**Proof.** Noting (3.11.26), we have

$$\begin{split} \langle \alpha', jm | \mathbf{J \cdot V} | \alpha, jm \rangle &= \langle \alpha', jm | (J_0 V_0 - J_{+1} V_{-1} - J_{-1} V_{+1}) | \alpha, jm \rangle \\ &= m \hbar \langle \alpha', jm | V_0 | \alpha, jm \rangle + \frac{\hbar}{\sqrt{2}} \sqrt{(j+m)(j-m+1)} \\ &\times \langle \alpha', jm - 1 | V_{-1} | \alpha, jm \rangle \\ &- \frac{\hbar}{\sqrt{2}} \sqrt{(j-m)(j+m+1)} \langle \alpha', jm + 1 | V_{+1} | \alpha, jm \rangle \\ &= c_{jm} \langle \alpha' j | | \mathbf{V} | | \alpha j \rangle \end{split}$$

by the Wigner-Eckart theorem (3.11.31), where  $c_{jm}$  is independent of  $\alpha$ ,  $\alpha'$ , and V, and the matrix elements of  $V_{0,\pm 1}$  are all proportional to the double-bar matrix element (sometimes also called the **reduced matrix element**). Furthermore,  $c_{jm}$  is independent of m because  $J \cdot V$  is a scalar operator, so we may as well write it as  $c_j$ . Because  $c_j$  does not depend on V, (3.11.42) holds even if we let  $V \rightarrow J$  and  $\alpha' \rightarrow \alpha$ ; that is,

$$\langle \alpha, jm | \mathbf{J}^2 | \alpha, jm \rangle = c_j \langle \alpha j | |\mathbf{J}| | \alpha j \rangle.$$
 (3.11.43)

Returning to the Wigner-Eckart theorem applied to  $\mathcal{V}_q$  and  $\mathcal{J}_q$ , we have

$$\frac{\langle \alpha', jm' | V_q | \alpha, jm \rangle}{\langle \alpha, jm' | J_q | \alpha, jm \rangle} = \frac{\langle \alpha' j | | \mathbf{V} | | \alpha j \rangle}{\langle \alpha j | | \mathbf{J} | | \alpha j \rangle}.$$
(3.11.44)

But we can write  $\langle \alpha', jm | \mathbf{J} \cdot \mathbf{V} | \alpha, jm \rangle / \langle \alpha, jm | \mathbf{J}^2 | \alpha, jm \rangle$  for the right-hand side of (3.11.44) by (3.11.42) and (3.11.43). Moreover, the left-hand side of (3.11.43) is

$$\langle \alpha', jm' | V_q | \alpha, jm \rangle = \frac{\langle \alpha', jm | \mathbf{J} \cdot \mathbf{V} | \alpha, jm \rangle}{\hbar^2 j (j+1)} \langle jm' | J_q | jm \rangle, \tag{3.11.45}$$

which proves the projection theorem.

We will give applications of the theorem in subsequent sections.

## **Problems**

- 3.1 Find the eigenvalues and eigenvectors of  $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ . Suppose an electron is in the spin state  $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ . If  $s_y$  is measured, what is the probability of the result  $\hbar/2$ ?
- 3.2 Find, by explicit construction using Pauli matrices, the eigenvalues for the Hamiltonian

$$H = -\frac{2\mu}{\hbar} \mathbf{S} \cdot \mathbf{B}$$

for a spin  $\frac{1}{2}$  particle in the presence of a magnetic field  $\mathbf{B} = B_x \hat{\mathbf{x}} + B_y \hat{\mathbf{y}} + B_z \hat{\mathbf{z}}$ .



## Chapter 3 Theory of Angular Momentum

3.3 Consider the  $2 \times 2$  matrix defined by

$$U = \frac{a_0 + i\boldsymbol{\sigma} \cdot \mathbf{a}}{a_0 - i\boldsymbol{\sigma} \cdot \mathbf{a}},$$

where  $a_0$  is a real number and a is a three-dimensional vector with real components.

- (a) Prove that U is unitary and unimodular.
- (b) In general, a  $2 \times 2$  unitary unimodular matrix represents a rotation in three dimensions. Find the axis and angle of rotation appropriate for U in terms of  $a_0$ ,  $a_1$ ,  $a_2$ , and  $a_3$ .

3.4 The spin-dependent Hamiltonian of an electron-positron system in the presence of a uniform magnetic field in the *z*-direction can be written as

$$H = A\mathbf{S}^{(e^{-})} \cdot \mathbf{S}^{(e^{+})} + \left(\frac{eB}{mc}\right) \left(S_{z}^{(e^{-})} - S_{z}^{(e^{+})}\right).$$

Suppose the spin function of the system is given by  $\chi_{+}^{(e^{-})}\chi_{-}^{(e^{+})}$ .

- (a) Is this an eigenfunction of H in the limit  $A \to 0$ ,  $eB/mc \ne 0$ ? If it is, what is the energy eigenvalue? If it is not, what is the expectation value of H?
- (b) Solve the same problem when  $eB/mc \rightarrow 0$ ,  $A \neq 0$ .

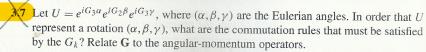
3.5 Consider a spin 1 particle. Evaluate the matrix elements of

$$S_z(S_z + \hbar)(S_z - \hbar)$$
 and  $S_x(S_x + \hbar)(S_x - \hbar)$ .

3.6 Let the Hamiltonian of a rigid body be

$$H = \frac{1}{2} \left( \frac{K_1^2}{I_1} + \frac{K_2^2}{I_2} + \frac{K_3^2}{I_3} \right),$$

where K is the angular momentum in the body frame. From this expression obtain the Heisenberg equation of motion for K, and then find Euler's equation of motion in the correspondence limit.



**3.8** What is the meaning of the following equation?

$$U^{-1}A_kU=\sum R_{kl}A_l,$$

where the three components of A are matrices. From this equation show that matrix elements  $\langle m|A_k|n\rangle$  transform like vectors.

3.9 Consider a sequence of Euler rotations represented by

$$\mathcal{D}^{(1/2)}(\alpha,\beta,\gamma) = \exp\left(\frac{-i\sigma_3\alpha}{2}\right) \exp\left(\frac{-i\sigma_2\beta}{2}\right) \exp\left(\frac{-i\sigma_3\gamma}{2}\right)$$

$$= \begin{pmatrix} e^{-i(\alpha+\gamma)/2}\cos\frac{\beta}{2} & -e^{-i(\alpha-\gamma)/2}\sin\frac{\beta}{2} \\ e^{i(\alpha-\gamma)/2}\sin\frac{\beta}{2} & e^{i(\alpha+\gamma)/2}\cos\frac{\beta}{2} \end{pmatrix}.$$

Because of the group properties of rotations, we expect that this sequence of operations is equivalent to a *single* rotation about some axis by an angle  $\theta$ . Find  $\theta$ .

- (3.10 (a) Consider a pure ensemble of identically prepared spin  $\frac{1}{2}$  systems. Suppose the expectation values  $\langle S_x \rangle$  and  $\langle S_z \rangle$  and the sign of  $\langle S_y \rangle$  are known. Show how we may determine the state vector. Why is it unnecessary to know the magnitude of  $\langle S_y \rangle$ ?
  - (b) Consider a mixed ensemble of spin  $\frac{1}{2}$  systems. Suppose the ensemble averages  $[S_x]$ ,  $[S_y]$ , and  $[S_z]$  are all known. Show how we may construct the  $2 \times 2$  density matrix that characterizes the ensemble.
  - **3.11** (a) Prove that the time evolution of the density operator  $\rho$  (in the Schrödinger picture) is given by

$$\rho(t) = \mathcal{U}(t, t_0) \rho(t_0) \mathcal{U}^{\dagger}(t, t_0).$$

- (b) Suppose we have a pure ensemble at t=0. Prove that it cannot evolve into a mixed ensemble as long as the time evolution is governed by the Schrödinger equation.
- **3.12** Consider an ensemble of spin 1 systems. The density matrix is now a  $3 \times 3$  matrix. How many independent (real) parameters are needed to characterize the density matrix? What must we know in addition to  $[S_x]$ ,  $[S_y]$ , and  $[S_z]$  to characterize the ensemble completely?
- 3.13 An angular-momentum eigenstate  $|j,m=m_{\max}=j\rangle$  is rotated by an infinitesimal angle  $\varepsilon$  about the y-axis. Without using the explicit form of the  $d_{m'm}^{(j)}$  function, obtain an expression for the probability for the new rotated state to be found in the original state up to terms of order  $\varepsilon^2$ .
- original state up to terms of order  $\varepsilon^2$ .

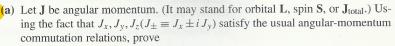
  3.14 Show that the  $3 \times 3$  matrices  $G_i (i = 1, 2, 3)$  whose elements are given by

$$(G_i)_{ik} = -i\hbar\varepsilon_{ijk},$$

where j and k are the row and column indices, satisfy the angular-momentum commutation relations. What is the physical (or geometric) significance of the transformation matrix that connects  $G_i$  to the more usual  $3 \times 3$  representations of the angular-momentum operator  $J_i$  with  $J_3$  taken to be diagonal? Relate your result to

$$\mathbf{V} \rightarrow \mathbf{V} + \hat{\mathbf{n}} \delta \phi \times \mathbf{V}$$

under infinitesimal rotations. (*Note:* This problem may be helpful in understanding the photon spin.)



$$\mathbf{J}^2 = J_z^2 + J_+ J_- - \hbar J_z.$$

(b) Using (a) (or otherwise), derive the "famous" expression for the coefficient  $c_{-}$  that appears in

$$J_{-}\psi_{jm} = c_{-}\psi_{j,m-1}.$$

- 3.16 Show that the orbital angular-momentum operator L commutes with both the operators  $\mathbf{p}^2$  and  $\mathbf{x}^2$ ; that is, prove (3.7.2).
- V(r) is given by

$$\psi(\mathbf{x}) = (x + y + 3z)f(r).$$

- (a) Is  $\psi$  an eigenfunction of  $L^2$ ? If so, what is the *l*-value? If not, what are the possible values of *l* that we may obtain when  $L^2$  is measured?
- (b) What are the probabilities for the particle to be found in various  $m_l$  states?
- (c) Suppose it is known somehow that  $\psi(\mathbf{x})$  is an energy eigenfunction with eigenvalue E. Indicate how we may find V(r).
- A particle in a spherically symmetrical potential is known to be in an eigenstate of  $L^2$  and  $L_z$  with eigenvalues  $\hbar^2 l(l+1)$  and  $m\hbar$ , respectively. Prove that the expectation values between  $|lm\rangle$  states satisfy

$$\langle L_x \rangle = \langle L_y \rangle = 0, \quad \langle L_x^2 \rangle = \langle L_y^2 \rangle = \frac{[l(l+1)\hbar^2 - m^2\hbar^2]}{2}.$$

- Interpret this result semiclassically.
- Suppose a half-integer l-value, say  $\frac{1}{2}$ , were allowed for orbital angular momentum. From

$$L_+Y_{1/2,1/2}(\theta,\phi) = 0,$$

we may deduce, as usual,

$$Y_{1/2,1/2}(\theta,\phi) \propto e^{i\phi/2} \sqrt{\sin\theta}$$
.

- Now try to construct  $Y_{1/2,-1/2}(\theta,\phi)$  by (a) applying  $L_-$  to  $Y_{1/2,1/2}(\theta,\phi)$ ; and (b) using  $L_-Y_{1/2,-1/2}(\theta,\phi)=0$ . Show that the two procedures lead to contradictory results. (This gives an argument against half-integer l-values for orbital angular momentum.)
- Consider an orbital angular-momentum eigenstate  $|l=2,m=0\rangle$ . Suppose this state is rotated by an angle  $\beta$  about the y-axis. Find the probability for the new state to be found in  $m=0,\pm 1$ , and  $\pm 2$ . (The spherical harmonics for l=0,1, and 2 given in Section B.5 in Appendix B may be useful.)
  - **3.21** The goal of this problem is to determine degenerate eigenstates of the three-dimensional isotropic harmonic oscillator written as eigenstates of  $L^2$  and  $L_z$ , in terms of the Cartesian eigenstates  $|n_x n_y n_z\rangle$ .
    - (a) Show that the angular-momentum operators are given by

$$L_i = i\hbar\varepsilon_{ijk}a_ja_k^{\dagger}$$

$$\mathbf{L}^2 = \hbar^2 \left[ N(N+1) - a_k^\dagger a_k^\dagger a_j a_j \right],$$

where summation is implied over repeated indices,  $\varepsilon_{ijk}$  is the totally antisymmetric symbol, and  $N \equiv a_j^{\dagger} a_j$  counts the total number of quanta.

- (b) Use these relations to express the states  $|qlm\rangle = |01m\rangle$ ,  $m = 0, \pm 1$ , in terms of the three eigenstates  $|n_x n_y n_z\rangle$  that are degenerate in energy. Write down the representation of your answer in coordinate space, and check that the angular and radial dependences are correct.
- (c) Repeat for  $|qlm\rangle = |200\rangle$ .
- (d) Repeat for  $|qlm\rangle = |02m\rangle$ , with m = 0, 1, and 2.
- **3.22** Follow these steps to show that solutions to Kummer's Equation (3.7.46) can be written in terms of Laguerre polynomials  $L_n(x)$ , which are defined according to a generating function as

$$g(x,t) = \frac{e^{-xt/(1-t)}}{1-t} = \sum_{n=0}^{\infty} L_n(x) \frac{t^n}{n!},$$

where 0 < t < 1. The discussion in Section 2.5 on generating functions for Hermite polynomials will be helpful.

- (a) Prove that  $L_n(0) = n!$  and  $L_0(x) = 1$ .
- (b) Differentiate g(x,t) with respect to x, show that

$$L'_n(x) - nL'_{n-1}(x) = -nL_{n-1}(x),$$

and find the first few Laguerre polynomials.

(c) Differentiate g(x,t) with respect to t and show that

$$L_{n+1}(x) - (2n+1-x)L_n(x) + n^2L_{n-1}(x) = 0.$$

(d) Now show that Kummer's Equation is solved by deriving

$$xL_n''(x) + (1-x)L_n'(x) + nL_n(x) = 0,$$

and associate n with the principal quantum number for the hydrogen atom.

What is the physical significance of the operators

$$K_{+} \equiv a_{+}^{\dagger} a_{-}^{\dagger}$$
 and  $K_{-} \equiv a_{+} a_{-}$ 

in Schwinger's scheme for angular momentum? Give the nonvanishing matrix elements of  $K_{+}$ .

We are to add angular momenta  $j_1 = 1$  and  $j_2 = 1$  to form j = 2, 1, and 0 states. Using either the ladder operator method or the recursion relation, express all (nine)  $\{j,m\}$  eigenkets in terms of  $|j_1j_2,m_1m_2\rangle$ . Write your answer as

$$|j = 1, m = 1\rangle = \frac{1}{\sqrt{2}}|+,0\rangle - \frac{1}{\sqrt{2}}|0,+\rangle,...,$$

where + and 0 stand for  $m_{1,2} = 1$ , 0, respectively.

3.25 (a) Evaluate

$$\sum_{m=-j}^{j} |d_{mm'}^{(j)}(\beta)|^2 m$$

for any j (integer or half-integer); then check your answer for  $j = \frac{1}{2}$ .

## Chapter 3 Theory of Angular Momentum

(b) Prove, for any j,

$$\sum_{m=-j}^{j} m^2 |d_{m'm}^{(j)}(\beta)|^2 = \frac{1}{2} j(j+1) \sin^2 \beta + m'^2 \frac{1}{2} (3\cos^2 \beta - 1).$$

[Hint: This can be proved in many ways. You may, for instance, examine the rotational properties of  $J_z^2$  using the spherical (irreducible) tensor language.]



**3.26**\(\(\mathbb{a}\)\) Consider a system with j = 1. Explicitly write

$$\langle j=1,m'|J_y|j=1,m\rangle$$

in  $3 \times 3$  matrix form.

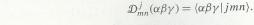
(b) Show that for j=1 only, it is legitimate to replace  $e^{-iJ_y\beta/\hbar}$  by

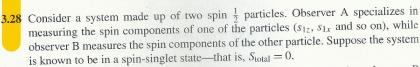
$$1 - i\left(\frac{J_y}{\hbar}\right)\sin\beta - \left(\frac{J_y}{\hbar}\right)^2 (1 - \cos\beta).$$

(c) Using (b), prove

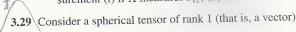
$$d^{(j=1)}(\beta) = \begin{pmatrix} \left(\frac{1}{2}\right)(1 + \cos\beta) & -\left(\frac{1}{\sqrt{2}}\right)\sin\beta & \left(\frac{1}{2}\right)(1 - \cos\beta) \\ \left(\frac{1}{\sqrt{2}}\right)\sin\beta & \cos\beta & -\left(\frac{1}{\sqrt{2}}\right)\sin\beta \\ \left(\frac{1}{2}\right)(1 - \cos\beta) & \left(\frac{1}{\sqrt{2}}\right)\sin\beta & \left(\frac{1}{2}\right)(1 + \cos\beta) \end{pmatrix}.$$

**3.27** Express the matrix element  $\langle \alpha_2 \beta_2 \gamma_2 | J_3^2 | \alpha_1 \beta_1 \gamma_1 \rangle$  in terms of a series in





- (a) What is the probability for observer A to obtain  $s_{1z} = \hbar/2$  when observer B makes no measurement? Solve the same problem for  $s_{1x} = \hbar/2$ .
- Observer B determines the spin of particle 2 to be in the  $s_{2z} = \hbar/2$  state with certainty. What can we then conclude about the outcome of observer A's measurement (i) if A measures  $s_{1z}$ ; (ii) if A measures  $s_{1x}$ ? Justify your answer.

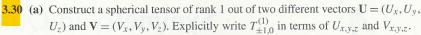


$$V_{\pm 1}^{(1)} = \mp \frac{V_x \pm i V_y}{\sqrt{2}}, \quad V_0^{(1)} = V_z.$$

Using the expression for  $d^{(j=1)}$  given in Problem 3.26, evaluate

$$\sum_{q'} d_{qq'}^{(1)}(\beta) V_{q'}^{(1)}$$

and show that your results are just what you expect from the transformation properties of  $V_{x,y,z}$  under rotations about the y-axis.



- (b) Construct a spherical tensor of rank 2 out of two different vectors **U** and **V**. Write down explicitly  $T_{\pm 2,\pm 1,0}^{(2)}$  in terms of  $U_{x,y,z}$  and  $V_{x,y,z}$ .
- 3.31 Consider a spinless particle bound to a fixed center by a central force potential.
  - (a) Relate, as much as possible, the matrix elements

$$\langle n', l', m' | \mp \frac{1}{\sqrt{2}} (x \pm iy) | n, l, m \rangle$$
 and  $\langle n', l', m' | z | n, l, m \rangle$ 

using *only* the Wigner-Eckart theorem. Make sure to state under what conditions the matrix elements are nonvanishing.

- **(b)** Do the same problem using wave functions  $\psi(\mathbf{x}) = R_{nl}(r)Y_l^m(\theta, \phi)$ .
- (a) Write xy, xz, and  $(x^2 y^2)$  as components of a spherical (irreducible) tensor of rank 2.
- (b) The expectation value

$$Q \equiv e\langle \alpha, j, m = j | (3z^2 - r^2) | \alpha, j, m = j \rangle$$

is known as the quadrupole moment. Evaluate

$$e\langle \alpha, j, m' | (x^2 - y^2) | \alpha, j, m = j \rangle,$$

where m' = j, j - 1, j - 2,..., in terms of Q and appropriate Clebsch-Gordan coefficients.

3.33 A spin  $\frac{3}{2}$  nucleus situated at the origin is subjected to an external inhomogeneous electric field. The basic electric quadrupole interaction may by taken to be

$$H_{\rm int} = \frac{eQ}{2s(s-1)\hbar^2} \left[ \left( \frac{\partial^2 \phi}{\partial x^2} \right)_0 S_x^2 + \left( \frac{\partial^2 \phi}{\partial y^2} \right)_0 S_y^2 + \left( \frac{\partial^2 \phi}{\partial z^2} \right)_0 S_z^2 \right],$$

where  $\phi$  is the electrostatic potential satisfying Laplace's equation, and the coordinate axes are chosen such that

$$\left(\frac{\partial^2 \phi}{\partial x \partial y}\right)_0 = \left(\frac{\partial^2 \phi}{\partial y \partial z}\right)_0 = \left(\frac{\partial^2 \phi}{\partial x \partial z}\right)_0 = 0.$$

Show that the interaction energy can be written as

$$A(3S_z^2 - \mathbf{S}^2) + B(S_+^2 + S_-^2),$$

and express A and B in terms of  $(\partial^2 \phi/\partial x^2)_0$  and so on. Determine the energy eigenkets (in terms of  $|m\rangle$ , where  $m=\pm\frac{3}{2},\pm\frac{1}{2}$ ) and the corresponding energy eigenvalues. Is there any degeneracy?