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# Random antiferromagnetic SU($N$) spin chains

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We analyze random isotropic antiferromagnetic SU($N$) spin chains using the real-space renormalization group. We find that they are governed at low energies by a universal infinite randomness fixed point *different* from the one of random spin-1/2 chains. We determine analytically the important exponents: the energy-length scale relation is $\Omega \sim \exp(-L^{\psi})$, where $\psi = 1/N$, and the mean correlation function is given by $\overline{C_{ij}} \sim (-1)^{i-j}/|i-j|^{\phi}$, where $\phi = 4/N$. Our analysis shows that the infinite-$N$ limit is unable to capture the behavior obtained at any finite $N$.

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The identification of several universality classes of disordered quantum systems in one dimension (1D) has seen enormous progress in recent years. Prominent among those is the case of random antiferromagnetic spin-1/2 chains, which have been shown to be controlled by an infinite randomness fixed point (IRFP) at low energies.[1,2] Many properties of this so-called random singlet phase have been obtained, e.g., the spin susceptibility $\chi \sim 1/T \log^2 T$, and the spin-correlation function $C_{ij} = \langle \mathbf{S}_i \cdot \mathbf{S}_j \rangle$ is such that its mean value $\overline{C_{ij}} \sim (-1)^{i-j}/|i-j|^2$, while the typical one $|C_{ij}|_{typ} \sim \exp(-\sqrt{|i-j|})$. Further studies of 1D spin systems have uncovered a wide variety of behaviors such as various Griffiths phases[3–9] and large spin phases.[9–11]

It is the purpose of this paper to extend these analyses by enlarging the symmetry group from SU (2) to SU($N$). We have several motivations for this. The inclusion of orbital degrees of freedom often leads to the enlargement SU(2)$\rightarrow$SU($N$). The strong spin-orbit interaction in rare-earth elements locks spin and orbital moments into a large multiplet with degeneracy $N$, whose description requires the enlargement from SU (2) to SU($N$).[12] Recently, a realization of a self-conjugate SU (4) spin chain has been proposed in a pillar array of semiconducting quantum dots, where the symmetry-breaking effect of the intradot electron-electron interaction is minimized due to the peculiarities of the dot potential.[13] Several other possible realizations of enlarged symmetry have been considered in the literature (see, for example, Ref. 14). In any of these cases, the effects of disorder are clearly of interest. Furthermore, the large-$N$ limit of SU($N$) spin models is of considerable interest. In this limit, many models can be solved by saddle-point methods and $1/N$ corrections can be obtained in a controlled manner.[15] The hope behind this approach is that the physics of $N=2$ is at least qualitatively captured as $N \rightarrow \infty$. The solution of a sequence of models *as a function of N*, though rarely possible, can determine the validity of this idea. We will show that the random antiferromagnetic SU($N$) chain provides just such a solution. Interestingly, the low-energy physics at finite $N$ is never captured at infinite $N$. We will show, however, that for any finite $N$, the system is governed by a *new universal IRFP* with characteristic exponents which we calculate and depend only on the group rank.

We will focus on the following Hamiltonian:

$$H = \sum_{i=1}^{L} J_i \mathbf{\Gamma}_i \cdot \mathbf{\Gamma}_{i+1}, \tag{1}$$

where $J_i$ are positive independent random variables distributed according to $P_0(J)$ and the components of $\mathbf{\Gamma}_i$, $\Gamma_i^a$ ($a = 1, \ldots, N^2-1$) are the generators of a representation of SU($N$). We will confine our analysis to totally antisymmetric irreducible representations. These correspond to Young tableaux with one column and $Q_i$ lines.[16] They are conveniently expressed with the help of auxiliary fermionic operators $c_{i\alpha}$ ($\alpha = 1, \ldots, N$) through $\Gamma_i^a = c_{i\alpha}^{\dagger} \Gamma_{\alpha\beta}^a c_{i\beta}$, where $\Gamma_{\alpha\beta}^a$ are the generator matrices of the fundamental representation of SU($N$). The fermions obey the constraint $\Sigma_{\alpha=1}^{N} c_{i\alpha}^{\dagger} c_{i\alpha} = Q_i$. We considered the cases where the $Q_i$'s are random and $Q_i = Q = \text{const}$.

To treat the Hamiltonian (1), we generalize the real-space renormalization group method introduced by Ma, Dasgupta, and Hu.[1] Our generalization is reminiscent of the treatment of random ferromagnetic and antiferromagnetic spin chains.[10] We first find the largest bond energy of the system, $\Omega = \max\{\Delta_i\}$. We define $\Delta_i$ as the energy difference between the ground and first excited multiplets of the $i$th bond. As $J_i > 0$, it can be shown that the ground-state multiplet is represented by a vertical Young tableau with $\tilde{Q}$ lines, where $\tilde{Q} = Q_i + Q_{i+1}$, if $Q_i + Q_{i+1} \leqslant N$, and $\tilde{Q} = Q_i + Q_{i+1} - N$, if $Q_i + Q_{i+1} > N$. The energies of ground and excited multiplets can be calculated from the Casimir's with the usual trick $2\mathbf{\Gamma}_i \cdot \mathbf{\Gamma}_{i+1} = (\mathbf{\Gamma}_i + \mathbf{\Gamma}_{i+1})^2 - \mathbf{\Gamma}_i^2 - \mathbf{\Gamma}_{i+1}^2$. The value of the Casimir's of the relevant tableaux is given in Ref. 17. We then decimate that bond by keeping only the ground multiplet and renormalizing the neighboring interactions in the following fashion. If $Q_i + Q_{i+1} = N$, the bond ground state is a singlet and is thus removed from the system. The new effective coupling between the neighboring spins $\mathbf{\Gamma}_{i-1}$ and $\mathbf{\Gamma}_{i+2}$ is, by second-order perturbation theory,

$$\tilde{J} = \frac{2 Q_i Q_{i+1} J_{i-1} J_{i+1}}{N^2 (N-1) J_i}. \tag{2}$$

Otherwise, the spin pair $\mathbf{\Gamma}_i$ and $\mathbf{\Gamma}_{i+1}$ is replaced by a new effective spin $\tilde{\mathbf{\Gamma}}$, which belongs to a totally antisymmetric representation with $\tilde{Q}$ lines as given above. It connects to the

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spins $\mathbf{\Gamma}_{i-1}$ and $\mathbf{\Gamma}_{i+2}$ through renormalized couplings given in *first-order* perturbation theory by

$$\tilde{J}_{i-1} = \xi_i J_{i-1}, \text{ and } \tilde{J}_{i+1} = (1-\xi_i) J_{i+1}, \tag{3}$$

respectively, where $\xi_i = Q_i/(Q_i+Q_{i+1})$, if $Q_i+Q_{i+1}<N$, and $\xi_i=(N-Q_i)/(2N-Q_i-Q_{i+1})$, otherwise. We point out the similarity with the case of the random chain with both ferromagnetic and antiferromagnetic interactions,[10] where both first and second-order decimations are generated. Unlike the latter, however, here the active (i.e., not yet decimated) spin clusters are always vertical tableaux and the procedure always maintains a totally antisymmetric spin chain. Moreover, the renormalized couplings are always smaller than the original ones. Thus, at every decimation step, the energy scale $\Omega$ is lowered.

An important feature of the decimation procedure is that it does not privilege any specific representation. When a spin pair $(Q_i, Q_{i+1})$ is decimated out, the new effective spin is never equal to any of the original ones $(\tilde{Q} \neq Q_i, \tilde{Q} \neq Q_{i+1})$. Thus, after an initial transient, each one of the $N-1$ totally antisymmetric representations is equally probable, *even if we start with* $Q_i = Q = \text{const.}$ (except for some special fine-tuned cases dealt with later). We have confirmed this numerically, as will be shown later. We have also checked that the distribution of representations becomes uncorrelated with the distribution of couplings. We thus focus on the flow of the coupling distribution, $P(J) \equiv P(J;\Omega)$, as the highest scale $\Omega$ is decreased.[1,2] We take $\Omega=1$ initially.

As will be shown later, similarly to the random spin-$1/2$ chain, $P(J)$ always flows to an extremely broad distribution. We are thus justified in neglecting the numerical prefactors in Eqs. (2) and (3), which are always less than unity and irrelevant asymptotically.[2] Furthermore, there are a total of $(N-1)^2$ possible decimation processes, all of them equally likely. Of these, $N-1$ are second-order, each with probability $p=1/(N-1)$, and the others are first order, with probability $q=1-p$. Thus, we can write a flow equation in the useful logarithmic variables $\Gamma=-\ln\Omega$ and $\zeta=\ln(\Omega/J)$,[2]

$$\frac{\partial}{\partial \Gamma}\rho(\zeta;\Gamma) = \frac{\partial}{\partial \zeta}\rho(\zeta;\Gamma) + q\rho(\zeta;\Gamma)\rho^0 + p\rho^0\rho\otimes\rho, \tag{4}$$

where $\rho(\zeta;\Gamma)\mathrm{d}\zeta = P(J;\Omega)\mathrm{d}J$, $\rho^0=\rho(0;\Gamma)$, and $\rho\otimes\rho = \int \mathrm{d}\zeta_1 \mathrm{d}\zeta_3 \rho(\zeta_1;\Gamma)\rho(\zeta_3;\Gamma)\delta(\zeta-\zeta_1-\zeta_3)$. The first term on the right-hand side is due to the fact that $\zeta$ changes when $\Gamma$ increases. The second one, absent in the random spin-$1/2$ chain, is due to first order decimation steps and only ensures the normalization of $\rho$. The last one is due to second-order steps, which strongly renormalize $\rho$ broadening it.

If $P_0(J)$ is not extremely singular, the flow Eq. (4) has only one stable fixed point solution,[2]

$$\rho^*(\zeta;\Gamma) = \frac{\theta(\zeta)}{p\Gamma} e^{-\zeta/(p\Gamma)}, \tag{5a}$$

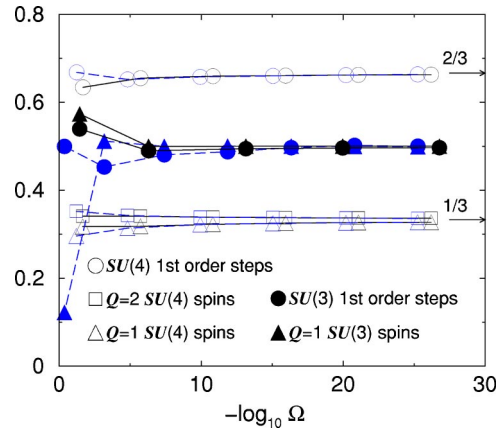


FIG. 1. (Color online) The fractions of first-order decimation steps, of spins in the fundamental ($Q=1$), and in the self-conjugate [$Q=2$, only for SU (4)] representations, all as a function of $\Omega$. For clarity, we only show data for chains $A$ (solid lines) and $E$ (dashed lines) (see text). The filled (open) symbols refer to the SU (3) [SU (4)] chains. The data error is about the size of the symbol.

$$P^*(J;\Omega) = \frac{\alpha}{\Omega}\left(\frac{\Omega}{J}\right)^{1-\alpha}\theta(\Omega-J), \tag{5b}$$

with $\alpha=1/(p\Gamma)=-(N-1)/\ln\Omega$. The fixed point distribution (5) broadens indefinitely in the limit $\Omega\to 0$, rendering the renormalization procedure increasingly more precise, and asymptotically exact.[2] The system is thus governed by an IRFP.

The relation between energy and length scales can be determined by finding the fraction of active spin clusters $n_\Gamma$ at the energy scale $\Gamma$.[2] If $\Gamma$ is increased by $d\Gamma$, a fraction $dn_\Gamma=(2p+q)n_\Gamma\rho(0;\Gamma)d\Gamma$ of active spin clusters is decimated. Thus, close to the fixed point, where $\rho(0;\Gamma)\approx\rho^*(0;\Gamma)$,[2]

$$L_\Gamma \sim n_\Gamma^{-1} \sim \Gamma^{1/\psi} = [\ln(1/\Omega)]^{1/\psi}, \tag{6}$$

where $\psi=p/(p+1)=1/N$. This type of "activated" dynamical scaling, corresponding to a dynamical exponent $z\to\infty$, arises here with an unexpected exponent $\psi$. When $N=2$, we recover the usual form found in the random spin-$1/2$ chains.[2,3]

In order to check the validity of the approximations leading up to Eq. (4), we have numerically implemented the full procedure. The data were generated by decimating chains with lengths up to $10^7$, averaging over 100 realizations of disorder. All the initial spins belong to the fundamental representation ($Q_i=1$, $\forall i$). We analyzed several initial distributions $P_0(J)$. Chains $A$, $B$, and $C$ had uniform distributions in the interval $x\leqslant J\leqslant 1$, with $x=0.9$, 0.5, and 0, respectively. In chains $D$, $E$, and $F$ we used initial power-law distributions $P_0\sim J^{-\beta}$, with $\beta=0.3$, 0.6, and 0.9, respectively. In Fig. 1, we show the fraction of first order decimation steps as a function of the energy scale $\Omega$, for the symmetry groups SU (3) and SU (4). As anticipated, it tends asymptotically to $q=(N-2)/(N-1)$. The figure also shows the fraction of $Q=1$ spins in the two cases, and the fraction of self-conjugate ($Q=2$) spins in the SU (4) chain. They all tend asymptotically to $1/(N-1)$, as expected.

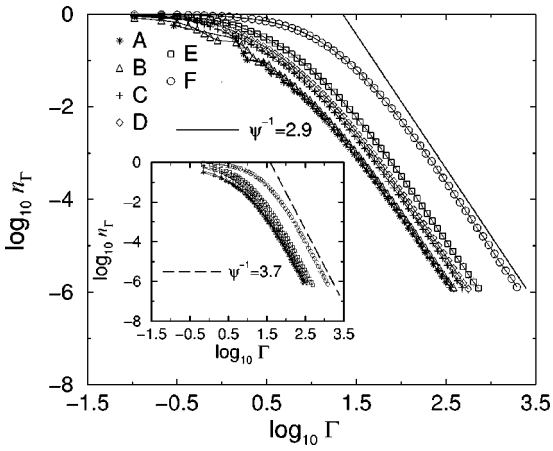


FIG. 2. Fraction of active spins $n_\Gamma$ as a function of the energy scale $\Gamma$, for the SU (3) (main) and SU (4) (inset) chains $A$ to $F$. The data error is smaller than the symbol size.

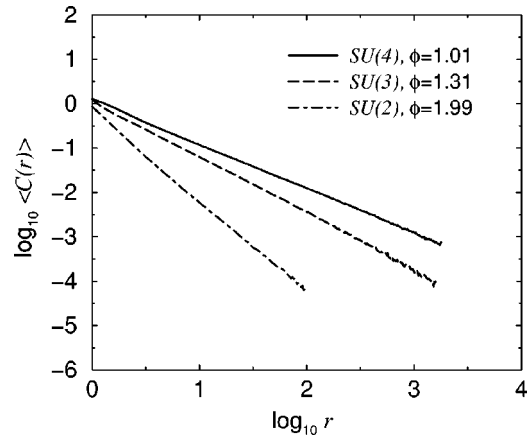


FIG. 3. Mean correlation function for groups SU (2), SU (3), and SU (4). The power-law dependence is evident. The exponents $\phi=1.99\pm0.03$, $1.31\pm0.03$, and $1.01\pm0.03$ are obtained by fitting the region $r>10$, and are in excellent agreement with the analytical value $\phi=4/N$ [see Eq. (8)].

In Fig. 2, we plot $n_\Gamma$ as a function of $\Gamma$ for SU (3) and SU (4). By fitting the asymptotic behavior, we confirm the universality of the exponent $\psi=1/N$, as predicted by Eq. (6). We point out that $\psi$ converges in a logarithmic manner, thus a more precise determination of $\psi$ demands the decimation of longer chains than the ones studied here. We see that as $N$ increases, so does the number of decimations needed for a given decrease in energy scale. This "delayed scaling" can be understood by realizing that only second-order processes are effective in lowering the energy scale, and these become less frequent as $N$ increases.

There are other IRFP's in addition to the one analyzed above. For example, the self-conjugate SU($2k$) spin chain (with integer $k>1$) flows towards an IRFP with $\psi=1/2$, since $Q_i=k$, $\forall i$, and only second-order decimation steps occur, like in the random spin-$1/2$ chain. Although these chains are gapful,[18] they are unstable against the introduction of weak disorder, due to the topological nature of their ground state, as explained for the random $J_1-J_2$ Heisenberg chain in Ref. 5. More importantly, this $\psi=1/2$ IRFP is unstable against the introduction of $Q\neq k$ spins. For a small concentration $n_i$ of such spins, the system will initially be governed by the $\psi=1/2$ IRFP, until the energy scale $\Gamma\sim n_i^{-1/2}$ is reached. Below that scale, the renormalization flow veers towards the IRFP of Eq. (5), with the characteristic exponent $\psi=1/2k$. Similar IRFP's exist for other SU($N$) chains, but they are equally unstable with respect to the introduction of "defect" spins.

We can easily calculate the asymptotic behavior of thermodynamic quantities using Eq. (6).[2] Since $P(J)$ becomes very broad at low energies, the active spins are approximately free at a low temperature $T=\Omega$, whereas the decimated ones do not contribute, since they are frozen in singlet states with excitation energies much greater than $T$. Hence, we find that the entropy density $\sigma\sim n_\Gamma\sim(-\ln T)^{-1/\psi}$ and the specific heat $c\sim(-\ln T)^{-(\psi+1)/\psi}$. Furthermore, it can be easily shown that the magnetic susceptibility of a single SU($N$) spin is Curie-like, from which it follows that for the whole system $\chi(T)\sim n_\Gamma/T=1/[T(-\ln T)^{1/\psi}]$.

We can also obtain the asymptotic behavior of the spin correlation function $C_{ij}\equiv\langle\mathbf{\Gamma}_i\cdot\mathbf{\Gamma}_j\rangle$.[2] Spins belonging to the same cluster develop $\mathcal{O}(1)$ correlations; otherwise, they are weakly correlated. Therefore, such spins dominate the mean correlation function. To find $\overline{C_{ij}}$, we need the probability that any two well-separated spins $\mathbf{\Gamma}_i$ and $\mathbf{\Gamma}_j$ are rigidly locked in the same spin cluster $\tilde{\mathbf{\Gamma}}$ when $|i-j|\sim n_\Gamma^{-1}$. First, we need to find $\mathcal{P}(t;n_\Gamma)$, the probability to find a spin cluster $\tilde{\mathbf{\Gamma}}$ composed of $t$ original spins at scale $n_\Gamma$. After many decimations, any spatial correlations between $Q$'s and $J$'s have vanished and any remaining bond is equally likely to be decimated. The fraction of clusters with $t$ spins at scale $n_\Gamma$ is $n_\Gamma\mathcal{P}(t;n_\Gamma)$. When $dN_{dec}$ decimations are performed, $n_\Gamma$ decreases by $dn_\Gamma=-(2p+q)dN_{dec}$ and

$$d[n_\Gamma\mathcal{P}(t)]=\left[-2\mathcal{P}(t)+q\sum_{t_1,t_2}\mathcal{P}(t_1)\mathcal{P}(t_2)\delta_{t_1+t_2,t}\right]dN_{dec},$$

where the two terms on the right-hand side give the fraction of decimated and added clusters with $t$ spins and we suppressed the $n_\Gamma$ dependence of $\mathcal{P}$ to lighten the notation. Hence,

$$n_\Gamma\frac{\partial\mathcal{P}(t)}{\partial n_\Gamma}=\frac{1-p}{1+p}\left[\mathcal{P}(t)-\sum_{t_1}\mathcal{P}(t_1)\mathcal{P}(t-t_1)\right], \tag{7}$$

whose solution is $\mathcal{P}(t;n_\Gamma)\sim n_\Gamma^\gamma\exp(-tn_\Gamma^\gamma)$ in the limit $n_\Gamma\to0$, with $\gamma=(1-p)/(1+p)=1-2/N$. Finally, the probability that $\mathbf{\Gamma}_i$ and $\mathbf{\Gamma}_j$ are active in the same cluster is equal to $(\bar{t}n_\Gamma)^2\sim(n_\Gamma^{1-\gamma})^2$, yielding

$$\overline{C_{ij}}\sim\frac{(-1)^{i-j}}{|i-j|^\phi}, \tag{8}$$

with $\phi=4/N$. The typical correlation function, however, is very different. Following Ref. 2, we note that it involves many factors of $\tilde{J}$ decimated at various scales $e^{-\Gamma}$. The scaling behavior is dominated by the smallest factor $\mathcal{O}(e^{-k|i-j|^\psi})$, where $k\sim\mathcal{O}(1)$ yielding typical correlations

$$|C_{ij}|_{typ} \sim \exp(-k|i-j|^{\psi}). \tag{9}$$

Figure 3 shows numerical results for the mean correlation function $\overline{C_{ij}}$ for groups SU (2), SU (3), and SU (4), averaged over 200 realizations of disorder for chain lengths up to $L=10^5$ and open boundary conditions. The numerical procedure consists of completely decimating a chain, and counting the fraction of spin pairs that become strongly correlated at the distance $|i-j|$.[7] Excellent agreement with the analytical prediction of $\phi$ is obtained. No significant dependence on initial disorder strength was observed, confirming the universal behavior.

In the large-$N$ limit, the mean correlation function decays extremely slowly. In this limit, the fraction of second-order processes is very small and the mean number of spins in a cluster diverges at low energies, all of them being strongly correlated. A $1/N$ expansion of Eq. (8) leads to $\overline{|C_{ij}|} \sim 1/\ln(|i-j|)$. Incidentally, this is the same behavior observed numerically in random ferromagnetic and antiferromagnetic spin chains.[19] This is no surprise, since both systems are dominated by similar first-order decimations *whose clustering rules are the same as* $N \to \infty$. Therefore, they are both described by Eq. (7) with $p=0$, hence the logarithmic dependence of the mean correlation function. This analytical explanation has not appeared before. However, we should stress that the asymptotic region governed by the IRFP is reached at energy scales which decrease with the increase of $N$, since the second-order processes become increasingly rare. Therefore, in the infinite-$N$ limit the universal behavior described above disappears and a direct infinite-$N$ approach fails to capture the physics at any finite $N$.

Interestingly, some multicritical points of random antiferromagnetic spin $S$ chains have been shown to exhibit a structure that is very similar to the generic SU($N$) IRFP described above.[20] In particular, the energy-length scale exponent is the same. In that case, $N$ is the number of phases meeting at the multicritical point.

In conclusion, we have identified in random antiferromagnetic SU($N$) chains an infinite randomness fixed point with exponents different than the ones previously found in spin-1/2 chains. An important question which we leave for future study is the stability of this phase against the introduction of anisotropy.

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