. In the absence of the Kondo effect or the RKKY interactions between these moments, they would give rise to a Curie response as shown in Milovanović *et al* (1989). In the presence of the itinerant electron fluid in the metallic phase, however, the Kondo effect may quench these moments. The scale at which this quenching occurs is the Kondo temperature $T_K \sim D\exp(-1/\rho J)$, where D is the conduction electron half-band width, ρ is the DOS at the Fermi level and J is the exchange coupling between the local moment and the conduction electron fluid. Spatial fluctuations lead to a distribution of couplings J and consequently to a distribution of Kondo temperatures. Thus, only those sites with $T_K < T$ will contribute significantly to the thermodynamic properties. The overall behaviour is still singular, though the corrections to the Curie law are very mild

$$\chi(T) \sim \frac{C(T)}{T} \sim \frac{1}{T}\exp\left\{-A\ln^d\left[\ln\left(\frac{T_0}{T}\right)\right]\right\}, \tag{40}$$

where A and T_0 are constants and d is the dimensionality. The effects of the RKKY interactions are stronger and lead, in the absence of ordering, to a divergence that is slower than any

power law

$$\chi(T) \sim \frac{C(T)}{T} \sim \exp\left\{B \ln^{1/d}\left(\frac{T_0}{T}\right)\right\}, \tag{41}$$

where again B is a constant. The picture that emerges is that of a random singlet phase as will be explained below in section 4.1.2. A phase transition into a spin-glass phase, however, may intervene at low temperatures.

This problem is further complicated by the fact that the Kondo temperature depends exponentially both on the exchange coupling J and the local conduction electron DOS ρ. The distribution of T_K calculated by Bhatt and Fisher did not take into account the fluctuations of the latter. We emphasize that ρ should be generalized, in a disordered system, to the *local* DOS $\rho(\boldsymbol{R})$ at the position $\boldsymbol{R}$ of the magnetic moment. This is not a self-averaging quantity and its fluctuations are expected to grow exponentially as the disorder-induced MIT is approached. These effects were analysed by Dobrosavljević *et al* (1992). These authors emphasized the universal log-normal form of the distribution of local DOS, which in turn leads to a universal T_K distribution. As usual for a broad distribution, the magnetic susceptibility can be written as $\chi(T) \sim n(T)/T$, where $n(T) \sim T^{\alpha(T)}$ is the temperature dependent number of unquenched spins at temperature T. Although the exponent $\alpha(T) \to 0$ as $T \to 0$, it does so in an extremely slow fashion and in practice, for reasonable values of the parameters and for a strongly disordered but still metallic system, $n(T) \sim 50\%$ for $T \sim 10^{-4}$–1 K.

4.1.2. Random singlet phases. Local moment formation is generically a crossover phenomenon which occurs at intermediate temperatures whose signature is a Curie–Weiss magnetic susceptibility. Possible low-temperature fates of these moments include some form of magnetic ordering, such as antiferromagnetism and ferromagnetism or spin-glass freezing. A novel kind of low temperature fixed point of disordered localized magnetic moments, however, was proposed from studies of doped semiconductors close to the localization transition, as was discussed in section 3.2 (Bhatt and Lee 1981, 1982). In these systems, the magnetic susceptibility shows no signs of any type of ordering but diverges with a non-trivial power law (Paalanen and Bhatt 1991, Sarachik 1995)

$$\chi(T) \sim T^{\alpha-1}, \tag{42}$$

$$\gamma(T) = \frac{C}{T} \sim T^{\alpha-1}. \tag{43}$$

The theory of Bhatt and Lee (1981, 1982) was based upon a generalization to three dimensions of a method introduced by Ma, Dasgupta and Hu for one-dimensional disordered AFM spin chains (Dasgupta and Ma 1980, Ma *et al* 1979). The Hamiltonian of such a random spin system is written as

$$H = \sum_{i,j} J_{ij} \boldsymbol{S}_i \cdot \boldsymbol{S}_j, \tag{44}$$

where $J_{ij} > 0$ is a random variable distributed according to some given distribution function $P_0(J_{ij})$. The method consists of looking for the strongest coupling of a given realization of the random lattice, say Ω. At energy scales much smaller than Ω, there is only a small probability for this pair of spins to be in its excited triplet state. We can then assume the pair is locked in its ground singlet state and is magnetically inert at the scale considered. The pair is then effectively 'removed' from the system. The remaining spins which had interactions with these locked spins develop additional interactions induced by the removed pair. If spins i and j form

the removed pair and spins k and l interact with i and j, respectively, the new interaction can be obtained in second order of perturbation theory

$$\Delta J_{kl} = \frac{J_{ik} J_{jl}}{2\Omega}. \tag{45}$$

If initially k and l do not interact they will do so after the decimation step. Their new coupling will be smaller than any of the removed ones. Therefore, as more and more spins are decimated, the largest couplings are progressively removed while new smaller ones are generated, thus changing the shape of the 'bare' distribution into an effective 'renormalized' one at lower energy scales. Formally, the RG flow starts at $P[J, \Omega_0] = P_0(J)$, where Ω_0 is the initial largest value of the distribution. After many decimations, Ω_0 will have been decreased to Ω and the distribution will be $P[J, \Omega]$. Bhatt and Lee implemented numerically the decimation procedure we have just outlined using realistic initial distributions of couplings appropriate for the insulating phase of doped semiconductors. They showed that these initial distributions flowed towards very broad distributions at small J (Bhatt and Lee 1981, 1982), leading to diverging thermodynamic responses of the form given in equation (43), in remarkable agreement with the experiments. The low energy state of the system showed no tendency towards ordering but was characterized by the successive formation of singlet pairs between widely separated spins at each decreasing energy scale. This novel disordered magnetic state was dubbed a 'random singlet phase'.

4.1.3. Phenomenological Kondo disorder model. Although the importance of the distribution of Kondo temperatures was first proposed with doped semiconductors in mind, the context of disordered heavy fermion materials, where the Kondo effect is of primary importance, seemed like a natural arena for its applications. Indeed, it was put to good use in the attempts to understand both the temperature dependence of the Cu NMR line-widths and the thermodynamic properties of the Kondo alloy $UCu_{5-x}Pd_x$ ($x = 1$, 1.5) by Bernal *et al* (1995). As was mentioned in section 3.3, this compound is known to exhibit NFL behaviour in many of its properties. Bernal *et al* analysed the broad Cu NMR lines in the following way. The Knight shift at a particular Cu site positioned at $\boldsymbol{R}$, $K(\boldsymbol{R})$, is proportional to the local spin susceptibility

$$K(\boldsymbol{R}) = a(\boldsymbol{R})\chi(\boldsymbol{R}), \tag{46}$$

where $a(\boldsymbol{R})$ is the hyperfine coupling. If $a(\boldsymbol{R})$ is assumed to have little variation in the sample, the spatial fluctuations of the Knight shift can then be ascribed to the spatial variations of the local susceptibility $\delta K = a\delta\chi$. If we use the following fairly accurate form for the susceptibility of a single Kondo impurity

$$\chi(T) \sim \frac{1}{T + \alpha T_K}, \tag{47}$$

where as usual $T_K \approx D\exp[-1/(\rho J)]$, we can use the distribution of Kondo temperatures $P(T_K)$ to find the variance of the susceptibility and hence the line-width. Furthermore, the bulk susceptibility and the specific heat can be obtained by performing the same kind of average procedure over single-impurity results. Bernal *et al* first fitted the susceptibility data by adjusting the mean and the variance of a Gaussian distribution of the bare quantity ρJ. Then, a fairly reasonable agreement with the NMR line-width and specific heat data were obtained without further adjustments, explaining in particular the strong anomalous temperature dependence of the line-widths. The distribution of ρJ used was not too broad. However, due to the exponential form of $T_K \sim D\exp[-1/(\rho J)]$, even a fairly narrow distribution of ρJ

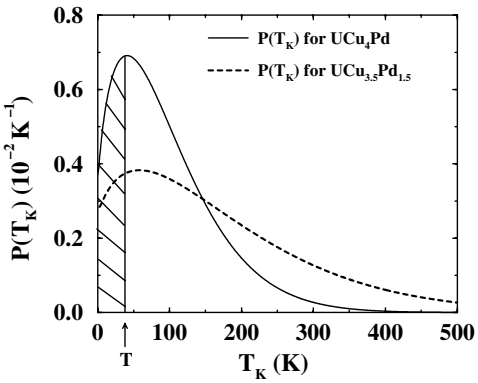

Figure 10. Distribution of Kondo temperatures obtained within the Kondo disorder model from an analysis of the disordered heavy fermion compounds $UCu_{5-x}Pd_x$ (Bernal *et al* 1995). Spins with $T_K < T$ (shaded area) dominate the thermodynamic response.

can lead to a very broad $P(T_K)$. In the case of Bernal *et al*, $P(T_K) \to$ const as $T_K \to 0$ (see figure 10), so that

$$\begin{aligned}\bar{\chi}(T) &\sim \int \mathrm{d}T_K \frac{P(T_K)}{T + \alpha T_K} \\ &\sim \int_0^{\Lambda} \mathrm{d}T_K \frac{P(0)}{T + \alpha T_K} + \text{const} \\ &\sim \ln\left(\frac{T_0}{T}\right), \end{aligned} \tag{48}$$

in good agreement with the experimental findings (see the solid lines of figure 8). The picture is again one where a few spins whose $T_K < T$ (shaded area in figure 10) dominate the thermodynamic response.

4.1.4. Dynamical mean-field theory. The phenomenological model of Bernal *et al* made the seemingly unjustified assumption of a collection of independent single Kondo impurities, even though the alloys $UCu_{5-x}Pd_x$ contained a concentrated fcc lattice of U ions. This assumption was put on a firmer theoretical basis by Miranda *et al* (1996, 1997a, 1997b). These authors used a DMFT approach to the disordered Kondo (or Anderson) lattice problem. The DMFT method has become a very useful tool in the study of strongly correlated materials. It is the natural analogue of the more familiar Weiss mean-field theory of magnetic systems, which is here generalized to describe quantum (especially fermionic) particles. Fairly complete reviews of the subject are available (Georges *et al* 1996, Pruschke *et al* 1995), and we will content ourselves with a brief description, emphasizing the physical aspects.

For definiteness, let us focus on a disordered Anderson lattice Hamiltonian in usual notation

$$\begin{aligned}H_{\mathrm{AND}} = -t \sum_{\langle ij\rangle\sigma} (c^{\dagger}_{i\sigma} c_{j\sigma} + \mathrm{H.c.}) + \sum_{j\sigma} \epsilon_j c^{\dagger}_{j\sigma} c_{j\sigma} + \sum_{j\sigma} E_{fj} f^{\dagger}_{j\sigma} f_{j\sigma} + U \sum_j f^{\dagger}_{j\uparrow} f_{j\uparrow} f^{\dagger}_{j\downarrow} f_{j\downarrow} \\ + \sum_{j\sigma} (V_j f^{\dagger}_{j\sigma} c_{j\sigma} + \mathrm{H.c.}), \end{aligned} \tag{49}$$

where the conduction, f-electron and hybridization energies ϵ_j, E_{fj} and V_j are, in principle, random variables distributed according to $P_{\epsilon}(\epsilon_j)$, $P_{E_f}(E_{f_j})$ and $P_V(V_J)$, respectively. As in

the Weiss theory, one starts by singling out a particular lattice site, say j, and writing out its effective action. A simplification is made here, by neglecting all non-quadratic terms in the local fermionic operators, except for the U-term. One gets

$$S_{\rm eff}^{\rm AND}(j) = S_c(j) + S_f(j) + S_{\rm hyb}(j), \tag{50}$$

$$S_c(j) = \sum_\sigma \int_0^\beta {\rm d}\tau \int_0^\beta {\rm d}\tau' c_{j\sigma}^\dagger(\tau)[\delta(\tau-\tau')(\partial_\tau + \epsilon_j - \mu) + \Delta_c(\tau-\tau')]c_{j\sigma}(\tau'), \tag{51}$$

$$S_f(j) = \int_0^\beta {\rm d}\tau \left[\sum_\sigma f_{j\sigma}^\dagger(\tau)(\partial_\tau + E_{fj} - \mu) f_{j\sigma}(\tau) + U f_{j\uparrow}^\dagger(\tau) f_{j\uparrow}(\tau) f_{j\downarrow}^\dagger(\tau) f_{j\downarrow}(\tau) \right], \tag{52}$$

$$S_{\rm hyb}(j) = \sum_\sigma \int_0^\beta {\rm d}\tau \, [V_j f_{j\sigma}^\dagger(\tau) c_{j\sigma}(\tau) + {\rm H.c.}]. \tag{53}$$

The site j 'talks' to the rest of the lattice only through the bath (or 'cavity') function $\Delta_{cj}(\tau)$ in equation (51). For simplicity, we particularize the formulation to the case of a Bethe lattice, in which the bath function is given by

$$\Delta_c(\tau) = t^2 \overline{G_c^{\rm loc}(\tau)}. \tag{54}$$

Here, the average local Green's function is obtained by first calculating the local conduction electron Green's functions governed by the actions (50)

$$G_c(j,\tau) = -\langle T[c_{j\sigma}(\tau) c_{j\sigma}^\dagger(0)]\rangle_{\rm eff} \tag{55}$$

and then averaging over all sites with the distributions of bare parameters P_ϵ, P_{E_f} and P_V. Thus, all local correlations are accounted for faithfully, while inter-site ones enter only through the bath function. As in the Weiss theory, the whole procedure can be shown to be exact in the limit of large coordination ('infinite dimensions') if appropriate scaling of the hopping is performed ($t \sim \tilde{t}/\sqrt{z}$, with $\tilde{t}$ held constant as $z \to \infty$). Unlike the Weiss theory, however, the 'order parameter' here, namely, $\Delta_c(\tau)$ is a *function* and not just a number (hence the name 'dynamical'). Correlation effects are incorporated in the step where the conduction electron Green's function (55) is calculated. This is equivalent to solving a single-impurity Anderson model (Georges and Kotliar 1992) and since the bare parameters are random, one has to solve a whole *ensemble* of these. The treatment of disorder within DMFT is equivalent to the well-known coherent potential approximation (CPA) (Elliott *et al* 1974).

The connection to the Kondo disorder model can now be made more apparent. The *ensemble* of single impurity problems described by the actions (50) is the equivalent of the collection of independent single Kondo impurities of the phenomenological approach. However, the single impurity problems are not really independent as each one 'sees' the same bath function (54), which in turn contains information from all the other sites. The fully self-consistent calculation of the bath function, however, shows that its precise form does not contain essential features being as it is an average over many different sites. Each Anderson impurity has its local Kondo temperature T_{Kj}, which in the Kondo limit can be written as (taking $U \to \infty$)

$$T_{Kj} = D \exp\left[-\frac{|E_{fj}|}{2\rho V_j^2} \right]. \tag{56}$$

The connection to the Kondo model is obtained from $J_j = 2V_j^2/|E_{fj}|$. A distribution of Kondo temperatures then follows. The full solution is able to produce, with a judicious choice

of bare parameters, a $P(T_K)$ very similar to the experimentally determined one for $UCu_{5-x}Pd_x$ based on the simple phenomenological Kondo disorder model (section 4.1.3).

Within the DMFT framework, one can proceed to the calculation of various physical properties. Bulk thermodynamic responses in the particular case of the Anderson/Kondo lattice, can be very accurately obtained through an *ensemble* average of the individual contributions from each site (Miranda *et al* 1996), thus justifying the procedure adopted by Bernal *et al*. The reason for this is the dominance of the f-site contributions over the conduction electron part (Miranda *et al* 1996). Transport properties, however, cannot be so easily calculated and require a special consideration of the coherence of the motion through the lattice (Tesanović 1986). This is particularly striking in the clean limit, in which it is well known that the f-ion contributions very accurately add up to give the thermodynamic properties but the onset of heavy fermion coherence in the resistivity is totally absent from a single impurity description.

It is precisely with regard to transport properties that the DMFT approach is able to go beyond the phenomenological Kondo disorder model. Assuming the same T_K distribution obtained experimentally by Bernal *et al*, one gets a resistivity which has a large residual value and decreases linearly with temperature with a negative slope (Miranda *et al* 1996, 1997a, 1997b), in complete agreement with the experiments on $UCu_{5-x}Pd_x$ (Andraka and Stewart 1993). The large residual value is due to the total destruction of coherence by disorder. The anomalous NFL linear dependence comes from the gradual unquenching of local moments as the temperature is raised.

Other physical properties were calculated within DMFT, in good agreement with experiments: the dynamical spin susceptibility (Aronson *et al* 1995, Miranda *et al* 1996), the magneto-resistance (Chattopadhyay *et al* 1998) and the optical conductivity (Chattopadhyay and Jarrell 1997, Degiorgi and Ott 1996). In all cases, NFL behaviour was tied to the finite weight of $P(T_K)$ as $T_K \to 0$.

One of the perceived deficiencies of this theory is its extreme sensitivity to the choice of bare distributions. This is due to the exponential dependence of the Kondo temperature on model parameters, which is hardly affected by the self-consistency introduced by DMFT. In the particular case of $UCu_{5-x}Pd_x$, the finite intercept $P(T_K = 0)$ is the most important feature of the distribution and only fine tuning of the bare distributions can provide the correct value. Besides, the thermodynamic properties of many other compounds are more accurately described by power laws (de Andrade *et al* 1998, Stewart 2001). These can be accommodated within the theory through a power-law distribution of Kondo temperatures $P(T_K) \sim T_K^{\alpha-1}$. However, once again fine-tuning is necessary if one wants to obtain such a distribution within DMFT.

4.1.5. Statistical DMFT: localization effects and results of numerical calculations. The problem of fine-tuning within DMFT was remedied in a more complete approach. An important assumption of DMFT is the averaging procedure contained in equation (54). It means that each site feels an *average* hybridization with its neighbours, which is why the procedure becomes increasingly more accurate, the larger the coordination. Fluctuations of this quantity, usually associated with Anderson localization effects, can, however, be incorporated in a so-called statistical DMFT (statDMFT) (Dobrosavljević and Kotliar 1997, 1998). A discussion as applied to the Anderson lattice can be found in Aguiar *et al* (2003). The assumption of retaining only on-site correlations, as in DMFT, is maintained in statDMFT. Formally, this means that one still has to solve an *ensemble* of single impurity problems defined by the actions (50). The only difference comes from the prescription for the bath function, which is

now site-dependent: $\Delta_c(\tau) \to \Delta_c(j,\tau)$. If the lattice coordination is z, we can write

$$\Delta_c(j,\tau) = t^2 \sum_{l=1}^{z} G_c^{(j)}(l,\tau), \tag{57}$$

where $G_c^{(j)}(l,\tau)$ is the local Green's function of nearest-neighbour site l, *after the removal of site* j. Once again, we are particularizing to the Bethe lattice, which simplifies the discussion considerably. The Green's function with a nearest-neighbour site removed is defined as in equation (55) but has to be calculated with a different action. This action has the same form as (50), but now the bath function $\Delta_c(\tau) \to \Delta_c^{(j)}(l,\tau)$, where

$$\Delta_c^{(j)}(l,\tau) = t^2 \sum_{m=1}^{z-1} G_c^{(l)}(m,\tau). \tag{58}$$

Note that in (58) the summation has only $z-1$ terms, as opposed to the z terms of (57), since site j has been removed. The self-consistency loop is closed by noting that the Green's functions on the right-hand side of equation (58) are the same (i.e. obey the same distributions) as the ones on the right-hand side of equation (57). We emphasize that the self-consistency problem is now no longer a simple algebraic equation but rather a set of stochastic equations in the functions $G_c(j,\tau)$ and $G_c^{(j)}(l,\tau)$, whose distributions have to be determined. If the interactions are turned off, the treatment of disorder we have described is equivalent to the self-consistent theory of localization (Abou-Chacra *et al* 1973). The latter is known to exhibit a disorder-induced Anderson MIT for coordination $z \geqslant 3$. The formulation above can be generalized to any lattice (Dobrosavljević and Kotliar 1998). However, the full set of stochastic equations has to be solved numerically, in which case it is advantageous to work in a Bethe lattice. In a realistic lattice, this approach offers the advantage of treating the disorder exactly, albeit numerically, while incorporating the local effects of correlations. In this sense, it is a natural generalization of the work of Milovanović *et al* (1989), to which it reduces if the single-impurity problems are treated within Anderson's mean-field theory (Anderson 1961). More sophisticated treatments of the single-impurity problems are, however, possible, especially in the low temperature region.

The ability to access full distribution functions of physical quantities is an especially appealing feature of statDMFT. In particular, the local DOS

$$\rho_c(j,\omega) = \frac{1}{\pi}\mathrm{Im}[G_c(j,\omega - i\delta)] \tag{59}$$

is intimately connected with the transport properties. It has the natural interpretation of an *escape rate* from site j and is expected to be finite if states at energy ω are extended and to vanish if they are localized, as first pointed out by Anderson in his ground-breaking work (Anderson 1958). Furthermore, the distribution of $\rho_c(j,\omega)$ can become very broad with increasing disorder. Although the average DOS $\overline{\rho_c(\omega)}$ remains finite for any disorder strength, its typical value $\rho_c^{\mathrm{typ}}(\omega)$ vanishes at the MIT and can serve as an order parameter for localization (Anderson 1958). A convenient measure of $\rho_c^{\mathrm{typ}}(\omega)$ is the geometric average $\exp\{\overline{\ln[\rho(\omega)]}\}$.

Furthermore, other local quantities of interest can also be computed, such as the distribution of Kondo temperatures $P(T_K)$ already familiar from DMFT. Recalling that $T_K \approx D\exp[-1/(\rho J)]$, we notice that the DOS ρ 'seen' by any particular site j is a fluctuating quantity in statDMFT and is thus a new source of T_K fluctuations. For the case of the f-electrons of the disordered Anderson lattice, for example

$$\rho_j = \frac{1}{\pi}\mathrm{Im}\left[\frac{V_j^2}{\omega - \epsilon_j + \mu - \Delta_c(j,\omega - i\delta)}\right]. \tag{60}$$

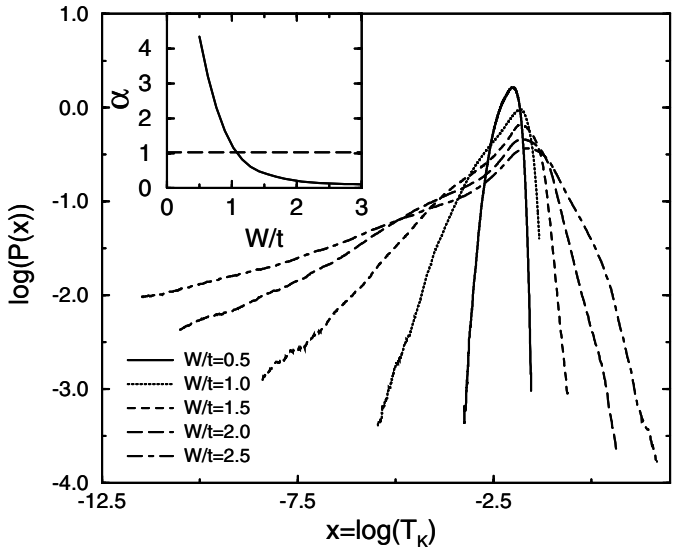

Figure 11. Distribution of Kondo temperatures within statDFMT, showing $P(T_K) \sim T_K^{\alpha-1}$, with $\alpha(W)$ continuously varying with the strength of disorder. The onset of NFL behaviour occurs at $\alpha \leqslant 1$. From Miranda and Dobrosavljević (2001b).

In particular, even if $J = 2V^2/|E_f|$ is not random (e.g. if only conduction electron diagonal disorder is present), there will be a distribution of Kondo temperatures. Moreover, in DMFT, a discrete distribution of J leads necessarily to a discrete distribution of T_K, as ρ is fixed. This is no longer true in statDMFT, however. Even if J (or, equivalently, V or E_f) is discrete, there will be a continuous distribution of ρ and consequently of T_K. This is because ρ_j at a particular site is influenced by fluctuations of other quantities at sites which may be very distant from j, due to the extended nature of the conduction electron wave function in the metallic phase.

4.1.6. Electronic Griffiths phase in disordered Kondo lattice models of dirty heavy-fermion materials. The application of statDMFT to the disordered Anderson lattice problem was described in references (Aguiar *et al* 2003, Miranda and Dobrosavljević 1999, 2001a, 2001b). The single-impurity problems were solved at $T = 0$ within the slave boson mean-field theory (Barnes 1976, Coleman 1984, 1987, Read and Newns 1983) or for $T \neq 0$ with second-order perturbation theory (Kajueter and Kotliar 1996, Meyer and Nolting 2000a, 2000b). Several important features deserve mention. The distribution of Kondo temperatures is generically log-normal for weak disorder but slowly evolves towards a power-law at small T_K,

$$P(T_K) \sim T_K^{\alpha-1}, \tag{61}$$

where the exponent α is a continuously varying function of the disorder strength W, as shown in figure 11. This generic form is fairly insensitive to the particular form of the bare distribution. This universality reflects the mixing of many single-impurity problems which are connected by the extended wave function of the conduction electrons within a correlation volume.

From the power-law distribution of energy scales, the usual phenomenology of a Griffiths phase follows. Although our discussion in this section is self-contained, a complete description of a generic quantum Griffiths phase is deferred to section 4.2. From the distribution (61), the

susceptibility and the specific heat coefficient, for example, are given by

$$\chi(T) \sim \frac{C(T)}{T} \sim \int \mathrm{d}T_K \frac{P(T_K)}{T + \alpha T_K} \sim \frac{1}{T^{1-\alpha}}. \tag{62}$$

For sufficient disorder, $\alpha < 1$ and NFL behaviour is observed. The marginal case $\alpha = 1$ corresponds to a log-divergence. Note that the point $\alpha = 1$ does not signal any phase transition. Other physical quantities such as the non-linear susceptibility $\chi^{(3)}(T)$ start to diverge at a *larger* value of α (corresponding to smaller disorder strength), since $\chi^{(3)}(0) \sim 1/T_K^3$. Several other physical quantities can be obtained as continuous functions of the exponent α. Such behaviour appears consistent with many observations in disordered heavy fermion materials which show NFL behaviour (de Andrade *et al* 1998, Stewart 2001) (section 3.3).

Griffiths phases are common in the proximity of magnetic phase transitions, as is explained in more detail in section 4.2. Here, however, the Griffiths phase occurs in the absence of any form of magnetic ordering. Rather, it is associated with the proximity to the disorder-induced MIT, which occurs for sufficiently large disorder ($W_{\mathrm{MIT}} \approx 12t$ for conduction electron diagonal disorder only (Miranda and Dobrosavljević 2001b)). This is somewhat reminiscent of the local moment phase close to the MIT in doped semiconductors, but is different in two important respects: (i) in a disordered Kondo/Anderson lattice, local moments are assumed stable even in the absence of disorder and (ii) the NFL divergences begin to show up at fairly weak randomness and persist for a wide range of disorder strength. Many other features of the statDMFT solution of the disordered Anderson lattice, particularly concerning transport properties, can be found in Aguiar *et al* (2003) and Miranda and Dobrosavljević (1999, 2001a, 2001b).

We should stress that the anomalous power laws obtained in this approach are a direct consequence of the power law in the distribution of Kondo temperatures. Interestingly, all that is needed is a power-law distribution of energy scales for the spin fluctuators, *not necessarily of Kondo origin or related to localization effects*. Given this form of distribution the associated Griffiths divergences follow immediately (see a discussion of generic properties of Griffiths phases in section 4.2.2). As a result, widely different microscopic mechanisms can give rise to the same macroscopic behaviour. This will be exemplified later when we discuss magnetic Griffiths phases (section 4.2), which are usually tied to a *magnetic* phase transition. Whereas this highlights an interesting universality of this phenomenology, it makes it hard to distinguish between different microscopic mechanisms based solely upon macroscopic measurements.

Some insight into the origin of the power law distribution of T_K was offered in Tanasković *et al* (2004b). In that work, it was shown how a power-law distribution of Kondo temperatures can be easily obtained within a DMFT treatment of the disordered Anderson lattice, if a Gaussian distribution of conduction electron on-site energies

$$P_\epsilon(\epsilon_j) = \frac{1}{\sqrt{2\pi}W} \exp\left(-\frac{\epsilon_j^2}{2W^2}\right) \tag{63}$$

is used, with fixed values of E_f and V. In this case, although the bath function is fixed at its average value (section 4.1.4), different values of ϵ_j generate different Kondo temperatures (cf equation (50)) according to

$$T_{Kj} = T_K^0 \exp(-A\epsilon_j^2), \tag{64}$$

where A is a constant and T_K^0 is the Kondo temperature for $\epsilon_j = 0$. It is then straightforward to show that, up to log corrections

$$P(T_K) \sim T_K^{\alpha-1}, \tag{65}$$

where $\alpha^{-1} = 2AW^2$. This type of argument, in which an energy scale depends exponentially on a random variable, which in turn occurs with an exponential probability, leading to a

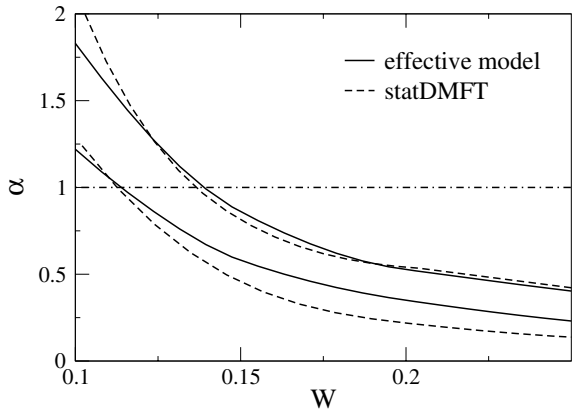

Figure 12. Comparison between the $\alpha(W)$ exponent determined within statDMFT and the one obtained in the effective model of Tanasković *et al* (2004b).

power law distribution of the energy scale is very common in, and perhaps generic to Griffiths phases (see section 4.2.2). Now, although the specific form of the distribution (63) is required within DMFT for the power law to emerge, in statDMFT *fluctuations of the bath function are generically Gaussian*, at least at weak disorder. By noting that the bath function enters additively to ϵ_j in the single-site action (cf equation (51)), it is clear that the appropriate statistics are naturally generated within this approach. Careful comparison between the two approaches confirms that this is indeed the case (see figure 12), thus clarifying the generic nature of the mechanism behind the Griffiths phase in these systems.

4.1.7. Electronic Griffiths phase near the Mott–Anderson MIT. The statDMFT approach can be usefully employed in the analysis of the disordered Hubbard model, which serves as a prototypical model for both disorder-induced (Anderson) as well as interaction-induced (Mott–Hubbard) MITs. The Hamiltonian in this case is

$$H_{\mathrm{HUB}} = -t \sum_{\langle ij\rangle\sigma} (c^{\dagger}_{i\sigma} c_{j\sigma} + \mathrm{H.c.}) + \sum_{j\sigma} \epsilon_j c^{\dagger}_{j\sigma} c_{j\sigma} + U \sum_{j} c^{\dagger}_{j\uparrow} c_{j\uparrow} c^{\dagger}_{j\downarrow} c_{j\downarrow}. \tag{66}$$

The statDMFT equations in this case are very similar to the Anderson lattice and we write them down here for completion. The auxiliary single-site actions read (Dobrosavljević and Kotliar 1997, 1998)

$$S^{\mathrm{HUB}}_{\mathrm{eff}}(j) = \sum_{\sigma} \int_0^{\beta} \mathrm{d}\tau \int_0^{\beta} \mathrm{d}\tau'\, c^{\dagger}_{j\sigma}(\tau)[\delta(\tau-\tau')(\partial_\tau + \epsilon_j - \mu) + \Delta_c(j, \tau-\tau')]c_{j\sigma}(\tau') + U \int_0^{\beta} \mathrm{d}\tau\, c^{\dagger}_{j\uparrow}(\tau) c_{j\uparrow}(\tau) c^{\dagger}_{j\downarrow}(\tau) c_{j\downarrow}(\tau), \tag{67}$$

where the bath function is once again given by equation (57). The Green's function with a nearest-neighbour site removed is obtained from an action of the form (67), with a bath function given by equation (58), which closes the self-consistency loop (again, in a Bethe lattice).

The analysis of the distributions of two local quantities which come out of the single-site actions (67) provide especially useful insights into the physics of the disordered Hubbard model. One is the local DOS at the Fermi level $\rho(j, 0)$, already defined in equation (59). The other is the local Kondo temperature of each single-site action T_{Kj}. The product of these

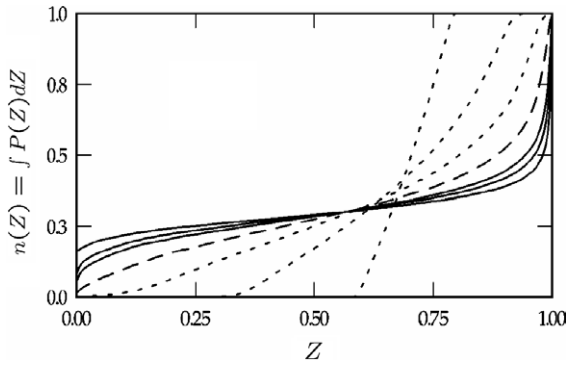

Figure 13. Distribution of Kondo temperatures (here named Z) within statDMFT for the disordered Hubbard model. From Dobrosavljević and Kotliar (1997).

two is the quasi-particle weight $Z_i = \rho(j, 0)T_{Kj}$. While the Kondo temperature governs the thermodynamic response of the system, the local DOS is related to the transport properties, as explained in section 4.1.5.

The results of statDMFT as applied to the disordered Hubbard model show that, similarly to the disordered Anderson lattice, a Griffiths phase emerges, with NFL behaviour, for intermediate values of disorder strength (Dobrosavljević and Kotliar 1997, 1998). This is signalled by a power-law distribution of Kondo temperatures $P(T_K) \sim T_K^{\alpha-1}$, where α varies continuously with disorder. The marginal value of $\alpha = 1$, at which the magnetic susceptibility diverges logarithmically with decreasing temperature, occurs at $W \approx 7t$, as shown in figure 13. This is reminiscent of the behaviour of doped semiconductors. In contrast to the Anderson lattice though, the NFL behaviour requires much stronger disorder. This can be rationalized by noting that in contrast to the latter, where local moments are stable in the clean limit, here they require quite a large amount of randomness to appear.

Moreover, for low conduction electron filling ($n \approx 0.3$), the typical local DOS goes to zero at a large value of disorder strength $W_{\mathrm{MIT}} \approx 11t$, signalling a MIT (Dobrosavljević and Kotliar 1997, 1998). As in the non-interacting case, the average value remains non-critical, actually diverging at W_{MIT}. The full distribution of $\rho(j, 0)$ is very close to a log-normal form, with a width which increases as disorder increases. Closer to half-filling ($n \approx 0.7$), however, the behaviour is quite different. At these larger densities the typical value of $\rho(j, 0)$ remains finite for very large values of disorder, suggesting that W_c is pushed up considerably (Dobrosavljević and Kotliar 1998). A similar 'screening' of the disorder potential by strong interactions has been also found in a DMFT study of the disordered Hubbard model (Tanasković *et al* 2003).

4.1.8. Incoherent metallic phase and the anomalous resistivity drop near the two-dimensional MIT. The puzzle of the two-dimensional MIT observed in several materials (Abrahams *et al* 2001) remains a largely unsolved problem. One of the most intriguing features of these systems is the large resistivity drop observed in a temperature range comparable to the Fermi energy. This suggests the importance of inelastic scattering processes, a feature usually absent in Fermi liquid based approaches, or at most included only in a perturbative fashion. An initial attempt at the incorporation of such inelastic effects was made within a DMFT framework (Aguiar *et al* 2004). A half-filled disordered Hubbard model was considered in that work. The associated single-impurity problems were solved within second-order perturbation theory (Kajueter and Kotliar 1996) and checked against Quantum Monte Carlo calculations. The results were also compared with a Hartree–Fock impurity solver, which does not include inelastic scattering.

The most important result of that work is the demonstration of the large contribution of inelastic processes to transport properties in a region of parameter space where kinetic energy, interaction and disorder strength are of comparable size. This is especially striking when one compares the results of DMFT with Hartree–Fock. Whereas in the former, a drop of order 10 in the total scattering rate is obtained, only a very weak drop is observed in the latter for the same values of disorder and interaction. This large difference is clarified when one separates the inelastic from the elastic parts. The Hartree–Fock scattering rate, which is completely elastic, is comparable to the elastic contribution in DMFT. However, for comparable values of disorder, Fermi energy and interaction strengths, the contribution of inelastic scattering is seen to completely overwhelm the weak elastic processes, in effect determining the size and temperature dependence of the total rate.

Furthermore, keeping the interaction and Fermi energy of equal size and decreasing randomness, one is able to enhance even more the size of the drop in the scattering rate. The more gradual drop for larger disorder is attributed to a wide distribution of coherence scales. These scales set the boundary between coherent transport at low temperatures and incoherent inelastic-process-dominated transport at higher temperatures. Once a distribution of these scales is generated, different spatial regions of the system become incoherent at different temperatures as the temperatures are raised, leading to a gradual rise in the overall scattering rate. This rise is approximately linear in the range where the various coherence scales are equally frequent. An approximately linear rise of the scattering rate is observed in Si MOSFETs (Abrahams *et al* 2001). Finally, the coherence scale is inversely proportional to the local effective mass, which in turn is tied to the thermodynamic response of the system. Thus, when the disorder is comparable to the interaction strength, a sufficiently broad distribution of coherence scales is generated such that very small scales have finite weight, leading to an enhanced overall effective mass and thermodynamic properties. Results similar to those DMFT findings were also obtained from exact numerical studies of finite size lattices (Denteneer and Scalettar 2003, Denteneer *et al* 1999, 2001), but more work is needed to gain a more definitive understanding of this regime.

4.2. Magnetic Griffiths phases

4.2.1. Quantum Griffiths phases in insulating magnets with disorder: random singlet formation and the IRFP. We have seen in section 4.1.2, how the Bhatt and Lee numerical study of random AFM spin systems (Bhatt and Lee 1981, 1982) led to the concept of a random singlet phase. Further insight into this kind of behaviour came from a rather complete analytical treatment of the nearest-neighbour random AFM chain (equation (44)) by Fisher (1994) using the renormalization group procedure invented by Ma, Dasgupta and Hu and described in section 4.1.2. Fisher showed that an essentially exact solution of the problem could be obtained asymptotically at low energies. He showed that the relevant fixed point distribution of couplings is infinitely broad and the system slowly approaches it as it is probed at lower and lower energies. The system is governed by an IRFP which reflects the true physical significance of the random singlet phase. Note that, if the distribution is very broad, the side couplings of the strongest bond Ω, J_{ik} and J_{jl} in equation (45), are almost certainly much smaller than Ω, rendering the perturbation theory result essentially exact. It is in this sense that the Fisher solution is asymptotically exact. Initial decimations will be inaccurate but if the system flows towards broader and broader distributions, then the results will become increasingly more accurate and asymptotically exact. Interestingly, results at weak disorder (Doty and Fisher 1992) showed that the clean system is unstable with respect to infinitesimal disorder and numerical results seem to confirm that practically any initial randomness is in the basin of attraction of the IRFP

found by Fisher. We will highlight some of the most important physical properties of this type of phase. This is a vast subject and we refer the reader to some recent reviews of experimental and theoretical results of disordered spin chains and ladders (Continentino *et al* 2004, Iglói and Monthus 2005).

When the system is governed by the IRFP it exhibits many unusual properties. In particular, the relation between energy (Ω) and length (L) scales is 'activated'

$$\Omega \sim \Omega_0 \exp(-L^{\psi}), \tag{68}$$

where $\psi = 1/2$ in the Heisenberg chain case. This should be contrasted with the more common dynamical scaling observed in quantum critical systems $\Omega \sim L^{-z}$, where z is the dynamical exponent (Hertz 1976) (see section 3.1.1). Formally, we can say that the 'activated dynamical scaling' of equation (68) corresponds to a divergent dynamical exponent $z \to \infty$. This has many important consequences. The magnetic susceptibility of the system at temperature T can be obtained by scaling down to $\Omega = T$. It can then be very accurately calculated through

$$\chi(T) \sim \frac{n(\Omega = T)}{T}, \tag{69}$$

where it is assumed that the spins that were decimated at larger scales form magnetically inert singlets while the remaining ones, whose density is $n(\Omega = T)$ are almost free and contribute a Curie-like term. This estimate is better, the broader the distribution of couplings. If we now write

$$n^{-1}(\Omega = T) \sim L(\Omega = T) \sim \left[\ln\left(\frac{\Omega_0}{T}\right)\right]^{1/\psi}, \tag{70}$$

where use was made of equation (68), we can write for the random Heisenberg chain

$$\chi(T) \sim \frac{1}{T \ln^{1/\psi}(\Omega_0/T)} \sim \frac{1}{T \ln^2(\Omega_0/T)}. \tag{71}$$

Similarly, the entropy density $s(T) \sim n(\Omega = T) \ln 2$, from which follows

$$\frac{C(T)}{T} \sim \frac{1}{T[\ln(\Omega_0/T)]^{(\psi+1)/\psi}} \sim \frac{1}{T \ln^3(\Omega_0/T)}. \tag{72}$$

Furthermore, the spin–spin correlation functions are very broadly distributed. Fisher showed that while the *average* correlation function decays as a power law ($C_{ij} \equiv \langle \boldsymbol{S}_i \cdot \boldsymbol{S}_j \rangle$)

$$\overline{C_{ij}} \sim \frac{(-1)^{i-j}}{|i-j|^2}, \tag{73}$$

the *typical* one, given by the geometric average, decays as a stretched exponential

$$\exp(\overline{\ln |C_{ij}|}) \sim \exp(-|i-j|^{\psi}) \sim \exp(-\sqrt{|i-j|}). \tag{74}$$

This large difference between average and typical correlations is a result of the fact that a typical pair of spins separated by a distance $L \sim |i-j|$ does *not* form a random singlet and is therefore very weakly correlated. Those rare pairs that do form a random singlet, however, are strongly correlated and dominate the average.

Further extensions of these studies deserve mention. Random chains with higher spins have been investigated. Integer spin chains are especially interesting because the clean system is gapped (Haldane gap (Haldane 1983a, 1983b)). It has been shown that the $S = 1$ chain (and presumably other integer S ones) are stable with respect to weak enough disorder, retaining a (pseudo)gap (Boechat *et al* 1996, Hida 1999, Hyman and Yang 1997, Monthus *et al* 1997, 1998, Saguia *et al* 2003a, Yang *et al* 1996). For larger disorder, the pseudogap is destroyed and a Griffiths phase is realized, with a diverging power-law susceptibility with a varying

non-universal exponent. At a sufficiently large disorder, a random singlet phase is obtained, governed by an IRFP. Similar results were also obtained in other systems for which the clean analogue is gapped: random chains in which the gap is induced by a dimerization of the couplings (Henelius and Girvin 1998, Hyman *et al* 1996, Yang *et al* 1996) and the two-leg ladder (Hoyos and Miranda 2004, Mélin *et al* 2002, Yusuf and Yang 2002). A Griffiths phase was also identified in a disordered Kondo necklace model (Doniach 1977), in which the spin sector of the one-dimensional Kondo lattice is mimicked by a double spin chain system (Rappoport *et al* 2003).

It has been proposed that the $S = 3/2$ AFM random chain realizes two kinds of random singlet phases separated by a quantum phase transition (Refael *et al* 2002). At weaker randomness, the *excitations* of the random singlet phase have $S = 1/2$, whereas in the stronger disorder phase they have $S = 3/2$. The QCP between these two phases can be viewed as a multicritical point in a larger parameter space which includes the effects of an added dimerization of the couplings (Damle and Huse 2002). The physics of this and other higher spin multicritical points has been shown to be governed by IRFPs with different exponents: for example, $\psi = 1/4$ for the $S = 3/2$ chain. The phase diagram of the $S = 3/2$ chain proposed in Refael *et al* (2002) has been recently disputed, however (Carlon *et al* 2004, Saguia *et al* 2003b).

Another important universality class of disordered spin chains is found in systems with random ferromagnetic and AFM interactions. In this case, it is possible to use a generalization of the decimation procedure described above for any spin size and both signs of couplings (Frischmuth and Sigrist 1997, Frischmuth *et al* 1999, Hida 1997, Hikihara *et al* 1999, Westerberg *et al* 1995, 1997). In general, the RG flow generates spins whose average size grows without limit as a power law. In fact, these large spin clusters grow in a fashion which can be described as a 'random walk in spin space', so that the total spin and the cluster size scale as

$$\overline{S^2} \sim L. \tag{75}$$

Moreover, the energy-length scale relation has a conventional power-law form $\Omega \sim L^{-z}$. For disorder distributions which are not too singular, the dynamical exponent has a universal value $z \approx 4.5$ (for stronger disorder, the value of z is non-universal). Employing similar arguments as outlined above for the random singlet phase one finds

$$\chi(T) \sim \frac{1}{T}, \tag{76}$$

$$\frac{C(T)}{T} \sim \frac{\ln(1/T)}{T^{1-1/z}}. \tag{77}$$

Interestingly, the z exponent does not show up in the susceptibility to leading order. This is because there is a cancellation of the temperature dependences of the number of spin clusters and squared spin of the clusters (equation (75))

$$\chi(T) \sim \frac{\overline{S^2}(\Omega = T)n(\Omega = T)}{T} \sim \frac{1}{T}. \tag{78}$$

This phase has been dubbed a 'large spin phase' and has been also identified in some spin ladder systems (Hoyos and Miranda 2004, Mélin *et al* 2002, Yusuf and Yang 2003a, 2003b). In fact, the larger connectivity of spin ladders and higher dimensional systems favours the formation of ferromagnetic couplings and large spins in the decimation procedure. This is easily seen if we consider the decimation of an antiferromagnetically coupled spin pair such that two other spins are initially antiferromagnetically coupled to the *same spin*. It is physically evident and easy to show that the effective coupling generated between the latter

spins is *ferromagnetic*. Thus, unless the lattice geometry is fine-tuned (an example of which is the two-leg spin ladder (Hoyos and Miranda 2004, Mélin *et al* 2002, Yusuf and Yang 2002)), there is a strong tendency towards a large spin phase formation. The singular behaviour first discovered by Bhatt and Lee in higher dimensions is thus only a crossover and gives way to a large spin phase behaviour at lower temperatures (Lin *et al* 2003, Motrunich *et al* 2001).

Generically, Griffiths phases seem to be a ubiquitous phenomenon in these random insulating magnets, usually in the vicinity of an IRFP. Systems with an Ising symmetry are especially interesting. Once again, a great deal of insight has been gained from an essentially exact analytical solution in the vicinity and at the QCP of the random transverse field Ising chain due to Fisher (1992, 1995) (see also Shankar and Murthy (1987)). The clean version of this system has a quantum phase transition between an Ising ferromagnet and a disordered paramagnet (Lieb *et al* 1961, Pfeuty 1970). The disordered version of the model is related to the McCoy–Wu model (McCoy 1969, McCoy and Wu 1968, 1969), in which context quantum Griffiths singularities were first studied (hence the name Griffiths–McCoy singularities sometimes used). It can be written as

$$H_{\mathrm{TFIM}} = -\sum_i J_i \sigma_i^z \sigma_{i+1}^z - \sum_i h_i \sigma_i^x. \tag{79}$$

It also has a quantum phase transition tuned by the difference between the average exchange ($\Delta_J = \overline{\ln J}$) and the average transverse field ($\Delta_h = \overline{\ln h}$). Through a generalization of the Ma–Dasgupta–Hu strong disorder RG procedure, Fisher was able to solve for the effective distributions of exchange and transverse field couplings in the quantum critical region. The QCP itself was shown to have all the features of an IRFP, including activated dynamical scaling (with $\psi = 1/2$), diverging thermodynamic responses

$$\chi(T) \sim \frac{[\ln(1/T)\,]^{2\phi-2}}{T}, \tag{80}$$

$$\frac{C(T)}{T} \sim \frac{1}{T \ln^3(1/T)} \tag{81}$$

and widely different average and typical correlation functions ($C^a(x) \equiv \langle \sigma_i^a \cdot \sigma_{i+x}^a \rangle$, $a = x, z$)

$$\overline{C^a}(x) \sim \frac{1}{x^{2-\phi}}, \tag{82}$$

$$\overline{\ln C^a}(x) \sim -\sqrt{x}, \tag{83}$$

where $\phi = (1+\sqrt{5})/2$, the golden mean. Note the similarity with the result for the Heisenberg case if ϕ is taken to be zero. The singular behaviour is attributed to the presence of large spin clusters at low energies which can tunnel between two different configurations with reversed magnetizations. The behaviour off but near criticality, however, is characteristic of a Griffiths phase. Defining

$$\delta = \frac{\Delta_h - \Delta_J}{\mathrm{var}(\ln h) + \mathrm{var}(\ln J)}, \tag{84}$$

where $\mathrm{var}(O) \equiv \overline{O^2} - \bar{O}^2$ is the variance of the variable O, it is found that, in the slightly disordered region ($\Delta_h \gtrsim \Delta_J$)

$$\chi(T) \sim \delta^{4-2\phi} \frac{[\ln(1/T)\,]^2}{T^{1-1/z}}, \tag{85}$$

$$\frac{C(T)}{T} \sim \delta^3 \frac{1}{T^{1-1/z}}, \tag{86}$$

whereas in the slightly ordered region ($\Delta_h \lesssim \Delta_J$)

$$\chi(T) \sim |\delta|^{2-2\phi} \frac{1}{T^{1+1/z}}, \tag{87}$$

$$\frac{C(T)}{T} \sim |\delta|^3 \frac{1}{T^{1-1/z}}, \tag{88}$$

where $z \sim 1/(2|\delta|)$ is a non-universal disorder-dependent exponent, typical of Griffiths phases. An exact expression for z in terms of the bare distributions was obtained in Iglói *et al* (2001). The typical and average correlations both decay exponentially at large distances but with different correlation lengths

$$\xi_{\rm typ} \sim \frac{1}{|\delta|}, \tag{89}$$

$$\xi_{\rm av} \sim \frac{1}{\delta^2}, \tag{90}$$

such that $\xi_{\rm av} \gg \xi_{\rm typ}$ and $\overline{C^z}(x) \gg \exp \overline{\ln C^z}(x)$. The exponent of the correlation length of the average correlation function is the one that satisfies (in fact, saturates) the bound $\nu \geqslant 2/d = 2$ of Chayes *et al* (1986), which is the generalization of the Harris criterion (Harris 1974) to a disordered critical point.

There has been independent (i.e. not based on the approximate RG scheme) numerical confirmation of Fisher's results on the one-dimensional random transverse field Ising model through the mapping to free fermions (Young and Rieger 1996). Furthermore, similar behaviour was also found in higher dimensions by Quantum Monte Carlo (Pich *et al* 1998) and by a numerical implementation of the above decimation procedure (Motrunich *et al* 2001). In addition, a random *dilute* Ising model in a transverse field (IMTF) has been shown to have these same general properties close to the percolation transition in any dimension $d > 1$ (Senthil and Sachdev 1996).

A similar scenario has also been discussed phenomenologically for a quantum Ising spin glass transition in higher dimensions (Thill and Huse 1995). The presence of a Griffiths phase in a model of a quantum Ising spin glass in a transverse field has been confirmed numerically in $d = 2$ and 3 (Guo *et al* 1994, 1996, Rieger and Young 1994, 1996), although the IRFP was neither confirmed nor ruled out at the quantum phase transition point due to the small lattice sizes used in those studies. However, it is believed that this QCP is also governed by an IRFP. The argument is based on the IRFP nature of the random transverse field Ising model QCP in higher dimensions (Motrunich *et al* 2001, Pich *et al* 1998) and the realization that frustration is *irrelevant* at the IRFP (Motrunich *et al* 2001). Indeed, at low energies the weakest link of a given loop of spins is almost certainly infinitely weaker than the other links. Therefore, one can neglect it and the loop is always unfrustrated.

4.2.2. General properties of quantum Griffiths phases. The general nature of Griffiths phases can be elucidated in a simple and general fashion as we now show. We have seen in section 4.1.6 how it arises in the context of the electronic Griffiths phase. In that case, the fluctuators are local moments coupled by a Kondo interaction J_i to the fluid of conduction electrons. The energy scale governing the local moment dynamics is the Kondo temperature, which depends exponentially on the combination $\rho_i J_i$, $T_{Ki} \approx D \exp(-1/\rho_i J_i)$, where ρ_i is the local conduction electron DOS. We showed how the exponential tail of the distribution of the quantity $1/\rho_i J_i$ leads to a power law distribution of Kondo temperatures, $P(T_K) \sim T_K^{\alpha-1}$. This exponential tail is naturally obtained from localization effects. The final result is a collection of essentially independent Kondo spins whose characteristic energy scales are distributed in a power law fashion.

The magnetic Griffiths phase arises in a wholly analogous fashion. In the vicinity of a quantum phase transition spatial fluctuations due to disorder generate droplets of the ordered phase inside a system which is on the whole in the disordered phase and vice versa. Let us for simplicity focus on the disordered side of the transition. For the simplest case of uncorrelated disorder, the statistics of these rare regions is Poissonian and the probability of a region of the ordered phase with volume L^d is

$$P(L^d) \sim \exp(-cL^d), \tag{91}$$

where c is a constant tuned by the disorder strength. This droplet of ordered phase has a net magnetic moment. In the case of a ferromagnetic transition, the moment is proportional to the droplet size, whereas in the case of an antiferromagnet the net droplet moment is $\propto L^{(d-1)/2}$ due to random incomplete spin cancellations on the surface of the droplet. These net moments are subject to quantum tunnelling between states with reversed magnetizations. Since the driving mechanism is single-spin tunnelling the total tunnelling rate of a droplet is exponential on the number of spins

$$\Delta \sim \omega_0 \exp(-bL^d), \tag{92}$$

where b is another constant related to the microscopic tunnelling mechanism and ω_0 is some frequency cutoff. The tunnelling rate gives the energy splitting between the lowest and first excited states of the droplet, higher excited states lying far above in energy. Now, the interesting thing is that the exponentially small probability of a large cluster is 'compensated' by the exponentially large tunnelling times between states, such that the distribution of energy splittings (gaps) is given by a *power law*

$$P(\Delta) \sim \int \mathrm{d}L^d P(L^d)\delta[\Delta - \omega_0 \exp(-bL^d)] \sim \Delta^{\alpha-1}, \tag{93}$$

where $\alpha = c/b$. We are then left with a distribution of independent droplets, whose frequencies are distributed according to a power law. Thus, although the microscopic origin of the magnetic and the electronic Griffiths phases are very different, the end result, namely, a collection of fluctuators whose energy scales are distributed in a power-law fashion, is common to both mechanisms.

Given the power-law distribution of energy scales, thermodynamic and dynamical properties follow immediately. For example, the spin susceptibility at temperature T can be calculated by assuming that all the droplets (Kondo spins) with $\Delta > T$ ($T_K > T$) are frozen in the non-magnetic ground state, whereas all the others are essentially free and give a Curie response. The error introduced by the borderline moments is small if the distribution is broad as in equation (93). Thus, the susceptibility is given by the number of free spins at temperature T, $n(T) = \int_0^T \mathrm{d}\Delta P(\Delta)$ times $1/T$

$$\chi(T) \sim \frac{n(T)}{T} \sim \frac{1}{T^{1-\alpha}}. \tag{94}$$

Note that, in the magnetic droplet case, we have neglected the droplet moment $\mu \propto L^{\phi} \propto \ln^{\phi/d} \Delta$ (where ϕ depends on the nature of the magnetic order), because it only gives rise to a negligible logarithmic correction to the power law (see, however, section 4.2.4). Analogously, the total entropy $S(T) \propto n(T)$, such that the specific heat is given by

$$\frac{C(T)}{T} = \frac{\mathrm{d}S(T)}{\mathrm{d}T} \sim P(T) \sim \frac{1}{T^{1-\alpha}}. \tag{95}$$

Similarly, the droplet dynamical susceptibility is

$$\chi''_{\mathrm{drop}}(\omega) \sim \tanh\left(\frac{\Delta}{2T}\right)\delta(\omega - \Delta). \tag{96}$$

Averaging over the droplet distribution

$$\chi''(\omega) \sim \omega^{\alpha-1} \tanh\left(\frac{\omega}{2T}\right). \tag{97}$$

Interestingly, if the fluctuator has relaxational dynamics (as in the case of a Kondo spin)

$$\chi''_{\rm rel}(\omega) \sim \frac{\omega\chi(T)\Gamma(T)}{\omega^2+\Gamma^2(T)}, \tag{98}$$

where $\Gamma(T)$ is the relaxation rate and $\chi(T)$ is the static susceptibility. Typically at high temperatures, $\Gamma(T) \sim T$, whereas as $T \to 0$, $\Gamma(T) \sim \Delta$. Since $\chi(T)\Gamma(T)$ is an almost constant smooth function of T, we can ignore it in equation (98). We thus obtain at low temperatures after averaging

$$\chi''(\omega) \sim \overline{\frac{\omega}{\omega^2+\Delta^2}} \sim \omega^{\alpha-1}, \tag{99}$$

which has the same form as equation (97).

Similar considerations apply to the magnetization as a function of magnetic field H. The fluctuator magnetization has the following limiting behaviours

$$M_{\rm fluc}(H,T,\Delta) \sim \begin{cases} \mu & H \gg T,\Delta, \\ H/\Delta & H \ll T \ll \Delta, \\ H/T & H \ll \Delta \ll T, \end{cases} \tag{100}$$

where μ is the fluctuator moment. This applies both to a magnetic droplet in a transverse field (Castro Neto and Jones 2000) and to a Kondo spin ($\Delta = T_K$) (Hewson 1993). Averaging over Δ with $P(\Delta)$ from equation (93), it is clear that the large field behaviour is dominated by the fluctuators with $\Delta \ll H$, such that

$$M(H,T) = \overline{M_{\rm fluc}(H,T,\Delta)} = \overline{\mu}\int_0^H {\rm d}\Delta P(\Delta), \tag{101}$$

$$= \overline{\mu} n(H) \propto H^{\alpha}, \tag{102}$$

where $\overline{\mu}$ is an average moment (we have again neglected the possible logarithmic contribution coming from the moment size). At lower fields, this crosses over to a linear behaviour, with a coefficient given by the (singular) susceptibility of equation (94).

The physical properties of Griffiths phases are therefore rather generic and are determined almost exclusively by the exponent of the power law distribution of energy scales, quite independent of the underlying mechanism. This makes it very difficult to distinguish microscopic models solely on the basis of this generic power law behaviour of macroscopic properties. We will, however, point out in section 4.2.4 some possible measurements which can distinguish between different microscopic mechanisms.

4.2.3. Possibility of quantum Griffiths phases in itinerant random magnets. The discovery of a plethora of quantum Griffiths phases in the vicinity of phase transitions of several models of disordered insulating magnets immediately suggests the possibility of similar phenomena in metallic compounds. As we have seen in section 3.3, many of the systems of interest have the required ingredients, namely, the presence of extrinsic disorder and/or some kind of magnetic order whose critical temperature can be tuned to zero by some external parameter such as external or chemical pressure. Furthermore, the diverging susceptibility with tunable exponents observed close to the MIT in doped semiconductors *on the metallic side* was found to have many similarities to Griffiths singularities (Bhatt and Fisher 1992, Bhatt and Lee 1981, 1982, Dobrosavljević *et al* 1992, Paalanen *et al* 1988).

Quantum Griffiths phase in the theory of Castro Neto, Castilla and Jones. The possibility of a magnetic Griffiths phase as the origin of NFL behaviour in Kondo alloys was first put forward by Castro Neto *et al* (1998). In that work, a disordered anisotropic Kondo lattice system was analysed, since heavy fermion systems are known to exhibit large spin–orbit interactions which tend to strongly break Heisenberg spin SU(2) symmetry. Due to the competition between the RKKY interaction and the Kondo effect in a disordered environment, it was argued that large spatial fluctuations would give rise to regions where either of the two competing tendencies would dominate. The local moments that belong to a region where the local Kondo temperature is especially large are efficiently quenched at low temperatures and contribute to the formation of a disordered heavy Fermi liquid. On the other hand, anomalously small Kondo temperature fluctuations give rise to local spins which, though still subject to weak Kondo spin-flip processes, are left to interact with other similar partners through the RKKY interaction. Assuming strong anisotropy, the magnetic interactions lead to a random Ising model for the unquenched spins. Furthermore, the Kondo spin-flip terms were shown to give rise to an effective transverse field, i.e. if the Ising model variables are σ_i^z, anisotropic Kondo scattering generates a local field in the x-direction. One is then left with an effective random IMTF, which was discussed in section 4.2.1. As we have seen, there is numerical evidence that the order–disorder quantum phase transition of this model is governed by an IRFP flanked on both sides by Griffiths phases with non-universal tunable exponents. In the picture proposed by Castro Neto *et al*, alloying tends to enhance quantum (Kondo) correlations, which act as a destabilizing mechanism on long-range magnetic order. Thus, a power law distribution of energy scales is immediately obtained as is generically expected within Griffiths phases, as explained in section 4.2.2. The calculation of physical quantities then follows from the arguments of that section and they obtained, in terms of a non-universal tunable parameter λ (which is the same as the α of section 4.2.2)

$$\chi(T) \sim \frac{C(T)}{T} \sim T^{\lambda-1}, \tag{103}$$

$$\chi^{(3)}(T) \sim T^{\lambda-3}, \tag{104}$$

$$\chi''_{\mathrm{loc}}(\omega) \sim \omega^{\lambda-1} \tanh\left(\frac{\omega}{T}\right), \tag{105}$$

$$T_1^{-1}(T) \sim \omega^{\lambda-2} T \tanh\left(\frac{\omega}{T}\right), \tag{106}$$

$$\frac{\delta\chi(T)}{\chi(T)} \sim T^{-\lambda/2}, \tag{107}$$

where $\chi^{(3)}(T)$ is the non-linear magnetic susceptibility, $\chi''_{\mathrm{loc}}(\omega)$ is the imaginary part of the local frequency-dependent susceptibility, $T_1(T)$ is the NMR spin relaxation time, and $\delta\chi(T)$ is the root mean square susceptibility due to the disorder fluctuations. As we have seen, $\lambda < 1$ implies NFL behaviour. Some of these predictions have found experimental support in the NFL Kondo alloys $Th_{1-x}U_xPd_2Al_3$, $Y_{1-x}U_xPd_3$ and $UCu_{5-x}M_x$ (M = Pd, Pt, where the specific heat and susceptibility values could be well fitted to the forms in equation (103) with λ ranging from about 0.6 to about 1 (de Andrade *et al* 1998). Some other systems have been reanalysed by Stewart in his review (Stewart 2001) in light of the above results and found to be describable by such power laws with appropriately chosen values of λ.

Here again, the anomalous power laws are a result of a power-law distribution of energy scales for the spin fluctuators (see the discussion in section 4.2.2). In the specific case here, in which the system teeters at the onset of magnetic order, these are large clusters of one phase (say, the ordered one) inside the other phase (the disordered one). Sample

averaging then leads to the results in equations (103)–(107). Although these are different from the low-T_K spins of the electronic Griffiths phase, the same phenomenology is obtained in either case.

Effects of dissipation by the conduction electrons. These early results of the magnetic Griffiths phase route towards NFL behaviour did not, however, take into account the effect of dissipation on the tunnelling of the Ising spins caused by the excitation of conduction electron particle–hole pairs of the metallic host. These were later incorporated in a more complete treatment by Castro Neto and Jones (2000). In that work, the authors considered essentially the same anisotropic Kondo lattice model. The same competition between the RKKY interaction and the Kondo effect in a disordered environment was argued to lead, in general, to the formation of clusters of spins locked together by their mutual interactions and with widely distributed sizes. These clusters are able to tunnel at low temperatures between two different configurations with reversed signs of magnetization (in the ferromagnetic case) or staggered magnetization (in the AFM case). The tunnelling is caused either by residual anisotropic RKKY interactions or by the Kondo spin-flip scattering with the conduction electrons. The latter was called a *cluster Kondo effect*. The picture is thus similar to the previous work but now there is a distribution of cluster sizes $P(N)$, which is argued to be given by percolation theory as

$$P(N) = \frac{N^{1-\theta}\mathrm{e}^{-N/N_\xi}}{\Gamma(2-\theta)N_\xi^{2-\theta}}. \tag{108}$$

Here, θ is a critical exponent and N_ξ is a correlation volume (given in terms of the number of spins enclosed by it) that diverges as $N \sim |p - p_c|^{-\nu}$ close to the percolation threshold p_c. The origin of percolation here is the alloying-induced growth of the quenched spin fluid, which acts as an inert pervasive heavy Fermi liquid.

For a general cluster with N spins, the *bare* tunnelling splitting between the two configurations scales like

$$\Delta_0(N) = \omega_0 \mathrm{e}^{-\gamma N}, \tag{109}$$

where γ is related to the single-spin-flip mechanism and ω_0 is an attempt frequency. Moreover, the authors also included the renormalization generated by dissipation due to the conduction electrons in the cluster tunnelling processes. The dissipation constant of a cluster was shown to scale like $\alpha(N) = (N/N_c)^\varphi$, where φ is an exponent dependent on the specific ordering wave vector or tunnelling mechanism. This dissipative two-level system is known to be governed by the scale (Leggett *et al* 1987, Lesage and Saleur 1998)

$$T_K(N) \sim \frac{\Lambda}{\alpha(N)}\left[\frac{\Delta_0(N)}{\Lambda}\right]^{1/[1-\alpha(N)]}, \tag{110}$$

where Λ is a non-universal high-energy cutoff. The renormalization acts to suppress the tunnelling rate, which should be viewed as an effective *cluster* Kondo temperature. When $\alpha(N) < 1$, the cluster exhibits damped oscillations between the two configurations, whereas for $\alpha(N) > 1$ dissipation completely suppresses tunnelling and the cluster is frozen, behaving like a superparamagnetic particle. From the form of $\alpha(N)$ it can be seen that tunnelling will cease to happen for sufficiently large clusters, i.e. for $N > N_c$, where $\alpha(N_c) = 1$. The authors then proceeded to treat this collection of clusters of different sizes by a generalization of the quantum droplet model of Thill and Huse (1995) to the dissipative case.

The T_K distribution for small values of T_K is a result of the exponentially rare large clusters imposed by equation (108) and the exponentially large tunnelling times given by equations (109) and (110) for large values of $N < N_c$. As usual (see the discussion in section 4.2.2), this yields the familiar Griffiths phase power law dependence $P(T_K) \sim T_K^{1-\lambda}$,

with a tunable λ, which in turn is at the origin of the singular thermodynamic properties. The new element here is the cutoff imposed by dissipation at $N = N_c$. It replaces the above power law, at very low T_K, by an extremely singular form

$$P(T_K) \sim \frac{1}{T_K \ln(T_K/\omega_0)}. \tag{111}$$

The power law behaviour is valid above a crossover scale $T_K \gtrsim T_K^*$, whereas equation (111) is obtained for $T_K \lesssim T_K^*$, where T_K^* can be related to N_c (Castro Neto and Jones 2000). Thus, for scales above T_K^* Griffiths effects should be observable. However, for temperatures below T_K^*, more singular responses are obtained, such as $\chi(T) \sim 1/[T \ln(\Lambda/T)]$ (superparamagnetism). The conclusion, then, is that the dissipation caused by a metallic environment has a dramatic effect on Griffiths phase behaviour, effectively suppressing it at low enough temperatures and confining it to an intermediate temperature crossover regime.

Theory of Millis, Morr and Schmalian. Another important work on the possibility of Griffiths singularities in metallic disordered systems is the one by Millis *et al* (2001, 2002). These authors considered the effects of disorder fluctuations on an almost critical system with Ising symmetry both with and without dissipation due to conduction electrons. The approach is based on a coarse-grained Landau action description, possibly modified by an ohmic dissipation term, in which disorder fluctuations couple to the quadratic term, locally changing the value of T_c. The authors confined themselves to situations in which the effective dimensionality in the Hertz sense (Hertz 1976) $d_{\text{eff}} = d + z$ is above the upper critical dimension 4 and the analysis can be carried out at the mean field level (see section 3.1.1). They then considered the behaviour of rare disorder fluctuations which are able to nucleate a locally ordered region ('droplet') inside the paramagnetic phase. In the first work (Millis *et al* 2001), the behaviour of a single droplet was analysed. Although they considered point, line and planar defects, we will focus here on the case of point droplets. It was shown that the order parameter decays very slowly ($\sim 1/r$) in the region right outside the droplet core (defined approximately as the region where the local $T_c(r) > T$) and not further from it than the clean correlation length ξ. This slow decay region, which can be very large close to criticality, is essential when it comes to dissipation. They show that these droplets will tunnel between two reversed-magnetization configurations at a rate given by the usual expression for a dissipative two-level system (Leggett *et al* 1987), analogous to equation (110). However, the dissipation constant is far greater than the critical value and actually *diverges* at criticality. Therefore, tunnelling will be totally suppressed sufficiently close to the QCP. Millis *et al* clarify this result by noting that the very large effective droplet size implies that many conduction electron angular momentum channels will be scattered by it, as opposed to the simple s-wave case of a point scatterer. Similar considerations were also made concerning the effect of magnetic impurities in a system close to a ferromagnetic QCP (Larkin and Melnikov 1972, Maebashi *et al* 2002).

In a subsequent paper, Millis *et al* (2002) extended this analysis and considered the collective effect of a distribution of droplets with different sizes and strengths. By relating the energetics and the size of a single droplet and using standard statistical methods, they determined the form of the droplet distribution function. Using the results for the tunnelling rate of a single droplet the overall response of the system could be calculated. It was shown how the usual Griffiths singularities of an insulating (i.e. non-dissipative) nearly critical magnet (as expounded in section 4.2.1) can be re-obtained within this macroscopic Landau-functional scheme. More importantly, in the metallic case, the suppression of tunnelling by dissipation was shown to completely destroy Griffiths behaviour. Indeed, since dissipation effectively freezes *most* droplets, their contribution is that of essentially classical degrees of freedom, giving rise to simple superparamagnetism $\chi(T) \sim 1/T$. Close to criticality, the largest

contribution to the thermodynamic properties comes from such frozen superparamagnetic droplets. Moreover, in the contribution of the unfrozen droplets (i.e. those that do tunnel) only those droplets *on the verge* of classicality, and consequently with very low tunnelling rates, contribute significantly, leading again to $\chi(T) \sim 1/T$. The results of Millis *et al* thus cast some doubts on the relevance of a magnetic Griffiths phase picture to the physics of NFL disordered heavy fermion compounds.

Even though differing in their detailed approach, the results obtained by Millis *et al* are, in general, in broad agreement with those of Castro Neto and Jones. In both cases, disorder fluctuations lead to the formation of large clusters of ordered material inside the disordered phase, which are able to tunnel between two magnetic configurations. Very large clusters are exponentially rare. Their tunnelling rates, however, are exponentially small, leading to a power-law distribution of small energy scales (tunnelling energy splittings) and quantum Griffiths singularities at intermediate temperatures. Finally, dissipation imposes an upper cutoff N_c on the fluctuating cluster sizes: above this threshold, the clusters are frozen and behave like superparamagnetic particles with a classical Curie response. The frozen cluster contribution eventually swamps the quantum droplets and kills the Griffiths singularities at the lowest temperatures. Their discrepant conclusions with regard to the applicability of their theories to Kondo alloys boil down to the assumptions made in each case about the strength of dissipation (see the recent exchange Castro Neto and Jones (2004a, 2004b), Millis *et al* (2004)). Millis *et al* argue that in heavy fermion systems the local moments are strongly coupled to the conduction electrons such that the most natural assumptions lead to a dissipation constant which is of the same order as the other energy scales. As a result, the largest fluctuating cluster N_c would be fairly small (of the order of a few sites) and the energy scale separating Griffiths behaviour from superparamagnetic response would be of order of the energy cutoff (e.g. the Kondo temperature of the clean system). This would leave only a rather small window of temperatures in which quantum Griffiths effects might be observable, effectively ruling out this mechanism as the source of NFL behaviour in disordered heavy fermion systems. Castro Neto and Jones, on the other hand, argue that the conduction electron DOS in the region of the ordered cluster is not renormalized by the Kondo effect and is small. The dissipation is therefore considerably reduced and N_c is correspondingly enhanced, leaving a sufficiently wide range of temperatures for the Griffiths singularities to be detectable in heavy fermion systems. Further progress in determining which of these pictures, if any, applies to heavy fermion alloys will likely come from a direct determination of the strength of dissipation in real systems, either theoretically or from experiments.

Vojta theory of QCP 'rounding'. The essential gist of the results of Millis *et al* can be put in a simpler, more general form and its consequences extended, as shown by Vojta (2003). The argument is based on the same coarse-grained Ising-symmetry Landau approach adopted by Millis *et al* Vojta pointed out that droplet regions at $T = 0$ are bounded in the spatial directions but infinite along the imaginary time dimension. Since the ohmic dissipation ($\sim|\omega_n|$) term of the Hertz action is equivalent to a $\sim 1/(\tau - \tau')^2$ kernel in the imaginary time direction, each droplet at $T = 0$ is equivalent to one-dimensional Ising models with $\sim 1/r^2$ interactions. These are known to undergo a phase transition at finite temperature (equivalent to a finite coupling constant in the quantum case, which means a large enough droplet), i.e. they are above their lower critical dimension (Thouless 1969). Therefore, sufficiently large droplets will be of order (freeze) close enough to the critical point. Interestingly, this is in complete agreement with results of Millis *et al* and lends a new, simpler perspective to the rather involved calculations of that work. Considering the statistics of the droplet sizes for different types of

disorder distributions, it was shown then that the clean quantum phase transition is 'rounded' by disorder. For a Gaussian disorder distribution, for example, the magnetization in d dimensions at $T = 0$ is *always finite*

$$M \sim \exp(-Bg^{2-d/\phi}), \tag{112}$$

where g measures deviations from the clean critical point, B is a constant and ϕ is the finite-size scaling shift exponent of the clean d-dimensional system. The term 'rounding' here relates to the fact that the onset of order is *not* a collective effect as in conventional phase transitions but is rather the result of a sum of many finite-size frozen droplet contributions. At finite temperatures, droplet–droplet interactions induce a conventional thermal phase transition. If these interactions are assumed to be exponential in the droplet separation, the critical temperature is a double exponential of g

$$\ln T_c \sim -\exp(Ag^{2-d/\phi}), \tag{113}$$

where A is a constant. Finally, for non-dissipative dynamics $|\omega_n|$ is replaced by ω_n^2. The equivalent one-dimensional Ising model is short-ranged and thus *at the lower critical dimension*. This leads naturally to the Griffiths phenomena already identified by other methods. The analysis of Vojta, therefore, puts a number of different results on a common setting by means of some simple and transparent arguments.

More recently, the analysis of the Ising symmetry has been extended to the Heisenberg (in fact, continuous, $O(N)$) case by Vojta and Schmalian (2004). Whereas the dissipative Ising droplet is above its lower critical dimension (when viewed as a one-dimensional system in the time direction with $1/r^2$ interactions), the Heisenberg analogue is *at its lower critical dimension* (Bruno 2001, Joyce 1969). Therefore, in the language of Vojta, the itinerant, disordered, nearly critical Heisenberg magnet is analogous to the insulating, disordered, nearly critical Ising system. It is no surprise then that Griffiths singularities should arise in this system as well. Indeed, this is the main result of Vojta and Schmalian (2004), which shows that a power-law distribution of cluster tunnelling scales is obtained in the by now familiar fashion: $P(\Delta) \sim \Delta^{d/z-1}$. In particular, the non-universal tunable dynamical exponent z could be calculated in a controlled $1/N$ expansion as a function of disorder strength. Similar calculations in an XY magnet were also performed in Loh *et al* (2005) with similar conclusions. The presence of Griffiths singularities in nearly critical metals with continuous order parameter symmetry suggests the exciting possibility that the associated critical point, by analogy with the insulating Ising case, is also governed by an IRFP.

Long-range RKKY interactions and the cluster glass phase. The Griffiths phase of itinerant systems with continuous symmetry was obtained by neglecting the effects of droplet–droplet interactions. These were incorporated recently in an extended DMFT fashion (Dobrosavljević and Miranda 2005). These interactions are generically long-ranged in metallic systems due to the RKKY interactions. The presence of disorder is known to lead to an exponential suppression of the *average* RKKY interaction, because it introduces a random phase in the oscillating part. However, the fluctuations around the average retain their long-ranged $\sim 1/r^d$ amplitude and are random in sign (Jagannathan *et al* 1988, Narozhny *et al* 2001). A particular droplet will therefore be coupled to a large number of other droplets and subject to their Weiss dynamical molecular field. A self-consistent treatment leads to a Weiss field dynamics set by the *average local* susceptibility of all the other droplets $\overline{\chi}_{\rm loc}(\omega_n)$. In the very dilute limit far from the critical point, the results of Vojta and Schmalian, which neglect droplet–droplet interactions, can be taken as a zeroth order approximation and an instability analysis can be performed. This has the usual power law form

$$\bar{\chi}_{\rm loc}(\omega_n) \sim \int \mathrm{d}\Delta\, \Delta^{d/z-1} \frac{1}{\Delta + |\omega_n|} \sim \overline{\chi}_{\rm loc}(0) - \widetilde{\gamma}|\omega_n|^{d/z-1} + \mathcal{O}(|\omega_n|). \tag{114}$$

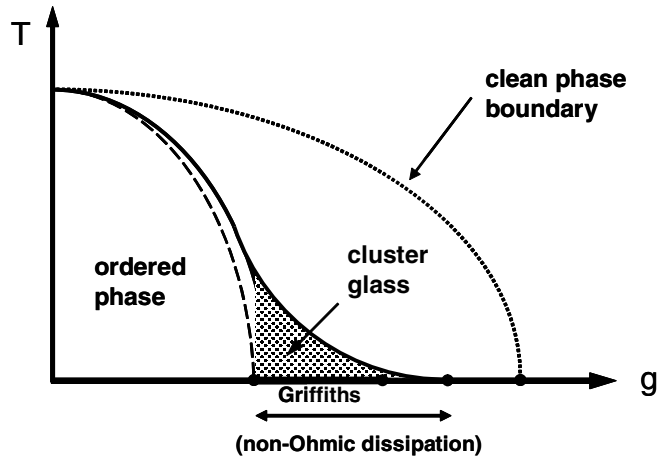

Figure 14. Phase diagram of an itinerant system with disorder, close to the clean QCP, following Dobrosavljević and Miranda (2005). Shown are the clean critical line (· · · · · ·) and that in the presence of disorder if non-ohmic dissipation is ignored (- - - -). In this case, a Griffiths phase emerges close to the magnetically ordered phase. When dissipation is present, sufficiently large droplets 'freeze', leading to the formation of the 'cluster glass' phase (shaded region), preceding the uniform ordering.

The Weiss dynamical field adds to the usual ohmic term ($\sim|\omega_n|$) and *dominates at low frequencies* if $d/z < 2$, which occurs even before the Griffiths singularity in the susceptibility (which requires $d/z < 1$). This sub-ohmic dissipation term places a single droplet *above* its lower critical dimension (Bruno 2001, Joyce 1969), leading again to freezing of sufficiently large droplets and eventually to ordering, in analogy with the Ising system. The conclusion is that, within this picture, long-ranged droplet–droplet interactions generate *additional dissipation*, over and above the one induced by the itinerant carriers, which is able to trigger magnetic order of a cluster-glass type. Interestingly, the existence of a very similar cluster glass phase was proposed in early work, over thirty years ago (Sherrington and Mihill 1974).

The Griffiths phase will be 'hidden' inside this ordered phase (figure 14). It should be stressed that this argument cannot be generalized to insulating systems, where the presence of short-ranged interactions casts doubts on the validity of the mean field treatment, at least in low dimensions. Likewise, for itinerant systems, this approach is likely to break down in low dimensions and the Griffiths phase scenario may survive. An interesting unsolved question is the value of the lower critical dimension for this class of disordered systems.

Internal quantum dynamics of droplets. Finally, we would like to mention the work which takes into account the internal quantum dynamics of droplets with a special geometry. Shah and Millis (2003) considered a 'necklace' of spins in a ring geometry coupled as a finite XXZ chain. They showed that internal quantum fluctuations of the necklace facilitate the reversal of the droplet magnetization, whose amplitude now decays as a power of the number of droplet spins, instead of an exponential. As a result, the nature of the droplet dynamics is closer to the multi-channel Kondo effect and the spin flips actually tend to proliferate at lower temperatures. The special geometry of a circular ring enforces a residual chiral symmetry of the necklace which protects a two-channel fixed point with its well-known anomalous properties (Affleck and Ludwig 1991a, 1991b, 1991c, Ludwig and Affleck 1991, Nozières and Blandin 1980). This symmetry is not expected to be present in the case of disorder-induced droplets. Kondo-type dynamics at low temperatures was also proposed for magnetic impurities with XY symmetry in a nearly ferromagnetic metal (Loh *et al* 2005).

4.2.4. On the applicability of the magnetic Griffiths phase phenomenology to metallic disordered systems. An important unsolved question that remains is whether a magnetic Griffiths phase (i.e. a Griffiths phase associated with a magnetic phase transition) can be realized in an observable temperature/frequency range in metallic systems. As already discussed in section 4.2.3, a great deal of controversy surrounds the question of whether dissipation due to the conduction electrons is weak enough to allow for Griffiths singularities to be observable in an appreciable temperature range. However, a number of additional points, which are independent of the answer to this question, are also worth mentioning regarding the applicability of this scenario to real metallic systems.

Less singular thermodynamic response of heavy fermion systems. We have seen that power law anomalies are observed in both doped semiconductors and heavy fermion materials which conform, in principle, with the Griffiths phase phenomenology (sections 3.2 and 3.3). There are some trends, however, which seem to distinguish these two classes of systems. Griffiths phases are characterized by power-law divergent thermodynamic properties such as

$$\chi(T) \sim \frac{1}{T^{1-\alpha}}, \tag{115}$$

$$\frac{C(T)}{T} \sim \frac{1}{T^{1-\alpha}}, \tag{116}$$

where the power-law exponent α is non-universal and tunable by the disorder strength. NFL behaviour is signalled by $\alpha < 1$. A significant feature of the phenomenology of the above compounds is the fact that α typically lies in the range 0.7–1.0 ($\alpha = 1$ implying a logarithmic divergence) in the case of heavy fermion compounds (with a few cases below 0.7 but always above 0.5, see Stewart (2001) and de Andrade *et al* (1998)), whereas doped semiconductors show a more singular response $\alpha \approx 0.3$–0.4 (Paalanen and Bhatt 1991, Sarachik 1995). A possible explanation to this interesting trend is the fact that localized magnetic moments are well formed and stable in heavy fermion systems, whereas they are induced by disorder fluctuations in doped semiconductors. Therefore, interactions among local moments are likely to be *stronger* in the former than in the latter. Such interactions, usually neglected in Griffiths phase theories, where the droplets are assumed dilute and independent, contribute to quench the local moments and may be at the origin of the less singular spin entropy available at low temperatures in heavy fermion materials.

Wilson ratio. As shown in section 4.2.2, the Griffiths phase thermodynamic responses can be obtained from the scaling of a few physical quantities with temperature. The results on the random quantum Ising model are very instructive in this respect (Fisher 1992, 1995). The spin susceptibility, for example, is given by

$$\chi(T) \sim \frac{\mu^2(T) n(T)}{T}, \tag{117}$$

where $\mu(T)$ is the average value of the magnetic moment per cluster and $n(T)$ is the number of active clusters, both at temperature T. The specific heat is obtained similarly from the entropy

$$S(T) \sim n(T) \ln 2, \tag{118}$$

$$\frac{C(T)}{T} \sim \frac{\mathrm{d}S}{\mathrm{d}T}. \tag{119}$$

As mentioned in section 4.2.1, in the disordered Griffiths phase of the one-dimensional random Ising chain in a transverse field,

$$\chi(T) \sim \delta^{4-2\phi} \frac{[\ln(1/T)]^2}{T^{1-1/z}}, \tag{120}$$

$$\frac{C(T)}{T} \sim \delta^3 \frac{1}{T^{1-1/z}}. \tag{121}$$

It can be seen that the Wilson ratio

$$R_W \propto \frac{T\chi(T)}{C(T)} \sim \left[\ln\left(\frac{1}{T}\right)\right]^2, \tag{122}$$

which is logarithmically divergent as $T \to 0$. The origin of this divergence can be traced back to the scaling of the magnetic moment of the ordered (ferromagnetic) clusters with size or energy scales, which leads to $\mu \sim \ln(\Omega_0/T)$. Thus, the Wilson ratio is a useful quantity to estimate the cluster moment at low temperatures, as the cluster number density which appears both in the susceptibility and the specific heat cancels out. These results are expected to be valid even in higher dimensions. Since we expect the cluster moments to grow with size, both in ferromagnetic and in AFM Griffiths phases (in the latter, $\mu \sim L^{(d-1)/2}$), the Wilson ratio should diverge, albeit slowly, as $T \to 0$. Although perhaps hard to determine, a careful examination of the Wilson ratio may be a useful guide to test the applicability of the magnetic Griffiths phase scenario.

Observed entropy and the size of Griffiths droplets. The generic quantum magnetic Griffiths phase is built upon large clusters which tunnel between two reversed (staggered or uniform) magnetization states. Therefore, the available entropy per cluster is $S_{\rm cl} = k_B \ln 2$, the other microscopic degrees of freedom being effectively frozen. On the other hand, the relevant clusters are generically assumed to be large, with a typical number of N_ξ spins per cluster. If the total number of clusters is of order $N_{\rm cl}$, the total entropy can be estimated as $S \approx N_{\rm cl} S_{\rm cl}$, and the entropy per spin is

$$\frac{S}{N_{\rm spins}} \approx \frac{N_{\rm cl} S_{\rm cl}}{N_{\rm spins}} \approx \frac{S_{\rm cl}}{N_\xi} \sim \frac{k_B \ln 2}{N_\xi}, \tag{123}$$

where $N_{\rm spins}$ is the total number of spins. We conclude that the entropy per mole of spins is *decreased from the typical value of* $R \ln 2$ *by a number of the order of the typical cluster size.* We thus expect that conventional magnetic Griffiths behaviour should be characterized by divergent power laws with *rather small amplitudes*. Conversely, if Griffiths phase behaviour with a sizable ($\sim R \ln 2$) molar spin entropy is observed, it necessarily follows that the relevant fluctuators involve a small, of order 1, number of spins. All the candidate heavy fermion compounds analysed if Stewart (2001) and de Andrade *et al* (1998) have molar spin entropies of the order of R, which seem to leave little room for large clusters (Aguiar *et al* 2003). Therefore, the electronic Griffiths phase (section 4.1.6), whose fluctuators are individual Kondo spins seems a much more viable explanation. Of course, if dissipation freezes clusters larger than a critical size N_c, these clusters will contribute a Curie-like term, which should be taken into account. However, it is clear that the above argument also limits the low temperature entropy of the frozen clusters, even though their contribution is more singular. The conclusion that $S_{\rm molar} \sim R$ implies small fluctuators seems therefore inescapable.

Temperature range of Griffiths phase behaviour. Another question relates to the range of temperatures where the Griffiths singularities may be observable. We have seen that the strength of dissipation generated by the conduction electrons is a very important input in the determination of this range (see section 4.2.3). Only very small dissipation rates are compatible with a wide range of Griffiths singularities. However, *even if the dissipation is negligibly weak*, the range of temperatures available for Griffiths anomalies is still fairly restricted. Let us consider the phenomenology of insulating disordered magnets discussed in section 4.2.1. As we have seen, all the known Griffiths phases in these systems occur in the vicinity of a phase or a point which is governed by an IRFP. Formally, this is usually seen through the divergence of the dynamical exponent z as the IRFP is approached. Now, as the system approaches the IRFP, there is a line of temperature crossover that approaches zero, as in any QCP. Above

the crossover line, the system is still governed by the IRFP, even off criticality. Only below the crossover line can the non-critical behaviour, in particular the quantum Griffiths singularities, be observed (Sachdev 1999). The functional form of this crossover line can be easily obtained for systems governed by an IRFP (Fisher 1995). The primary feature of the IRFP is the 'activated dynamical scaling' which relates energy and length scales (see section 4.2.1)

$$\Omega \sim \Omega_0 \exp(-L^{\psi}). \tag{124}$$

Close to criticality, there is a correlation length ξ which sets the scale beyond which equation (124) no longer holds and conventional dynamical scaling is recovered. As shown by Fisher (1995), this is the 'true' correlation length in the sense of the criterion of Chayes *et al* (1986): beyond this scale, the system 'knows' it is non-critical and most magnetic clusters can be treated as effectively independent, leading to typical Griffiths singularities. This 'true' correlation length diverges as a power law as the IRFP is approached ($\nu \to 0$)

$$\xi \sim \frac{1}{\delta^{\nu}}, \tag{125}$$

where ν satisfies the Harris criterion $\nu d \geqslant 2$ (Chayes *et al* 1986, Harris 1974). This is the correlation length of the average correlation function. By plugging this characteristic length into equation (124) and setting $\Omega = T_{\text{cross}}$ we can determine the crossover line

$$T_{\text{cross}} \sim \Omega_0 \exp\left(-\frac{1}{\delta^{\nu\psi}}\right). \tag{126}$$

In the particular case of the one-dimensional IMTF $\nu\psi = 1$. We thus see that, quite generically, quantum Griffiths behaviour is expected to occur below an energy scale that is *exponentially smaller* than the natural energy scales of the problem. Above this scale, the more singular universal behaviour characteristic of the IRFP should be observed. This represents a severe restriction on the range of temperatures where Griffiths effects could be observed. Apparently, the diluted Ising model analysed in Senthil and Sachdev (1996) is an exception to this kind of behaviour. This is probably due to the peculiar percolating character of the transition in that case. Since metallic systems are characterized by long-ranged RKKY interactions, it is reasonable to expect that percolation is less likely to play a significant role and the more generic behaviour of equation (126) should apply.

4.3. Itinerant quantum glass phases and their precursors

4.3.1. Inherent instability of the electronic Griffiths phases to spin-glass ordering. So far, we have discussed the electronic and the magnetic Griffiths phase scenarios for disorder-induced NFL behaviour. Both pictures envision the formation of rare regions with anomalously slow dynamics, which under certain conditions dominate the low temperature properties. Neither picture, however, seems satisfactory for the following key reason: in both cases the resulting NFL behaviour is characterized by power law anomalies, with non-universal, rapidly varying exponents. In contrast, most experimental data seem to show reasonably weak anomalies, close to MFL behaviour (Stewart 2001).

Physically, it is clear what is missing from the theory. Similarly as magnetic Griffiths phases, the electronic Griffiths phase is characterized (Miranda and Dobrosavljević 2001b, Tanasković *et al* 2004b) by a broad distribution $P(T_K) \sim (T_K)^{\alpha-1}$ of local energy scales (Kondo temperatures), with the exponent $\alpha \sim W^{-2}$ rapidly decreasing with disorder W. At any given temperature, the local moments with $T_K(i) < T$ remain unscreened. As disorder increases, the number of such unscreened spins rapidly proliferates. Within the existing theory (Miranda and Dobrosavljević 2001b, Miranda *et al* 1996, 1997a, Tanasković *et al* 2004b) these

unscreened spins act essentially as free local moments and provide a very large contribution to the thermodynamic response. In a more realistic description, however, even the Kondo-unscreened spins are *not* completely free, since the metallic host generates long-ranged RKKY interactions even between relatively distant spins.

In a disordered metal, impurity scattering introduces random phase fluctuations in the usual periodic oscillations of the RKKY interaction, which, however, retains its power law form (although its *average* value decays exponentially (Jagannathan *et al* 1988, Narozhny *et al* 2001)). Hence, such an interaction acquires a random amplitude J_{ij} of zero mean but finite variance $\langle J_{ij}^2(R)\rangle \sim R^{-2d}$ (Jagannathan *et al* 1988, Narozhny *et al* 2001). As a result, in a disordered metallic host, a given spin is effectively coupled with random but long-range interactions to many other spins, often leading to spin-glass freezing at the lowest temperatures. How this effect is particularly important in Griffiths phases can also be seen from the mean-field stability criterion (Bray and Moore 1980) for spin glass ordering, which takes the form

$$J\chi_{\rm loc}(T) = 1. \tag{127}$$

Here, J is a characteristic interaction scale for the RKKY interactions, and $\chi_{\rm loc}(T)$ is the disorder average of the local spin susceptibility. As we generally expect $\chi_{\rm loc}(T)$ to diverge within a Griffiths phase, this argument strongly suggests that in the presence of RKKY interactions such systems should have an inherent instability to finite (even if very low) temperature spin glass ordering.

Similar to other forms of magnetic order, the spin glass ordering is typically reduced by quantum fluctuations (e.g. the Kondo effect) which are enhanced by the coupling of the local moments to itinerant electrons. Sufficiently strong quantum fluctuations can completely suppress spin-glass ordering even at $T = 0$, leading to a QCP separating metallic spin glass from the conventional Fermi liquid ground state. As in other QCPs, one expects the precursors to magnetic ordering to emerge even before the transition is reached and produce NFL behaviour within the corresponding quantum critical region. Since many systems where disorder-driven NFL behaviour is observed are not too far from incipient spin-glass ordering, it is likely that these effects play an important role, and should be theoretically examined in detail.

From the theoretical point of view, a number of recent works have examined the general role of quantum fluctuations in glassy systems and the associated quantum critical behaviour. Most of the results obtained so far have concentrated on the behaviour within the mean-field picture (i.e. in the limit of large coordination), where a consistent description of the QCP behaviour has been obtained for several models. In a few cases (Read *et al* 1995), corrections to mean-field theory have been examined, but the results appear inconclusive and controversial at this time. In the following, we briefly review the most important results obtained within the mean-field approaches.

4.3.2. Quantum critical behaviour in insulating and metallic spin glasses

Ising spin glass in a transverse field. The simplest framework to study the quantum critical behaviour of spin glasses is provided by localized spin models such as the infinite-range IMTF with random exchange interactions J_{ij} of zero mean and variance J^2/N ($N \to \infty$ is the number of lattice sites).

$$H_{\rm TFIM} = -\sum_{ij} J_{ij}\sigma_i^z\sigma_j^z - \Gamma\sum_i \sigma_i^x. \tag{128}$$

In the classical limit ($\Gamma = 0$), this model reduces to the well-studied Sherrington–Kirkpatrick model (Mézard *et al* 1986), where spins freeze with random orientations below a critical

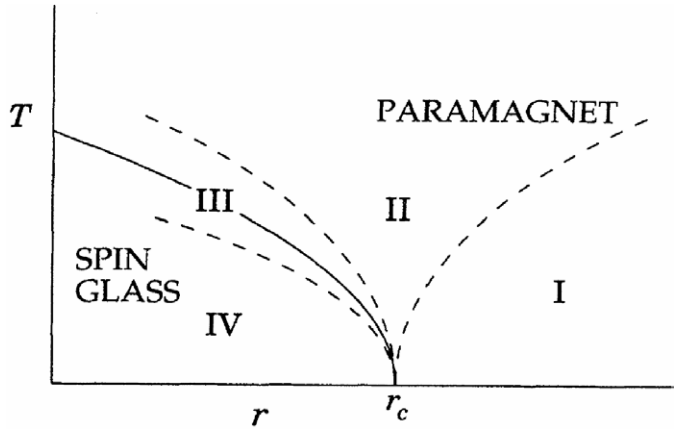

Figure 15. Generic phase diagram (following Read *et al* (1995)) of the quantum critical behaviour for spin glasses. The parameter r, which measures the quantum fluctuations, can represent the transverse field for localized spin models or the Fermi energy in metallic spin glasses.

temperature $T_{SG}(\Gamma = 0) = J$. Quantum fluctuations are introduced by turning on the transverse field, which induces up–down spin flips with tunnelling rate $\sim\Gamma$. As Γ grows, the critical temperature $T_{SG}(\Gamma)$ decreases, until the QCP is reached at $\Gamma = \Gamma_c \approx 0.731J$, signalling the $T = 0$ transition from a spin-glass to a quantum-disordered paramagnetic state (figure 15).

Similarly as in DMFT theories for electronic systems, such infinite range models can be formally reduced to a self-consistent solution of an appropriate quantum impurity problem, as first discussed in the context of quantum spin glasses by Bray and Moore (1980). Early work quickly established the phase diagram (Dobrosavljević and Stratt 1987) of this model, but the dynamics near the QCP proved more difficult to unravel, even when the critical point is approached from the quantum-disordered side. Here, the problem reduces to solving for the dynamics of a single Ising spin in a transverse field, described by an effective Hamiltonian (Miller and Huse 1993) of the form

$$H = -\frac{1}{2}J^2 \iint \mathrm{d}\tau \mathrm{d}\tau' \, \sigma^z(\tau)\chi(\tau - \tau')\sigma^z(\tau') + \Gamma \int \mathrm{d}\tau \, \sigma^x(\tau).$$

Physically, the interaction of the considered spin with the spin fluctuations of its environment generates the retarded interaction described by the 'memory kernel' $\chi(\tau - \tau')$. An appropriate self-consistency condition relates the memory kernel to the disorder-averaged local dynamical susceptibility of the quantum spin

$$\chi(\tau - \tau') = \overline{\langle T\sigma^z(\tau)\sigma^z(\tau')\rangle}.$$

A complete solution of the quantum critical behaviour can be obtained, as first established in a pioneering work by Miller and Huse (1993). These authors have set up a diagrammatic perturbation theory for the dynamic susceptibility, showing that the leading loop approximation already captures the exact quantum critical behaviour, as the higher order corrections provide only quantitative renormalizations. The dynamical susceptibility takes the general form

$$\chi(\omega_n) = \chi_0 + (\omega_n^2 + \Delta^2)^{1/2},$$

where the local static susceptibility χ_0 remains finite throughout the critical regime, and the spin excitations exist above a gap

$$\Delta \sim \left(\frac{r}{|\ln r|}\right)^{1/2}$$

which vanishes as the transition is approached from the paramagnetic side (here $r = (\Gamma - \Gamma_c)/\Gamma_c$ measures the distance from the critical point).

The quantum spin glass phase and the replicon mode. The validity of this solution was confirmed by a generalization (Ye *et al* 1993) to the M-component rotor model (the Ising model belongs to the same universality class as the $M = 2$ rotor model), which can be solved in closed form in the large M limit. This result, which proves to be exact to all orders in the $1/M$ expansion, could be extended even to the spin glass phase, where a full replica symmetric solution was obtained. Most remarkably, the spin excitation spectrum remains gapless ($\Delta = 0$) throughout the ordered phase. Such gapless excitations commonly occur in ordered states with broken continuous symmetry, but are generally not expected in classical or quantum models with a discrete symmetry of the order parameter. In glassy phases (at least within mean-field solutions), however, gapless excitations generically arise for both classical and quantum models. Here, they reflect the marginal stability (Mézard *et al* 1986) found in the presence of replica symmetry breaking (RSB), a phenomenon which reflects the high degree of frustration in these systems. The role of the Goldstone mode in this case is played by the so-called 'replicon' mode, which describes the collective low energy excitations characterizing the glassy state.

A proper treatment of the low energy excitations in this regime requires special attention to the role of RSB in the $T \to 0$ limit. The original work (Ye *et al* 1993) suggested that RSB is suppressed at $T = 0$, so that the simpler replica symmetric solution can be used at low temperatures. Later work (Georges *et al* 2001), however, established that the full RSB solution must be considered before taking the $T \to 0$ limit, and only then can the correct form of the leading low temperature corrections (e.g. the linear T-dependence of the specific heat) be obtained.

Physical content of the mean-field solution. In appropriate path-integral language (Dobrosavljević and Stratt 1987, Ye *et al* 1993), the problem can be shown to reduce to solving a one-dimensional classical Ising model with long-range interactions, the form of which must be self-consistently determined. Such classical spin chains with long-range interactions in general can be highly non-trivial. Some important examples are the Kondo problem (Anderson and Yuval 1969, Anderson *et al* 1970, Yuval and Anderson 1970) and the dissipative two-level system, both of which map to an Ising chain with $1/\tau^2$ interactions. Quantum phase transitions in these problems correspond to the Kosterlitz–Thouless transition found in the Ising chain (Kosterlitz 1976), the description of which required a sophisticated renormalization-group analysis. Why then is the solution of the quantum Ising spin glass model so simple? The answer was provided in the paper by Ye *et al* (1993), which emphasized that the critical state (and the RSB spin-glass state) does not correspond to the critical point, but rather to the high-temperature phase of the equivalent Ising chain, where a perturbative solution is sufficient. In Kondo language, this state corresponds to the Fermi-liquid solution characterized by a finite Kondo scale, as demonstrated by a quantum Monte Carlo calculation of Rozenberg and Grempel (1998), which also confirmed other predictions of the analytical theory.

From a more general perspective, the possibility of obtaining a simple analytical solution for quantum critical dynamics has a simple origin. It follows from the fact that all corrections

to Gaussian (i.e. Landau) theory are irrelevant above the upper critical dimension, as first established by the Hertz–Millis theory (Hertz 1976, Millis 1993) for conventional quantum criticality. The mean-field models become exact in the limit of infinite dimensions, hence the Gaussian solution of Miller and Huse (1993) and Ye *et al* (1993) becomes exact. Leading corrections to mean-field theory for rotor models were examined by an ε-expansion below the upper critical dimension $d_c = 8$ for the rotor models by Read, Sachdev and Ye, but these studies found run-away flows, presumably indicating non-perturbative effects that require more sophisticated theoretical tools. Most likely these include Griffiths-phase phenomena controlled by the IRFP, as already discussed in section 4.2.1.

Metallic spin glasses. A particularly interesting role of the low-lying excitations associated with the spin-glass phase is found in metallic spin glasses. Here the quantum fluctuations are provided by the Kondo coupling between the conduction electrons and local moments, and therefore can be tuned by controlling the Fermi energy in the system. The situation is again the simplest for Ising spins where an itinerant version of the rotor model of Sengupta and Georges (1995) can be considered, and similar results have been obtained for the 'spin-density glass' model of Sachdev *et al* (1995). The essential new feature in these models is the presence of itinerant electrons which, as in the Hertz–Millis approach (Hertz 1976, Millis 1993), have to be formally integrated out before an effective order-parameter theory can be obtained. This is justified *provided* that the quasi-particles remain well-defined at the QCP, i.e. the quasi-particle weight $Z \sim T_K$ remains finite and the Kondo effect remains operative. The validity of these assumptions is by no means obvious, and led to considerable controversy before a detailed quantum Monte Carlo solution of the model became available (Rozenberg and Grempel 1999), confirming the proposed scenario.

Under these assumptions, the theory can again be solved in closed form, and we only quote the principal results. Physically, the essential modification is that the presence of itinerant electron now induces Landau damping, which creates dissipation for the collective mode. As a result, the dynamics is modified, and the local dynamic susceptibility now takes the following form

$$\chi(\omega_n) = \chi_0 + (|\omega_n| + \omega^*)^{1/2}. \tag{129}$$

The dynamics is characterized by the crossover scale $\omega^* \sim r$ (r measures the distance to the transition) which defines a crossover temperature $T^* \sim \omega^*$ separating the Fermi liquid regime (at $T \ll T^*$) from the quantum critical regime (at $T \gg T^*$). At the critical point $\chi(\omega_n) = \chi_0 + |\omega_n|^{1/2}$, leading to NFL behaviour of all physical quantities, which acquire a leading low-temperature correction of the $T^{3/2}$ form. This is a rather mild violation of Fermi liquid theory, since both the static spin susceptibility and the specific heat coefficient remain finite at the QCP. A more interesting feature, which is specific to glassy systems, is the persistence of such quantum critical NFL behaviour *throughout* the metallic glass phase, reflecting the role of the replicon mode.

4.3.3. Spin-liquid behaviour, destruction of the Kondo effect by bosonic dissipation, and fractionalization

Quantum Heisenberg spin glass and the spin-liquid solution. Quantum spin glass behaviour proves to be much more interesting in the case of Heisenberg spins, where the Berry phase term (Fradkin 1991) plays a highly non-trivial role, completely changing the dynamics even within the paramagnetic phase. While the existence of a finite temperature spin-glass transition was established even in early work (Bray and Moore 1980), solving for the details of the dynamics proved difficult until the remarkable work of Sachdev and Ye (1993). By a clever use of large-N methods, these authors identified a striking *spin-liquid* solution within the paramagnetic phase.

In contrast to the non-singular behaviour of Ising or rotor quantum spin glasses, the dynamical susceptibility now displays a logarithmic singularity at low frequency. On the real axis it takes the form

$$\chi(\omega) \sim \ln\left(\frac{1}{|\omega|}\right) + \mathrm{i}\frac{\pi}{2}\mathrm{sign}(\omega).$$

A notable feature of this solution is that it is precisely of the form postulated for 'MFL' phenomenology (Varma *et al* 1989) of doped cuprates. The specific heat is also found to assume a singular form $C \sim \sqrt{T}$, which was shown (Georges *et al* 2001) to reflect a non-zero extensive entropy if the spin liquid solution is extrapolated to $T = 0$. Of course, the spin-liquid solution becomes unstable at a finite ordering temperature, and the broken symmetry state has to be examined to discuss the low temperature properties of the model.

Subsequent work (Georges *et al* 2001) demonstrated that this mean-field solution remains valid for all finite N and generalized the solution to the spin-glass (ordered) phase. A closed set of equations describing the low temperature thermodynamics in the spin glass phase was obtained, which was very recently re-examined in detail (Camjayi and Rozenberg 2003) to reveal fairly complicated behaviour.

Metallic Heisenberg spin glasses and fractionalization. Even more interesting is the fate of this spin liquid solution in itinerant systems, where an additional Kondo coupling is added between the local moments and the conduction electrons. The mean-field approach can be extended to this interesting situation by examining a Kondo–Heisenberg spin glass model (Burdin *et al* 2002) with the Hamiltonian

$$H_{\mathrm{KH}} = -t\sum_{\langle ij\rangle\sigma}(c^{\dagger}_{i\sigma}c_{j\sigma} + \mathrm{H.c.}) + J_K\sum_i \boldsymbol{S}_i\cdot\boldsymbol{s}_i + \sum_{\langle ij\rangle} J_{ij}\boldsymbol{S}_i\cdot\boldsymbol{S}_j. \tag{130}$$

In the regime where the scale of the RKKY interaction $J = \langle J_{ij}^2\rangle^{1/2}$ is small compared to the Kondo coupling J_K, one expects Kondo screening to result in standard Fermi liquid behaviour. In the opposite limit, however, the spin fluctuations associated with the retarded RKKY interactions may be able to adversely affect the Kondo screening, and novel metallic behaviour could emerge. This intriguing possibility can be precisely investigated in the mean-field (infinite range) limit, where the problem reduces to a single-impurity action of the form (Burdin *et al* 2002)

$$S^{\mathrm{KH}}_{\mathrm{eff}} = \sum_\sigma \int_0^\beta \mathrm{d}\tau\, c^{\dagger}_\sigma(\tau)(\partial_\tau - \mu + v_j)c_\sigma(\tau) - t^2\sum_\sigma \int_0^\beta \mathrm{d}\tau \int_0^\beta \mathrm{d}\tau'\, c^{\dagger}_\sigma(\tau)G_c(\tau-\tau')c_\sigma(\tau')$$
$$+ J_K\int_0^\beta \mathrm{d}\tau\, \boldsymbol{S}(\tau)\cdot\boldsymbol{s}(\tau) - \frac{J^2}{2}\int_0^\beta \mathrm{d}\tau\int_0^\beta \mathrm{d}\tau'\,\chi(\tau-\tau')\boldsymbol{S}(\tau)\cdot\boldsymbol{S}(\tau'). \tag{131}$$

Such a single-impurity action (131) describes the so-called Bose–Fermi Kondo (BFK) impurity model (Sengupta 2000, Si and Smith 1996, Zaránd and Demler 2002, Zhu and Si 2002) where, in addition to the coupling to the fermionic bath of conduction electrons, the Kondo spin also interacts with a bosonic bath of spin fluctuations, with local spectral density $\chi(\omega_n)$. Because the same BFK model also appears in 'extended' DMFT theories (Smith and Si 2000) of quantum criticality in clean systems (Sengupta 2000, Si and Smith 1996, Si *et al* 2001, Zaránd and Demler 2002, Zhu and Si 2002), its properties have been studied in detail and are by now well understood.

In the absence of the RKKY coupling ($J = 0$), the ground state of the impurity is a Kondo singlet for any value of $J_K \neq 0$. By contrast, when $J > 0$, the dissipation induced by the bosonic bath tends to destabilize the Kondo effect. For a bosonic bath of 'ohmic' form ($\chi(\omega_n) = \chi_0 - C|\omega_n|$), this effect only leads to a finite decrease of the Kondo

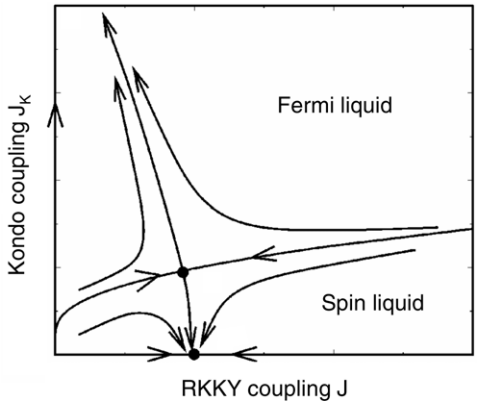

Figure 16. Phase diagram of the BFK model in the presence of a sub-ohmic bosonic bath (Sengupta 2000, Si and Smith 1996, Zaránd and Demler 2002, Zhu and Si 2002). Kondo screening is destroyed for sufficiently large dissipation (RKKY coupling to spin fluctuations).

temperature, but the Fermi liquid behaviour persists. In contrast, for 'sub-ohmic' dissipation ($\chi(\omega_n) = \chi_0 - C|\omega_n|^{1-\epsilon}$ with $\epsilon > 0$) two different phases exist, and for sufficiently large RKKY coupling the Kondo effect is destroyed. The two regimes are separated by a quantum phase transition (see figure 16).

Of course, in the considered Kondo lattice model with additional RKKY interactions, the form of the bosonic bath $\chi(\omega_n)$ is self-consistently determined, and can take different forms as the RKKY coupling J is increased. The model was analytically solved within a large N approach by Burdin *et al* (2002), who calculated the evolution of the Fermi liquid coherence scale T^*, and the corresponding quasi-particle weight Z in the presence of RKKY interactions. Within the paramagnetic phase both $T^*(J)$ and $Z(J)$ are found to decrease with J until the Kondo effect (and thus the Fermi liquid) is destroyed at $J = J_c \approx 10T_K^0$ (here T_K^0 is the $J = 0$ Kondo temperature), where both scales vanish (Burdin *et al* 2002, Tanasković *et al* 2004a). At $T > T^*(J)$ (and of course at any temperature for $J > J_c$) the spins effectively decouple from conduction electrons, and spin liquid behaviour, essentially identical to that of the insulating model, is established. Thus, sufficiently strong and frustrating RKKY interactions are able to suppress Fermi liquid behaviour, and MFL behaviour emerges in a metallic system. This phenomenon, corresponding to spin-charge separation resulting from the destruction of the Kondo effect, is sometimes called 'fractionalization' (Coleman and Andrei 1987, Demler *et al* 2002, Kagan *et al* 1992, Senthil *et al* 2003, 2004). Such behaviour has often been advocated as an appealing scenario for exotic phases of strongly correlated electrons, but with the exception of the described model, there are very few well established results and model calculations to support its validity. Finally, we should mention related work (Parcollet and Georges 1999) on doped Mott insulators with random exchanges, with many similarities with the above picture.

We should note, however, that this exotic solution is valid only within the paramagnetic phase, which is generally expected to become unstable to magnetic (spin glass) ordering at sufficiently low temperatures. Since fractionalization emerges only for sufficiently large RKKY coupling (in the large N model $J_c \approx 10T_K^0$), while, in general, one expects magnetic ordering to take place already at $J \sim T_K^0$ (according to the famous Doniach criterion (Doniach 1977)), one expects (Burdin *et al* 2002) the system to magnetically order much before the Kondo temperature vanishes. If this is true, then one expects the quantum critical behaviour to be very similar to metallic Ising spin glasses, i.e. to assume the conventional Hertz–Millis form, at least the for mean-field spin glass models we discussed here. The precise relevance

of this paramagnetic spin liquid solution thus remains unclear, at least for systems with weak or no disorder in the conduction band.

Fractionalized two-fluid behaviour of electronic Griffiths phases. The situation seems more promising in the presence of sufficient amounts of disorder, where the electronic Griffiths phase forms. As we have seen in section 4.1.6, here the disorder generates a very broad distribution of local Kondo temperatures, making the system much more sensitive to RKKY interactions. This mechanism has recently been studied within an extended DMFT approach (Tanasković *et al* 2004a), which is able to incorporate both the formation of the Griffiths phase, and the effects of frustrating RKKY interactions leading to spin-glass dynamics. At the local impurity level, the problem still reduced to the BFK model, but the presence of conduction electron disorder qualitatively modifies the self-consistency conditions determining the form of $\chi(\omega_n)$.

To obtain a sufficient condition for decoupling, we examine the stability of the Fermi liquid solution, by considering the limit of infinitesimal RKKY interactions. To leading order we replace $\chi(\omega_n) \to \chi_0(\omega_n) \equiv \chi(\omega_n; J = 0)$, and the calculation reduces to the 'bare model' of Tanasković *et al* (2004a). In this case, from equation (61), $P(T_K) \sim T_K^{\alpha-1}$, where $\alpha \sim 1/W^2$ and

$$\chi_0(\omega_n) \sim \int \mathrm{d}T_K\, P(T_K)\chi(\omega_n, T_K) \sim \chi_0(0) - C_0|\omega_n|^{1-\epsilon}, \tag{132}$$

where $\epsilon = 2 - \alpha$. Thus, for sufficiently strong disorder (i.e. within the electronic Griffiths phase), even the 'bare' bosonic bath is sufficiently singular to generate decoupling. The critical value of W will be modified by self-consistency, but it is clear that decoupling will occur for sufficiently large disorder.

Once decoupling is present, the system is best viewed as composed of two fluids, one made up of a fraction n of decoupled spins and the other of a fraction $(1 - n)$ of Kondo screened spins. The self-consistent $\chi(\omega_n)$ acquires contributions from both fluids

$$\chi(\omega_n) = n\chi_{\mathrm{dc}}(\omega_n) + (1 - n)\chi_s(\omega_n). \tag{133}$$

A careful analysis (Tanasković *et al* 2004a) shows that, for a bath characterized by an exponent ϵ

$$\chi_{\mathrm{dc}}(\omega_n) \sim \chi_{\mathrm{dc}}(0) - C|\omega_n|^{1-(2-\epsilon)}, \tag{134}$$

$$\chi_s(\omega_n) \sim \chi_s(0) - C'|\omega_n|^{1-(2-\epsilon-1/\nu)}, \tag{135}$$

where $\nu = \nu(\epsilon)$ is a critical exponent governing how the Kondo scale vanishes at the QCP of the Bose–Fermi model. Since $\nu > 0$, the contribution of the decoupled fluid is more singular and dominates at lower frequencies. Self-consistency then yields $\epsilon = 1$, as in the familiar spin liquid state of Sachdev and Ye (1993). For $\epsilon = 1$, the local susceptibility is logarithmically divergent (both in ω_n and T). This does not necessarily mean that the bulk susceptibility, which is the experimentally relevant quantity, behaves in the same manner (Parcollet and Georges 1999). More work remains to be done to determine the precise low temperature form of this and other physical quantities, and to assert the relevance of this mechanism for specific materials.

As in the case where conduction electron disorder is absent, the spin liquid state is unstable towards spin-glass ordering at sufficiently low temperatures. However, numerical estimates for the Griffiths phase model (Tanasković *et al* 2004a) suggest a surprisingly wide temperature window where the marginal behaviour should persist above the ordering temperature. Figure 17 represents the predicted phase diagram of this model. For weak disorder the system is in the Fermi liquid phase, while for $W > W_c$ the MFL phase emerges. The crossover temperature (dashed line) delimiting this regime can be estimated from the frequency up to which the logarithmic behaviour of the local dynamical susceptibility $\chi(i\omega)$ is observed. The spin glass phase, obtained from equation (127), appears only at the lowest temperatures, well below the

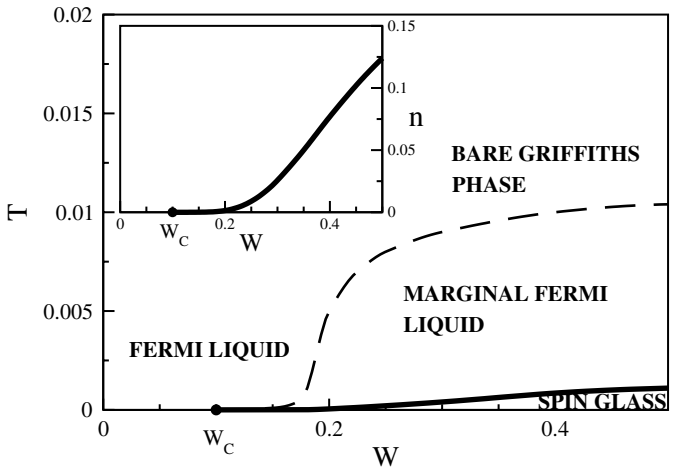

Figure 17. Phase diagram of the electronic Griffiths phase model with RKKY interactions (Tanasković *et al* 2004a). The inset shows the fraction of decoupled spins as a function of the disorder strength W.

MFL boundary. Interestingly, recent experiments (MacLaughlin *et al* 2001) have indeed found evidence of dynamical spin freezing in the millikelvin temperature range for the same Kondo alloys that display normal phase NFL behaviour in a much broader temperature window.

The two-fluid phenomenology of the disordered Kondo lattice we have described above is very reminiscent of earlier work on the clean Kondo lattice, where the conduction electrons effectively decouple from the local moments, the latter forming a spin liquid state (Coleman and Andrei 1987, Demler *et al* 2002, Kagan *et al* 1992, Senthil *et al* 2003, 2004). The major difference between the results presented in this section and these other cases is that here local spatial disorder fluctuations lead to an *inhomogeneous* coexistence of the two fluids, as each site decouples or not from the conduction electrons depending on its local properties. The discussed mean-field models should be considered as merely the first examples of this fascinating physics. The specific features of the spin liquid behaviour that were obtained from these models may very well prove to be too restrictive and perhaps even inaccurate. For example, the specific heat enhancements may well be overestimated, reflecting the residual $T = 0$ entropy of the mean-field models. Nevertheless, the physics of Kondo screening being destroyed by the interplay of disorder and RKKY interactions will almost certainly play a central role in determining the properties of many NFL systems, and clearly needs to be investigated in more detail in the future.

4.3.4. Electron glasses, freezing in the charge sector and the quantum AT line. Glassy behaviour in disordered electronic systems is not limited to phenomena associated with the freezing of spins. In fact, glassy physics in the charge sector was already envisioned a long time ago in the pioneering works of Efros and Shklovskii (Efros and Shklovskii 1975, Shklovskii and Efros 1984) and Pollak (1984) on disordered insulators. It is expected to result from the competition of the long-ranged Coulomb interaction and disorder. The physical picture that has emerged from these works is easy to understand. In absence of disorder, the Coulomb repulsion tends to keep the electrons as far from each other as possible. If quantum or thermal fluctuations are sufficiently small, the electrons tend to assume a periodic pattern, leading to the formation of a charge density wave (e.g. a Wigner crystal). In contrast, disorder tends

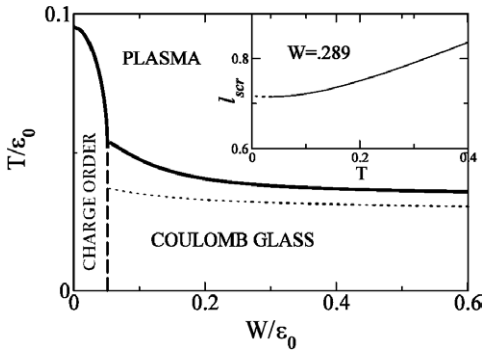

Figure 18. Phase diagram of the three-dimensional classical Coulomb glass, as obtained from the EDMFT approach, following Pankov and Dobrosavljević (2005). The full horizontal line indicates the glass transition temperature, and the dotted line shows where the entropy of the fluid solution turns negative. The inset shows the temperature dependence of the screening length as a function of temperature.

to arrange the electrons in a random fashion, opposing the periodic pattern favoured by the Coulomb interaction. When both effects are comparable the system is frustrated: there now exist many different low energy configurations separated by potential barriers, leading to glassy dynamics.

Of course, these glassy effects are most pronounced in disordered insulators, where quantum fluctuations are minimized by electron localization. Over the last 30 years a large number of theoretical studies have concentrated on the physics of the Coulomb glass. In this review we will not attempt to describe in detail this large body of work, since many of these results do not directly touch upon the NFL physics observed in metals. We will limit our attention to those works that concentrated on the manifestations of electronic glass behaviour on the metallic side of the MIT. The original works of Efros and Shklovskii (Efros and Shklovskii 1975, Shklovskii and Efros 1984) and Pollak (1984) concentrated on classical models for the electron glass, as did the work of many followers. Because most workers used numerical approaches to investigate the problem, it proved difficult to incorporate the effects of quantum fluctuations into this picture, which cannot be avoided close to the MIT.

More recently, extended DMFT approaches (EDMFT) (Chitra and Kotliar 2000, Smith and Si 2000) were shown to capture many relevant aspect of the classical Coulomb glasses, such as the formation of the Coulomb gap (Pankov and Dobrosavljević 2005, Pastor and Dobrosavljević 1999) (figure 18). In addition, these theories were able to discuss the effects of quantum fluctuations (Dobrosavljević *et al* 2003, Pastor and Dobrosavljević 1999) in the vicinity of the MIT. In the following we briefly describe the physical picture of the Coulomb glass emerging from these theories.

Coulomb gap, the replicon mode and self-organized criticality. The classical Coulomb glass model of Efros and Shklovskii (ES) (Efros and Shklovskii 1975, Shklovskii and Efros 1984) is given by the Hamiltonian

$$H = \sum_i v_i n_i + \frac{1}{2} \sum_{ij} \frac{e^2}{r_{ij}} (n_i - K)(n_j - K). \tag{136}$$

Here $n_i = 0, 1$ is the electron occupation number and v_i is a Gaussian distributed random potential of variance W^2. In their classic work on this model ES presented convincing evidence that at $T = 0$ a soft 'Coulomb gap' emerges in the single-particle DOS which, in arbitrary

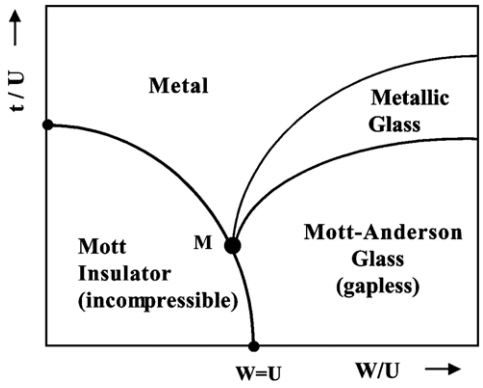

Figure 19. Global phase diagram for the disordered Hubbard model (Dobrosavljević *et al* 2003), as a function of the hopping element t and the disordered strength W, both expressed in units of the on-site interaction U. The size of the metallic glass phase is determined by the strength of the inter-site Coulomb interaction V.

dimension d, takes the form

$$g(\varepsilon) \sim \varepsilon^{d-1}. \tag{137}$$

From a general point of view this result is quite surprising. It indicates a power-law distribution of excitation energies, i.e. the absence of a characteristic energy scale for excitations above the ground state. Such behaviour is common in models with broken continuous symmetry, where it reflects the corresponding Goldstone modes, but is generally not expected in discrete symmetry models, such as the one used by ES. Within the EDMFT theory of the Coulomb glass (Muller and Ioffe 2004, Pankov and Dobrosavljević 2005, Pastor and Dobrosavljević 1999) this puzzling feature finds a natural explanation: it reflects the emergence of the soft 'replicon' mode, similarly to the classical and quantum spin-glass models we have already discussed. As a result, again in direct analogy with spin glass models, the ordered state of the Coulomb glass is expected to display self-organized criticality (Pazmandi *et al* 1999) and hysteresis behaviour (Pastor *et al* 2002) characterized by avalanches on all scales.

Another interesting feature of the Coulomb glass is worth mentioning. As explicitly demonstrated by examining the broken replica symmetric solution of the mean-field equations (Muller and Ioffe 2004), the screening length is found to diverge as $\ell_{\rm scr} \sim 1/T$ as a result of the vanishing zero-field cooled compressibility (Pastor and Dobrosavljević 1999) within the glassy phase. This result is significant because it explains the absence of screening in the ground state of the Coulomb glass, in agreement with the assumptions of the ES theory. It also indicates the divergence of the effective coordination number in the $T \to 0$ limit, giving further credence to the predictions of the mean-field approach.

Quantum Coulomb glass and the metallic glass phase. When quantum fluctuations are introduced in the model by including hopping amplitudes t_{ij} between lattice sites, one must examine the $T = 0$ transition where glassy freezing of electrons is suppressed. Such quantum melting of the electron glass was examined in Dobrosavljević *et al* (2003) and Pastor and Dobrosavljević (1999). Interestingly, the theory shows (Dobrosavljević *et al* 2003) that the approach to an Anderson-like insulator is a singular perturbation to the stability of the glassy phase. Physically, this reflects the fact that the quantum fluctuations in this model arise from the mobility of itinerant electrons—which is expected to vanish precisely at the MIT. As a result, the glass transition is predicted to generically *precede* the MIT (figure 19), giving rise

to an intermediate metallic glass phase, consistent with some recent experiments (Bogdanovich and Popović 2002).

Critical behaviour along the quantum AT line. As we can see from equation (136), the Coulomb glass Hamiltonian is essentially identical to that of an AFM random field Ising model (RFIM) with long-range interactions. The random potential v_i plays the role of a random symmetry-breaking field, and as a result, the finite temperature glass transition in our model assumes the character of a de Almeida–Thouless (AT) line (de Almeida and Thouless 1978, Mézard *et al* 1986). The $T = 0$ transition separating the Fermi liquid from a NFL metallic glass phase thus assumes the character of a 'quantum AT line'. A complete description of this QCP can be obtained within the mean-field formulation, which has been studied in detail in Dalidovich and Dobrosavljević (2002). Similarly as in metallic spin glasses (see section 4.3.2), within the quantum critical regime and the entire metallic glass phase, the resistivity acquires a NFL $T^{3/2}$ form, as seen in some experiments (Bogdanovich and Popović 2002). In contrast to the metallic spin glass scenario, the approach to the quantum AT line is characterized (Dalidovich and Dobrosavljević 2002) by a crossover scale

$$T^* \sim \delta t^2, \tag{138}$$

where $\delta t = (t - t_c)/t_c$ measures the distance to the critical point. This indicates a much broader quantum critical region in this case, which may explain the apparent suppression of weak localization (Fermi liquid) corrections in two dimensional electron systems near the MIT (Abrahams *et al* 2001).

5. Conclusions and open problems

We have seen throughout this review that many routes can lead to disorder-induced NFL behaviour. In this concluding section, we would like to pause to analyse general trends of what has been done so far, what is still missing and what are the most promising directions for future work.

We have seen how the phenomenology of quantum Griffiths singularities seems to afford a natural description for many of the anomalies seen in several systems, from doped semiconductors to disordered heavy fermion systems. On the other hand, we have listed several outstanding points which still need to (and should) be clarified. More generally, however, we would like to stress a different question. Griffiths phases rely on the existence of rare disorder configurations which generate essentially non-interacting fluctuators. It is the slow quantum dynamics of these fluctuators which eventually leads to the known singularities. This situation is often described as being one in which quantum mechanics (tunnelling) is a 'dangerously irrelevant operator' responsible for changing the scaling dimension of otherwise classical (and non-interacting) objects. On the other hand, we have described how droplet–droplet interactions can dramatically change this picture and be one possible destabilizing factor of Griffiths physics (section 4.2.3). This effect represents essentially the *delocalization* of the local (droplet) spin modes which are at the heart of the Griffiths phases. An important outstanding issue in this context is the determination of the lower critical dimension for spin mode localization. This is especially important in the case of metallic systems where the spin interactions are long-ranged.

At or below this lower critical dimension, there appears the question of the existence of an IRFP in metallic systems with continuous symmetry. These represent the best candidates so far for a quantum Griffiths phase which can survive down to zero temperature, as discussed in section 4.2.3. Below the (still unknown) lower critical dimension for spin mode localization and based on the well-understood case of insulating systems, we would expect that this Griffiths

phase would be a precursor of a phase transition governed by an IRFP. In fact, this is probably the only known route for such a point that does not require fine-tuning in systems with continuous symmetry. Indeed, as explained in section 4.2.1, except for a few special model systems (the nearest-neighbour random AFM Heisenberg and XX chains being the most prominent examples), almost all other perturbations, such as increased connectivity and/or ferromagnetic interactions, seem to lead to a large spin phase. These include the important case studied by Bhatt and Lee in the context of doped semiconductors. Therefore, the rigorous establishment of the stability of a Griffiths phase and an associated IRFP in systems with continuous symmetry and dissipation would be an important achievement.

Above the lower critical dimension, we expect glassy dynamics to take centre stage. Here, very interesting possibilities exist for non-trivial physics even in the paramagnetic phase, as hinted by the results and references quoted in section 4.3.3. Other questions which arise are: (i) is there a Griffiths phase close to the spin glass QCP? (ii) is it governed by an IRFP or is it characterized by finite disorder? (iii) how is the charge transport affected by the spin dynamics? (iv) what is the role of dissipation?

The exploration of glassiness in the charge sector also deserves further attention. Here again long-range Coulomb interactions play a crucial role. The interplay with the spin sector is still largely unexplored. This seems especially important in the case of the disorder-induced MIT both in three and in two dimensions.

We hope that addressing and answering some of these questions will provide important clues as to the origin of NFL behaviour in disordered systems.

Acknowledgments

We have benefitted over the years from discussions with many colleagues and collaborators. Some deserve special mention: E Abrahams, M C O Aguiar, M Aronson, R N Bhatt, A H Castro Neto, P Coleman, M A Continentino, D Dalidovich, A Georges, L P Gor'kov, D R Grempel, J A Hoyos, G Kotliar, D E MacLaughlin, A J Millis, D K Morr, J A Mydosh, C Panagopoulos, D Popović, M J Rozenberg, S Sachdev, J Schmalian, Q Si, G Stewart, S Sülow, D Tanasković, C M Varma, A P Vieira, T Vojta, K Yang and G Zaránd.

This work was supported by FAPESP through grant 01/00719-8 (EM), by CNPq through grant 302535/02-0 (EM), by the NSF through grant NSF-0234215 (VD) and the National High Magnetic Field Laboratory (VD and EM).

References

Abou-Chacra R, Anderson P W and Thouless D J 1973 *J. Phys. C: Solid State Phys.* **6** 1734
Abrahams E, Anderson P W, Licciardello D C and Ramakrishnan T V 1979 *Phys. Rev. Lett.* **42** 673
Abrahams E, Kravchenko S V and Sarachik M P 2001 *Rev. Mod. Phys.* **73** 251
Abrikosov A A 1962 *Sov. Phys. JETP* **14** 408
Abrikosov A A, Gor'kov L P and Dzyaloshinskii I E 1975 *Methods of Quantum Field Theory in Statistical Physics* (New York: Dover)
Abrikosov A A and Khalatnikov I M 1958 *Sov. Phys. JETP* **6** 888
Aeppli G and Fisk Z 1992 *Comment. Condens. Matter Phys.* **16** 155
Affleck I and Ludwig A W W 1991a *Nucl. Phys.* B **360** 641
Affleck I and Ludwig A W W 1991b *Nucl. Phys.* B **352** 849
Affleck I and Ludwig A W W 1991c *Phys. Rev. Lett.* **67** 161
Aguiar M C O, Miranda E and Dobrosavljević V 2003 *Phys. Rev.* B **68** 125104
Aguiar M C O, Miranda E, Dobrosavljević V, Abrahams E and Kotliar G 2004 *Europhys. Lett.* **67** 226
de Almeida J R L and Thouless D J 1978 *J. Phys.* A **11** 983
Altshuler B L and Aronov A B 1979 *JETP Lett.* **50** 968

Anderson P and Yuval G 1969 *Phys. Rev. Lett.* **23** 89
Anderson P, Yuval G and Hamman D 1970 *Phys. Rev.* B **1** 4464
Anderson P W 1958 *Phys. Rev.* **109** 1492
Anderson P W 1961 *Phys. Rev.* **124** 41
de Andrade M C *et al* 1998 *Phys. Rev. Lett.* **81** 5620
Andraka B 1994 *Phys. Rev.* B **49** 3589
Andraka B and Stewart G R 1993 *Phys. Rev.* B **47** 3208
Andraka B and Tsvelik A M 1991 *Phys. Rev. Lett.* **67** 2886
Aronson M C, Obsorn R, Robinson R A, Lynn J W, Chau R, Seaman C L and Maple M B 1995 *Phys. Rev. Lett.* **75** 725
Aronson M C, Osborn R, Chau R, Maple M B, Rainford B D and Murani A P 2001 *Phys. Rev. Lett.* **87** 197205
Arrachea L, Dalidovich D, Dobrosavljević V and Rozenberg M J 2004 *Phys. Rev.* B **69** 064419
Balakirev F F, Betts J B, Migliori A, Ono S, Ando Y and Boebinger G S 2003 *Nature* **424** 912
Barnes S E 1976 *J. Phys. F: Met. Phys.* **7** 1375
Bauer E D, Booth C H, Kwei G H, Chau R and Maple M B 2002 *Phys. Rev.* B **65** 245114
Baym G and Pethick C 1991 *Landau Fermi Liquid Theory: Concepts and Applications* (New York: Wiley)
Béal-Monod M T and Maki K 1975 *Phys. Rev. Lett.* **34** 1461
Belitz D and Kirkpatrick T R 1994 *Rev. Mod. Phys.* **66** 261
Beloborodov I S, Efetov K B, Lopatin A and Vinokur V 2003 *Phys. Rev. Lett.* **91** 246801
Benfatto G and Gallavotti G 1990 *Phys. Rev.* B **42** 9967
Berk N F and Schrieffer J R 1966 *Phys. Rev. Lett.* **17** 433
Bernal O O, MacLaughlin D E, Lukefahr H G and Andraka B 1995 *Phys. Rev. Lett.* **75** 2023
Bhatt R N and Fisher D S 1992 *Phys. Rev. Lett.* **68** 3072
Bhatt R N and Lee P A 1981 *J. Appl. Phys.* **52** 1703
Bhatt R N and Lee P A 1982 *Phys. Rev. Lett.* **48** 344
Boebinger G S, Ando Y, Passner A, Kimura T, Okuya M, Shimoyama J, Kishio K, Tamasaku K, Ichikawa N and Uchida S 1996 *Phys. Rev. Lett.* **77** 5417
Boechat B, Saguia A and Continentino M A 1996 *Solid State Commun.* **98** 411
Bogdanovich S and Popović D 2002 *Phys. Rev. Lett.* **88** 236401
Bogenberger B and Löhneysen H v 1995 *Phys. Rev. Lett.* **74** 1016
Booth C H, MacLaughlin D E, Heffner R H, Chau R, Maple M B and Kwei G H 1998 *Phys. Rev. Lett.* **81** 3960
Booth C H, Scheidt E-W, Killer U, Weber A and Kehrein S 2002 *Phys. Rev.* B **66** 140402
Bray A J and Moore M A 1980 *J. Phys. C: Solid State Phys.* **13** L655
Bruno P 2001 *Phys. Rev. Lett.* **87** 137203
Burdin S, Grempel D R and Georges A 2002 *Phys. Rev.* B **66** 045111
Büttgen N, Trinkl W, Weber J-E, Hemberger J, Loidl A and Kehrein S 2000 *Phys. Rev.* B **62** 11545
Buyers W J L 1996 *Physica* B **223–224** 9
Camjayi A and Rozenberg M J 2003 *Phys. Rev. Lett.* **90** 217202
Carlon E, Lajkó P, Rieger H and Iglói F 2004 *Phys. Rev.* B **69** 144416
Castellani C and Castro C D 1986 *Phys. Rev.* B **34** 5935
Castellani C, Castro C D, Lee P A and Ma M 1984 *Phys. Rev.* B **30** 527
Castellani C, Kotliar B G and Lee P A 1987 *Phys. Rev. Lett.* **56** 1179
Castro Neto A H, Castilla G and Jones B A 1998 *Phys. Rev. Lett.* **81** 3531
Castro Neto A H and Jones B A 2000 *Phys. Rev.* B **62** 14975
Castro Neto A H and Jones B A 2004a *Preprint* cond-mat/0412020
Castro Neto A H and Jones B A 2004b *Preprint* cond-mat/0411197
Chattopadhyay A and Jarrell M 1997 *Phys. Rev.* B **56** 2920
Chattopadhyay A, Jarrell M, Krishnamurthy H R, Ng H K, Sarrao J and Fisk Z 1998 *Preprint* cond-mat/9805127
Chau R and Maple M B 1996 *J. Phys.: Condens. Matter* **8** 9939
Chayes J T, Chayes L, Fisher D S and Spencer T 1986 *Phys. Rev. Lett.* **57** 2999
Chitra R and Kotliar G 2000 *Phys. Rev. Lett.* **84** 3678
Chou F C, Belk N, Kastner M, Birgeneau R and Aharony A 1995 *Phys. Rev. Lett.* **75** 2204
Coleman P 1984 *Phys. Rev.* B **29** 3035
Coleman P 1987 *Phys. Rev.* B **35** 5072
Coleman P and Andrei N 1987 *J. Phys.: Condens. Matter* **1** 4057
Coleman P, Pépin C, Si Q and Ramazashvili R 2001 *J. Phys.: Condens. Matter* **13** R723
Continentino M A 1994 *Phys. Rep.* **239** 179
Continentino M A 2001 *Quantum Scaling in Many-Body Systems* (Singapore: World Scientific)

Continentino M A, Fernandes J C, Guimaraes R B, Boechat B and Saguia A 2004 *Magnetic Materials* ed A Narlikar (Germany: Springer)
Continentino M A, Japiassu G M and Troper A 1989 *Phys. Rev.* B **39** 9734
Cox D L 1987 *Phys. Rev. Lett.* **59** 1240
Dagotto E 2002 *Nanoscale Phase Separation and Colossal Magnetoresistance* (Berlin: Springer)
Dalidovich D and Dobrosavljević V 2002 *Phys. Rev.* B **66** 081107
Damle K and Huse D A 2002 *Phys. Rev. Lett.* **89** 277203
Dasgupta C and Ma S k 1980 *Phys. Rev.* B **22** 1305
Degiorgi L 1999 *Rev. Mod. Phys.* **71** 687
Degiorgi L and Ott H R 1996 *J. Phys.: Condens. Matter* **8** 9901
Degiorgi L, Ott H R and Hulliger F 1995 *Phys. Rev.* B **52** 42
Degiorgi L, Wachter P, Maple M B, de Andrade M C and Herrmann J 1996 *Phys. Rev.* B **54** 6065
Demler E, Nayak C, Kee H-Y, Kim Y-B and Senthil T 2002 *Phys. Rev.* B **65** 155103
Denteneer P J H and Scalettar R T 2003 *Phys. Rev. Lett.* **90** 246401
Denteneer P J H, Scalettar R T and Trivedi N 1999 *Phys. Rev. Lett.* **83** 4610
Denteneer P J H, Scalettar R T and Trivedi N 2001 *Phys. Rev. Lett.* **87** 146401
Dobrosavljević V and Kotliar G 1993 *Phys. Rev. Lett.* **71** 3218
Dobrosavljević V and Kotliar G 1994 *Phys. Rev.* B **50** 1430
Dobrosavljević V and Kotliar G 1997 *Phys. Rev. Lett.* **78** 3943
Dobrosavljević V and Kotliar G 1998 *Phil. Trans. R. Soc. Lond.* A **356** 57
Dobrosavljević V, Kirkpatrick T R and Kotliar B G 1992 *Phys. Rev. Lett.* **69** 1113
Dobrosavljević V and Miranda E 2005 *Phys. Rev. Lett.* **94** 187203
Dobrosavljević V and Stratt R M 1987 *Phys. Rev.* B **36** 8484
Dobrosavljević V, Tanasković D and Pastor A A 2003 *Phys. Rev. Lett.* **90** 016402
Doniach S 1977 *Physica* B **91** 231
Doniach S and Engelsberg S 1966 *Phys. Rev. Lett.* **17** 750
Doty C A and Fisher D S 1992 *Phys. Rev.* B **45** 2167
Efros A L and Shklovskii B I 1975 *J. Phys. C: Solid State Phys.* **8** L49
Elliott R J, Krumhansl J A and Leath P L 1974 *Rev. Mod. Phys.* **46** 465
Finkel'stein A M 1983 *Zh. Eksp. Teor. Fiz.* **84** 168
Finkel'stein A M 1983 *Sov. Phys. JETP* **57** 97
Finkel'stein A M 1984 *Zh. Eksp. Teor. Fiz.* **86** 367
Finkel'stein A M 1983 *Sov. Phys. JETP* **59** 212
Fisher D S 1992 *Phys. Rev. Lett.* **69** 534
Fisher D S 1994 *Phys. Rev.* B **50** 3799
Fisher D S 1995 *Phys. Rev.* B **51** 6411
Fradkin E 1991 *Field Theories of Condensed Matter Systems* (Redwood City, CA: Adisson-Wesley)
Frischmuth B and Sigrist M 1997 *Phys. Rev. Lett.* **79** 147
Frischmuth B, Sigrist M, Ammon B and Troyer M 1999 *Phys. Rev.* B **60** 3388
Gajewski D A, Dilley N R, Chau R and Maple M B 1996 *J. Phys.: Condens. Matter* **8** 9793
Galitskii V M 1958 *Sov. Phys. JETP* **7** 104
Georges A and Kotliar G 1992 *Phys. Rev.* B **45** 6479
Georges A, Kotliar G, Krauth W and Rozenberg M J 1996 *Rev. Mod. Phys.* **68** 13
Georges A, Parcollet O and Sachdev S 2001 *Phys. Rev.* B **63** 134406
Goldenfeld N 1992 *Lectures on Phase Transitions and the Renormalization Group* (Reading, MA: Addison-Wesley)
Gor'kov L P and Sokol A V 1987 *JETP Lett.* **46** 420
Grewe N and Steglich F 1991 *Handbook on the Physics and Chemistry of Rare Earths* ed K A Geschneider Jr and L Eyring vol 14 (Amsterdam: Elsevier) p 343
Griffiths R B 1969 *Phys. Rev. Lett.* **23** 17
Guo M, Bhatt R N and Huse D A 1994 *Phys. Rev. Lett.* **72** 4137
Guo M, Bhatt R N and Huse D A 1996 *Phys. Rev.* B **54** 3336
Haldane F D M 1983a *Phys. Lett.* A **93** 464
Haldane F D M 1983b *Phys. Rev. Lett.* **50** 1153
Harris A B 1974 *J. Phys. C: Solid State Phys.* **7** 1671
Heffner R H and Norman M R 1996 *Comment. Condens. Matter Phys.* **17** 361
Henelius P and Girvin S M 1998 *Phys. Rev.* B **57** 11457
Hertz J A 1976 *Phys. Rev.* B **14** 1165
Hewson A C 1993 *The Kondo Problem to Heavy Fermions* (Cambridge: Cambrige University Press)

Hida K 1997 *J. Phys. Soc. Japan* **62** 330
Hida K 1999 *Phys. Rev. Lett.* **83** 3297
Hikihara T, Furusaki A and Sigrist M 1999 *Phys. Rev.* B **60** 12116
Hirsch M J, Holcomb D F, Bhatt R N and Paalanen M A 1992 *Phys. Rev. Lett.* **68** 1418
Hlubina R and Rice T M 1995 *Phys. Rev.* B **51** 9253
Hoyos J A and Miranda E 2004 *Phys. Rev.* B **69** 214411
Hyman R A and Yang K 1997 *Phys. Rev. Lett.* **78** 1783
Hyman R A, Yang K, Bhatt R N and Girvin S M 1996 *Phys. Rev. Lett.* **76** 839
Iglói F, Juhász R and Lajkó P 2001 *Phys. Rev. Lett.* **86** 1343
Iglói F and Monthus C 2005 *Preprint* cond-mat/0502448
Imada M, Fujimori A and Tokura Y 1998 *Rev. Mod. Phys.* **70** 1039
Itoh K M, Watanabe M, Ootuka Y, Haller E E and Ohtsuki T 2004 *J. Phys. Soc. Japan* **73** 173
Izuyama T, Kim D J and Kubo R 1963 *J. Phys. Soc. Japan* **18** 1925
Jagannathan A, Abrahams E and Stephen M J 1988 *Phys. Rev.* B **37** 436
Jaroszyński J, Popović D and Klapwijk T M 2002 *Phys. Rev. Lett.* **89** 276401
Jaroszynski J, Popovic D and Klapwijk T M 2004 *Phys. Rev. Lett.* **92** 226403
Joyce G. S 1969 *J. Phys. C: Solid State Phys.* **2** 1531
Kagan Y, Kikoin K A and Prokof'ev N V 1992 *Physica* B **182** 201
Kajueter H and Kotliar G 1996 *Phys. Rev. Lett.* **77** 131
Kanigel A, Keren A, Eckstein Y, Knizhnik A, Lord J and Amato A 2002 *Phys. Rev. Lett.* **88** 137003
Kivelson S A, Bindloss I P, Fradkin E, Oganesyan V, Tranquada J M, Kapitulnik A and Howald C 2003 *Rev. Mod. Phys.* **75** 1201
Kondo J 1964 *Prog. Theor. Phys.* **32** 37
Kosterlitz J M 1976 *Phys. Rev. Lett.* **37** 1577
Kotliar G 1987 *Anderson Localization* (Berlin: Springer)
Lakner M, Löhneysen H V, Langenfeld A and Wölfle P 1994 *Phys. Rev.* B **50** 17064
Landau L D 1957a, *Sov. Phys. JETP* **5** 101
Landau L D 1957b *Sov. Phys. JETP* **3** 920
Landau L D 1959 *Sov. Phys. JETP* **8** 70
Larkin A I and Melnikov V I 1972 *Sov. Phys. JETP* **34** 656
Lee P A and Ramakrishnan T V 1985 *Rev. Mod. Phys.* **57** 287
Lee P A, Rice T M, Serene J W, Sham L J and Wilkins J W 1986 *Comment. Condens. Matter Phys.* **12** 99
Leggett A J 1975 *Rev. Mod. Phys.* **47** 331
Leggett A J, Chakravarty S, Dorsey A T, Fisher M P A, Garg A and Zwerger W 1987 *Rev. Mod. Phys.* **59** 1
Lesage F and Saleur H 1998 *Phys. Rev. Lett.* **80** 4370
Li D X, Shiokawa Y, Homma Y, Uesawa A, Dönni A, Suzuki T, Haga Y, Yamamoto E, Honma T and Onuki Y 1998 *Phys. Rev.* B **57** 7434
Lieb E, Schultz T and Mattis D 1961 *Ann. Phys.* **16** 407
Limelette P, Wzietek P, Florens S, Georges A, Costi T A, Pasquier C, Jerome D, Meziere C and Batail P 2003 *Phys. Rev. Lett.* **91** 016401
Lin Y-C, Mélin R, Rieger H and Iglói F 2003 *Phys. Rev.* B **68** 024424
Liu C-Y, MacLaughlin D E, Neto A H C, Lukefahr H G, Thompson J D, Sarrao J L and Fisk Z 2000 *Phys. Rev.* B **61** 432
Loh Y L, Tripathi V and Turkalov M 2005 *Phys. Rev.* B **71** 024429
Ludwig A W W and Affleck I 1991 *Phys. Rev. Lett* **67** 3160
Luttinger J M, 1960 *Phys. Rev.* **119** 1153
Ma S k, Dasgupta C and Hu C k 1979 *Phys. Rev. Lett.* **43** 1434
MacLaughlin D E, Bernal O O, Heffner R H, Nieuwenhuys G J, Rose M S, Sonier J E, Andraka B, Chau R and Maple M B 2001 *Phys. Rev. Lett.* **87** 066402
MacLaughlin D E, Bernal O O and Lukefahr H G 1996 *J. Phys.: Condens. Matter* **8** 9855
MacLaughlin D E, Bernal O O, Sonier J E, Heffner R H, Taniguchi T and Miyako Y 2002a *Phys. Rev.* B **65** 184401
MacLaughlin D E, Heffner R H, Bernal O O, Ishida K, Sonier J E, Nieuwenhuys G J, Maple M B and Stewart G R 2004 *J. Phys.: Condens. Matter* **16 S**4479–98
MacLaughlin D E, Heffner R H, Bernal O O, Nieuwenhuys G J, Sonier J E, Rose M S, Chau R, Maple M B and Andraka B 2002b *Physica* B **312–313** 453
MacLaughlin D E, Heffner R H, Nieuwenhuys G J, Luke G M, Fudamoto Y, Uemura Y J, Chau R, Maple M B and Andraka B 1998 *Phys. Rev.* B **58** 11849
MacLaughlin D E *et al* 2000 *Physica* B **289–290** 15

MacLaughlin D E, Rose M S, Young B-L, Bernal O O, Heffner R H, Morris G D, Ishida K, Nieuwenhuys G J and Sonier J E 2003 *Physica* B **326** 381
Maebashi H, Miyake K and Varma C M 2002 *Phys. Rev. Lett.* **88** 226403
Maple M B, de Andrade M C, Herrmann J, Dalichaouch Y, Gajewski D A, Seaman C L, Chau R, Movshovich R, Aronson M C and Osborn R 1995 *J. Low. Temp. Phys.* **99** 223
Mathur N D, Grosche F M, Julian S R, Walker I R, Freye D M, Haselwimmer R K W and Lonzarich G G 1998 *Nature* **394** 39
Mayr F, Blanckenhagen G-F v and Stewart G R 1997 *Phys. Rev.* B **55** 947
McCoy B M 1969 *Phys. Rev.* B **188** 1014
McCoy B M and Wu T T 1968 *Phys. Rev.* B **176** 631
McCoy B M and Wu T T 1969 *Phys. Rev.* B **188** 982
Mélin R, Lin Y-C, Lajkó P, Rieger H and Iglói F, 2002 *Phys. Rev.* B **65** 104415
Meyer D and Nolting W 2000a *Phys. Rev.* B **62** 5657
Meyer D and Nolting W 2000b *Phys. Rev.* B **61** 13465
Mézard M, Parisi G and Virasoro M A 1986 *Spin Glass Theory and Beyond* (Singapore: World Scientific)
Migdal A B 1958 *Sov. Phys. JETP* **7** 996
Migdal A B 1967 *Theory of Finite Fermi Systems and Applications to Atomic Nuclei* (London: Pergamon)
Miller J and Huse D A 1993 *Phys. Rev. Lett.* **70** 3147
Millis A J 1993 *Phys. Rev.* B **48** 7183
Millis A J 1999 *Physica* B **259–261** 1169
Millis A J, Morr D K and Schmalian J 2001 *Phys. Rev. Lett.* **87** 167202
Millis A J, Morr D K and Schmalian J 2002 *Phys. Rev.* B **66** 174433
Millis A J, Morr D K and Schmalian J 2004 *Preprint* cond-mat/0411738
Milovanović M, Sachdev S and Bhatt R N 1989 *Phys. Rev. Lett.* **63** 82
Miranda E and Dobrosavljević V 1999 *Physica* B **259–261** 359
Miranda E and Dobrosavljević V 2001a *J. Magn. Magn. Mater.* **226–230** 110
Miranda E and Dobrosavljević V 2001b *Phys. Rev. Lett.* **86** 264
Miranda E, Dobrosavljević V and Kotliar G 1996 *J. Phys.: Condens. Matter* **8** 9871
Miranda E, Dobrosavljević V and Kotliar G 1997a *Phys. Rev. Lett.* **78** 290
Miranda E, Dobrosavljević V and Kotliar G 1997b *Physica* B **230** 569
Monthus C, Golinelli O and Jolicoeur T 1997 *Phys. Rev. Lett.* **79** 3254
Monthus C, Golinelli O and Jolicoeur T 1998 *Phys. Rev.* B **58** 805
Moryia T 1985 *Spin Fluctuations in Itinerant Electron Magnetism* (Berlin: Springer)
Motrunich O, Mau S-C, Huse D A and Fisher D S 2001 *Phys. Rev.* B **61** 1160
Mott N F 1990 *Metal–Insulator Transition* (London: Taylor and Francis)
Muller M and Ioffe L B 2004 *Phys. Rev. Lett.* **93** 256403
Naqib S H, Cooper J R, Tallon J L and Panagopoulos C 2003 *Physica* C **387** 365
Narozhny B N, Aleiner I L and Larkin A I 2001 *Phys. Rev.* B **62** 14898
Nozières P 1974 *J. Low Temp. Phys.* **17** 31
Nozières P and Blandin A 1980 *J. Phys.* **41** 193
Ott H R, Felder E and Bernasconi A 1993 *Physica* B **186–188** 207
Ovadyahu Z and Pollak M 1997 *Phys. Rev. Lett.* **79** 459
Paalanen M A and Bhatt R N 1991 *Physica* B **169** 231
Paalanen M A, Graebner J E, Bhatt R N and Sachdev S 1988 *Phys. Rev. Lett.* **61** 597
Paalanen M A, Rosenbaum T F, Thomas G A and Bhatt R N 1982 *Phys. Rev. Lett.* **48** 1284
Panagopoulos C and Dobrosavljević V 2005 *Phys. Rev.* B **72** 014536
Panagopoulos C, Tallon J L, Rainford B D, Cooper J R, Scott C A and Xiang T 2003 *Solid State Commun.* **126** 47–55 (Special issue ed A J Millis and Y Uemura) and references therein
Panagopoulos C, Tallon J L, Rainford B D, Xiang T, Cooper J R and Scott C A 2002 *Phys. Rev.* B **66** 064501
Pankov S and Dobrosavljević V 2005 *Phys. Rev. Lett.* **94** 046402
Parcollet O and Georges A 1999 *Phys. Rev.* B **59** 5341
Pastor A A and Dobrosavljević V 1999 *Phys. Rev. Lett.* **83** 4642
Pastor A A, Dobrosavreljević V and Horbach M L 2002 *Phys. Rev.* B **66** 014413
Pazmandi F, Zarand G and Zimanyi G T 1999 *Phys. Rev. Lett.* **83** 1034
Pfeuty P 1970 *Ann. Phys.* **57** 79
Pich C, Young A P, Rieger H and Kawashima N 1998 *Phys. Rev. Lett.* **81** 5916
Pietri R, Andraka B, Troc R and Tran V H 1997 *Phys. Rev.* B **56** 14505
Pines D and Nozières P 1965 *The Theory of Quantum Liquids* (New York: Benjamin)

Pollak M 1984 *Phil. Mag.* B **50** 265
Pruschke T, Jarrell M and Freericks J K 1995 *Adv. Phys.* **44** 187
Quirt J D and Marko J R 1971 *Phys. Rev. Lett.* **26** 318
Rappoport T G, Saguia A, Boechat B and Continentino M A 2003 *Europhys. Lett.* **61** 831
Read N and Newns D M 1983 *J. Phys. C: Solid State Phys.* **16** L1055
Read N, Sachdev S and Ye J 1995 *Phys. Rev.* B **52** 384
Refael G, Kehrein S and Fisher D S 2002 *Phys. Rev.* B **66** 060402(R)
Rieger H and Young A P 1994 *Phys. Rev. Lett.* **72** 4141
Rieger H and Young A P 1996 *Phys. Rev.* B **54** 3328
Rosch A 1999 *Phys. Rev. Lett.* **82** 4280
Rosch A 2000 *Phys. Rev.* B **62** 4945
Rosch A, Schröder A, Stockert O and Löhneysen H v 1997 *Phys. Rev. Lett.* **79** 159
Rosenbaum T F, Andres K, Thomas G A and Bhatt R N 1980 *Phys. Rev. Lett.* **45** 1423
Rozenberg M J and Grempel D 1998 *Phys. Rev. Lett.* **81** 2550
Rozenberg M J and Grempel D 1999 *Phys. Rev.* B **60** 4702
Sachdev S 1999 *Quantum Phase Transitions* (Cambridge: Cambridge University Press)
Sachdev S and Read N 1996 *J. Phys.: Condens. Matter* **8** 9723
Sachdev S, Read N and Oppermann R 1995 *Phys. Rev.* B **52** 10286
Sachdev S and Ye J 1993 *Phys. Rev. Lett.* **70** 3339
Saguia A, Boechat B and Continentino M A 2003a *Phys. Rev. Lett.* **89** 117202
Saguia A, Boechat B and Continentino M A 2003b *Phys. Rev.* B **68** 020403(R)
Sanna S, Allodi G, Concas G, Hillier A H and Renzi R D 2004 *Preprint* cond-mat/0403608
Sarachik M P 1995 *Metal–Insulator Transitions Revisited* ed P Edwards and C N R Rao (London: Taylor and Francis)
Schaffer L and Wegner F 1980 *Phys. Rev.* B **38** 113
Schlager H G and Lohneysen H V, 1997 *Europhys. Lett.* **40** 661
Schmalian J and Wolynes P G 2000 *Phys. Rev. Lett.* **85** 836
Schröder A, Aeppli G, Bucher E, Ramazashvili R and Coleman P 1998 *Phys. Rev. Lett.* **80** 5623
Schröder A, Aeppli G, Coldea R, Adams M, Stockert O, Löhneysen H, Bucher E, Ramazashvili R and Coleman P 2000 *Nature* **407** 351
Seaman C, Maple M B, Lee B W, Ghamaty S, Torikachvili M S, Kang J S, Liu L Z, Allen J W and Cox D L 1991 *Phys. Rev. Lett.* **67** 2882
Sengupta A M 2000 *Phys. Rev.* B **61** 4041
Sengupta A M and Georges A 1995 *Phys. Rev.* B **52** 10295
Senthil T and Sachdev S 1996 *Phys. Rev. Lett.* **77** 5292
Senthil T, Sachdev S and Vojta M 2003 *Phys. Rev. Lett.* **90** 216403
Senthil T, Vojta M and Sachdev S 2004 *Phys. Rev.* B **69** 035111
Shah N and Millis A J 2003 *Phys. Rev. Lett.* **91** 147204
Shankar R 1994 *Rev. Mod. Phys.* **66** 129
Shankar R and Murthy G 1987 *Phys. Rev.* B **36** 536
Sherrington D and Mihill K 1974 *J. Physique Coll.* **5** 199
Shklovskii B and Efros A 1984 *Electronic Properties of Doped Semiconductors* (Berlin: Springer)
Si Q, Rabello S, Ingersent K and Smith J L 2001 *Nature* **413** 804
Si Q and Smith J L 1996 *Phys. Rev. Lett.* **77** 3391
Silin V P 1958a *Sov. Phys. JETP* **6** 985
Silin V P 1958b *Sov. Phys. JETP* **6** 387
Smith J L and Si Q 2000 *Phys. Rev.* B **61** 5184
Stewart G R 1984 *Rev. Mod. Phys.* **56** 755
Stewart G R 2001 *Rev. Mod. Phys.* **73** 797
Stockert O, Löhneysen H v, Rosch A, Pyka N and Loewenhaupt M 1998 *Phys. Rev. Lett.* **80** 5627
Stupp H, Hornung M, Lakner M, Madel O and Löhneysen H V 1993 *Phys. Rev. Lett.* **71** 2634
Süllow S, Mentink S A M, Mason T E, Feyerherm R, Nieuwenhuys G J, Menovsky A A and Mydosh J A 2000 *Phys. Rev.* B **61** 8878
Takagi H, Batlogg B, Kao H L, Kwo J, Cava R J, Krajewski J J and Peck J W F 1992 *Phys. Rev. Lett.* **69** 2975
Tanasković D, Dobrosavljević V, Abrahams E and Kotliar G 2003 *Phys. Rev. Lett.* **91** 066603
Tanasković, D, Dobrosavljević V and Miranda E 2004a *Phys. Rev. Lett. Preprint* cond-mat/0412100 submitted
Tanasković D, Miranda E and Dobrosavljević V 2004b *Phys. Rev.* B **70** 205108
Tesanović Z 1986 *Phys. Rev.* B **34** 5212
Thill M J and Huse D A 1995 *Physica* A **214** 321

Thouless D J 1969 *Phys. Rev.* **187** 732
de la Torre M A L, Ellerby M, Watmough M and McEwen K A 1998 *J. Magn. Magn. Mater.* **177–181** 445
Tusch M A and Logan D E 1993 *Phys. Rev.* B **48** 14843
Ue H and Maekawa S 1971 *Phys. Rev.* **12** 4232
Vaknin A, Ovadyahu Z and Pollak M 1998 *Phys. Rev. Lett.* **81** 669
Vaknin A, Ovadyahu Z and Pollak M 2000 *Phys. Rev. Lett.* **84** 3402
Varma C M, Littlewood P B, Schmitt-Rink S, Abrahams E and Ruckenstein A E 1989 *Phys. Rev. Lett.* **63** 1996
Vojta T 2003 *Phys. Rev. Lett.* **90** 107202
Vojta T and Schmalian J 2004 *Preprint* cond-mat/0405609
Vollmer R, Pietrus T, Löhneysen H v, Chau R and Maple M B 2000 *Phys. Rev.* B **61** 1218
Waffenschmidt S, Pfleiderer C and Löhneysen H V 1999 *Phys. Rev. Lett.* **83** 3005
Weber A, Körner S, Scheidt E-W, Kehrein S and Stewart G R 2001 *Phys. Rev.* B **63** 205116
Wegner F 1976 *Z. Phys.* B **25** 327
Wegner F 1979 *Phys. Rev.* B **35** 783
Wegner F 1981 *Z. Phys.* B **44** 9
Westerberg E, Furusaki A, Sigrist M and Lee P A 1995 *Phys. Rev. Lett.* **75** 4302
Westerberg E, Furusaki A, Sigrist M and Lee P A 1997 *Phys. Rev.* B **55** 12578
Wilson K G 1975 *Rev. Mod. Phys.* **47** 773
Yang K, Hyman R A, Bhatt R N and Girvin S M 1996 *J. Appl. Phys.* **79** 5096
Ye J, Sachdev S and Read N 1993 *Phys. Rev. Lett.* **70** 4011
Young A P and Rieger H 1996 *Phys. Rev.* B **53** 8486
Yusuf E and Yang K 2002 *Phys. Rev.* B **65** 224428
Yusuf E and Yang K 2003a *Phys. Rev.* B **67** 144409
Yusuf E and Yang K 2003b *Phys. Rev.* B **68** 024425
Yuval G and Anderson P W 1970 *Phys. Rev.* B **1** 1522
Zaránd G and Demler E 2002 *Phys. Rev.* B **66** 024427
Zhu L and Si Q 2002 *Phys. Rev.* B **66** 024426