Proof. Noting (3.11.26), we have

$$\begin{aligned}\langle\alpha', jm|\mathbf{J}\cdot\mathbf{V}|\alpha, jm\rangle &= \langle\alpha', jm|(J_0V_0 - J_{+1}V_{-1} - J_{-1}V_{+1})|\alpha, jm\rangle \\ &= m\hbar\langle\alpha', jm|V_0|\alpha, jm\rangle + \frac{\hbar}{\sqrt{2}}\sqrt{(j+m)(j-m+1)} \\ &\quad\times\langle\alpha', jm-1|V_{-1}|\alpha, jm\rangle \\ &\quad -\frac{\hbar}{\sqrt{2}}\sqrt{(j-m)(j+m+1)}\langle\alpha', jm+1|V_{+1}|\alpha, jm\rangle \\ &= c_{jm}\langle\alpha' j||\mathbf{V}||\alpha j\rangle\end{aligned} \tag{3.11.42}$$

by the Wigner-Eckart theorem (3.11.31), where c_{jm} is independent of α, α', and $\mathbf{V}$, and the matrix elements of $V_{0,\pm1}$ are all proportional to the double-bar matrix element (sometimes also called the **reduced matrix element**). Furthermore, c_{jm} is independent of m because $\mathbf{J}\cdot\mathbf{V}$ is a scalar operator, so we may as well write it as c_j. Because c_j does not depend on $\mathbf{V}$, (3.11.42) holds even if we let $\mathbf{V}\to\mathbf{J}$ and $\alpha'\to\alpha$; that is,

$$\langle\alpha, jm|\mathbf{J}^2|\alpha, jm\rangle = c_j\langle\alpha j||\mathbf{J}||\alpha j\rangle. \tag{3.11.43}$$

Returning to the Wigner-Eckart theorem applied to V_q and J_q, we have

$$\frac{\langle\alpha', jm'|V_q|\alpha, jm\rangle}{\langle\alpha, jm'|J_q|\alpha, jm\rangle} = \frac{\langle\alpha' j||\mathbf{V}||\alpha j\rangle}{\langle\alpha j||\mathbf{J}||\alpha j\rangle}. \tag{3.11.44}$$

But we can write $\langle\alpha', jm|\mathbf{J}\cdot\mathbf{V}|\alpha, jm\rangle/\langle\alpha, jm|\mathbf{J}^2|\alpha, jm\rangle$ for the right-hand side of (3.11.44) by (3.11.42) and (3.11.43). Moreover, the left-hand side of (3.11.43) is just $j(j+1)\hbar^2$. So

$$\langle\alpha', jm'|V_q|\alpha, jm\rangle = \frac{\langle\alpha', jm|\mathbf{J}\cdot\mathbf{V}|\alpha, jm\rangle}{\hbar^2 j(j+1)}\langle jm'|J_q|jm\rangle, \tag{3.11.45}$$

which proves the projection theorem.

We will give applications of the theorem in subsequent sections.

Problems

L5 8

L6 4

3.1 Find the eigenvalues and eigenvectors of $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$. Suppose an electron is in the spin state $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$. If s_y is measured, what is the probability of the result $\hbar/2$?

3.2 Find, by explicit construction using Pauli matrices, the eigenvalues for the Hamiltonian

$$H = -\frac{2\mu}{\hbar}\mathbf{S}\cdot\mathbf{B}$$

for a spin $\frac{1}{2}$ particle in the presence of a magnetic field $\mathbf{B} = B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}} + B_z\hat{\mathbf{z}}$.

3.3 Consider the 2×2 matrix defined by

$$U = \frac{a_0 + i\boldsymbol{\sigma}\cdot\mathbf{a}}{a_0 - i\boldsymbol{\sigma}\cdot\mathbf{a}},$$

where a_0 is a real number and $\mathbf{a}$ is a three-dimensional vector with real components.

(a) Prove that U is unitary and unimodular.

(b) In general, a 2×2 unitary unimodular matrix represents a rotation in three dimensions. Find the axis and angle of rotation appropriate for U in terms of a_0, a_1, a_2, and a_3.

*

3.4 The spin-dependent Hamiltonian of an electron-positron system in the presence of a uniform magnetic field in the z-direction can be written as

$$H = A\mathbf{S}^{(e^-)}\cdot\mathbf{S}^{(e^+)} + \left(\frac{eB}{mc}\right)\left(S_z^{(e^-)} - S_z^{(e^+)}\right).$$

Suppose the spin function of the system is given by $\chi_+^{(e^-)}\chi_-^{(e^+)}$.

(a) Is this an eigenfunction of H in the limit $A \to 0, eB/mc \neq 0$? If it is, what is the energy eigenvalue? If it is not, what is the expectation value of H?

(b) Solve the same problem when $eB/mc \to 0, A \neq 0$.

3.5 Consider a spin 1 particle. Evaluate the matrix elements of

$$S_z(S_z + \hbar)(S_z - \hbar) \quad \text{and} \quad S_x(S_x + \hbar)(S_x - \hbar).$$

3.6 Let the Hamiltonian of a rigid body be

$$H = \frac{1}{2}\left(\frac{K_1^2}{I_1} + \frac{K_2^2}{I_2} + \frac{K_3^2}{I_3}\right),$$

where $\mathbf{K}$ is the angular momentum in the body frame. From this expression obtain the Heisenberg equation of motion for $\mathbf{K}$, and then find Euler's equation of motion in the correspondence limit.

3.7 Let $U = e^{iG_3\alpha}e^{iG_2\beta}e^{iG_3\gamma}$, where (α, β, γ) are the Eulerian angles. In order that U represent a rotation (α, β, γ), what are the commutation rules that must be satisfied by the G_k? Relate $\mathbf{G}$ to the angular-momentum operators.

3.8 What is the meaning of the following equation?

$$U^{-1}A_kU = \sum R_{kl}A_l,$$

where the three components of $\mathbf{A}$ are matrices. From this equation show that matrix elements $\langle m|A_k|n\rangle$ transform like vectors.

*

3.9 Consider a sequence of Euler rotations represented by

$$\begin{aligned}
\mathcal{D}^{(1/2)}(\alpha,\beta,\gamma) &= \exp\left(\frac{-i\sigma_3\alpha}{2}\right)\exp\left(\frac{-i\sigma_2\beta}{2}\right)\exp\left(\frac{-i\sigma_3\gamma}{2}\right)\\
&= \begin{pmatrix} e^{-i(\alpha+\gamma)/2}\cos\frac{\beta}{2} & -e^{-i(\alpha-\gamma)/2}\sin\frac{\beta}{2} \\ e^{i(\alpha-\gamma)/2}\sin\frac{\beta}{2} & e^{i(\alpha+\gamma)/2}\cos\frac{\beta}{2} \end{pmatrix}.
\end{aligned}$$

Because of the group properties of rotations, we expect that this sequence of operations is equivalent to a *single* rotation about some axis by an angle θ. Find θ.

* **3.10** **(a)** Consider a pure ensemble of identically prepared spin $\frac{1}{2}$ systems. Suppose the expectation values $\langle S_x \rangle$ and $\langle S_z \rangle$ and the sign of $\langle S_y \rangle$ are known. Show how we may determine the state vector. Why is it unnecessary to know the magnitude of $\langle S_y \rangle$?

(b) Consider a mixed ensemble of spin $\frac{1}{2}$ systems. Suppose the ensemble averages $[S_x]$, $[S_y]$, and $[S_z]$ are all known. Show how we may construct the 2×2 density matrix that characterizes the ensemble.

3.11 **(a)** Prove that the time evolution of the density operator ρ (in the Schrödinger picture) is given by

$$\rho(t) = \mathcal{U}(t, t_0)\rho(t_0)\mathcal{U}^\dagger(t, t_0).$$

(b) Suppose we have a pure ensemble at $t = 0$. Prove that it cannot evolve into a mixed ensemble as long as the time evolution is governed by the Schrödinger equation.

3.12 Consider an ensemble of spin 1 systems. The density matrix is now a 3×3 matrix. How many independent (real) parameters are needed to characterize the density matrix? What must we know in addition to $[S_x]$, $[S_y]$, and $[S_z]$ to characterize the ensemble completely?

3.13 An angular-momentum eigenstate $|j, m = m_{\text{max}} = j\rangle$ is rotated by an infinitesimal angle ε about the y-axis. Without using the explicit form of the $d^{(j)}_{m'm}$ function, obtain an expression for the probability for the new rotated state to be found in the original state up to terms of order ε^2.

3.14 Show that the 3×3 matrices $G_i (i = 1, 2, 3)$ whose elements are given by

$$(G_i)_{jk} = -i\hbar\varepsilon_{ijk},$$

where j and k are the row and column indices, satisfy the angular-momentum commutation relations. What is the physical (or geometric) significance of the transformation matrix that connects G_i to the more usual 3×3 representations of the angular-momentum operator J_i with J_3 taken to be diagonal? Relate your result to

$$\mathbf{V} \to \mathbf{V} + \hat{\mathbf{n}}\delta\phi \times \mathbf{V}$$

under infinitesimal rotations. (*Note:* This problem may be helpful in understanding the photon spin.)

3.15 **(a)** Let $\mathbf{J}$ be angular momentum. (It may stand for orbital $\mathbf{L}$, spin $\mathbf{S}$, or $\mathbf{J}_{\text{total}}$.) Using the fact that $J_x, J_y, J_z (J_\pm \equiv J_x \pm iJ_y)$ satisfy the usual angular-momentum commutation relations, prove

$$\mathbf{J}^2 = J_z^2 + J_+J_- - \hbar J_z.$$

(b) Using (a) (or otherwise), derive the "famous" expression for the coefficient c_- that appears in

$$J_-\psi_{jm} = c_-\psi_{j,m-1}.$$

3.16 Show that the orbital angular-momentum operator $\mathbf{L}$ commutes with both the operators $\mathbf{p}^2$ and $\mathbf{x}^2$; that is, prove (3.7.2).

3.17 The wave function of a particle subjected to a spherically symmetrical potential $V(r)$ is given by

$$\psi(\mathbf{x}) = (x + y + 3z) f(r).$$

(a) Is ψ an eigenfunction of $\mathbf{L}^2$? If so, what is the l-value? If not, what are the possible values of l that we may obtain when $\mathbf{L}^2$ is measured?

(b) What are the probabilities for the particle to be found in various m_l states?

(c) Suppose it is known somehow that $\psi(\mathbf{x})$ is an energy eigenfunction with eigenvalue E. Indicate how we may find $V(r)$.

3.18 A particle in a spherically symmetrical potential is known to be in an eigenstate of $\mathbf{L}^2$ and L_z with eigenvalues $\hbar^2 l(l+1)$ and $m\hbar$, respectively. Prove that the expectation values between $|lm\rangle$ states satisfy

$$\langle L_x \rangle = \langle L_y \rangle = 0, \quad \langle L_x^2 \rangle = \langle L_y^2 \rangle = \frac{[l(l+1)\hbar^2 - m^2\hbar^2]}{2}.$$

Interpret this result semiclassically.

3.19 Suppose a half-integer l-value, say $\frac{1}{2}$, were allowed for orbital angular momentum. From

$$L_+ Y_{1/2,1/2}(\theta,\phi) = 0,$$

we may deduce, as usual,

$$Y_{1/2,1/2}(\theta,\phi) \propto e^{i\phi/2}\sqrt{\sin\theta}.$$

Now try to construct $Y_{1/2,-1/2}(\theta,\phi)$ by (a) applying L_- to $Y_{1/2,1/2}(\theta,\phi)$; and (b) using $L_- Y_{1/2,-1/2}(\theta,\phi) = 0$. Show that the two procedures lead to contradictory results. (This gives an argument against half-integer l-values for orbital angular momentum.)

3.20 Consider an orbital angular-momentum eigenstate $|l = 2, m = 0\rangle$. Suppose this state is rotated by an angle β about the y-axis. Find the probability for the new state to be found in $m = 0, \pm 1$, and ± 2. (The spherical harmonics for $l = 0$, 1, and 2 given in Section B.5 in Appendix B may be useful.)

3.21 The goal of this problem is to determine degenerate eigenstates of the three-dimensional isotropic harmonic oscillator written as eigenstates of $\mathbf{L}^2$ and L_z, in terms of the Cartesian eigenstates $|n_x n_y n_z\rangle$.

(a) Show that the angular-momentum operators are given by

$$L_i = i\hbar\varepsilon_{ijk} a_j a_k^\dagger$$

$$\mathbf{L}^2 = \hbar^2 \left[N(N+1) - a_k^\dagger a_k^\dagger a_j a_j \right],$$

where summation is implied over repeated indices, ε_{ijk} is the totally antisymmetric symbol, and $N \equiv a_j^\dagger a_j$ counts the total number of quanta.

(b) Use these relations to express the states $|qlm\rangle = |01m\rangle$, $m = 0, \pm 1$, in terms of the three eigenstates $|n_x n_y n_z\rangle$ that are degenerate in energy. Write down the representation of your answer in coordinate space, and check that the angular and radial dependences are correct.

(c) Repeat for $|qlm\rangle = |200\rangle$.

(d) Repeat for $|qlm\rangle = |02m\rangle$, with $m = 0$, 1, and 2.

3.22 Follow these steps to show that solutions to Kummer's Equation (3.7.46) can be written in terms of Laguerre polynomials $L_n(x)$, which are defined according to a generating function as

$$g(x,t) = \frac{e^{-xt/(1-t)}}{1-t} = \sum_{n=0}^{\infty} L_n(x)\frac{t^n}{n!},$$

where $0 < t < 1$. The discussion in Section 2.5 on generating functions for Hermite polynomials will be helpful.

(a) Prove that $L_n(0) = n!$ and $L_0(x) = 1$.

(b) Differentiate $g(x,t)$ with respect to x, show that

$$L_n'(x) - nL_{n-1}'(x) = -nL_{n-1}(x),$$

and find the first few Laguerre polynomials.

(c) Differentiate $g(x,t)$ with respect to t and show that

$$L_{n+1}(x) - (2n+1-x)L_n(x) + n^2 L_{n-1}(x) = 0.$$

(d) Now show that Kummer's Equation is solved by deriving

$$xL_n''(x) + (1-x)L_n'(x) + nL_n(x) = 0,$$

and associate n with the principal quantum number for the hydrogen atom.

3.23 What is the physical significance of the operators

$$K_+ \equiv a_+^\dagger a_-^\dagger \quad \text{and} \quad K_- \equiv a_+ a_-$$

in Schwinger's scheme for angular momentum? Give the nonvanishing matrix elements of $K_\pm$.

3.24 We are to add angular momenta $j_1 = 1$ and $j_2 = 1$ to form $j = 2$, 1, and 0 states. Using either the ladder operator method or the recursion relation, express all (nine) $\{j,m\}$ eigenkets in terms of $|j_1 j_2; m_1 m_2\rangle$. Write your answer as

$$|j=1, m=1\rangle = \frac{1}{\sqrt{2}}|+,0\rangle - \frac{1}{\sqrt{2}}|0,+\rangle, \ldots,$$

where + and 0 stand for $m_{1,2} = 1, 0$, respectively.

3.25 **(a)** Evaluate

$$\sum_{m=-j}^{j} |d_{mm'}^{(j)}(\beta)|^2 m$$

for *any* j (integer or half-integer); then check your answer for $j = \frac{1}{2}$.

(b) Prove, for any j,

$$\sum_{m=-j}^{j} m^2 |d^{(j)}_{m'm}(\beta)|^2 = \frac{1}{2} j(j+1)\sin^2\beta + m'^2 \frac{1}{2}(3\cos^2\beta - 1).$$

[*Hint:* This can be proved in many ways. You may, for instance, examine the rotational properties of J_z^2 using the spherical (irreducible) tensor language.]

3.26 **(a)** Consider a system with $j = 1$. Explicitly write

$$\langle j = 1, m' | J_y | j = 1, m \rangle$$

in 3×3 matrix form.

(b) Show that for $j = 1$ only, it is legitimate to replace $e^{-iJ_y\beta/\hbar}$ by

$$1 - i\left(\frac{J_y}{\hbar}\right)\sin\beta - \left(\frac{J_y}{\hbar}\right)^2 (1 - \cos\beta).$$

(c) Using (b), prove

$$d^{(j=1)}(\beta) = \begin{pmatrix} \left(\frac{1}{2}\right)(1+\cos\beta) & -\left(\frac{1}{\sqrt{2}}\right)\sin\beta & \left(\frac{1}{2}\right)(1-\cos\beta) \\ \left(\frac{1}{\sqrt{2}}\right)\sin\beta & \cos\beta & -\left(\frac{1}{\sqrt{2}}\right)\sin\beta \\ \left(\frac{1}{2}\right)(1-\cos\beta) & \left(\frac{1}{\sqrt{2}}\right)\sin\beta & \left(\frac{1}{2}\right)(1+\cos\beta) \end{pmatrix}.$$

3.27 Express the matrix element $\langle \alpha_2\beta_2\gamma_2 | J_3^2 | \alpha_1\beta_1\gamma_1 \rangle$ in terms of a series in

$$\mathcal{D}^j_{mn}(\alpha\beta\gamma) = \langle \alpha\beta\gamma | jmn \rangle.$$

3.28 Consider a system made up of two spin $\frac{1}{2}$ particles. Observer A specializes in measuring the spin components of one of the particles (s_{1z}, s_{1x} and so on), while observer B measures the spin components of the other particle. Suppose the system is known to be in a spin-singlet state—that is, $S_{\text{total}} = 0$.

(a) What is the probability for observer A to obtain $s_{1z} = \hbar/2$ when observer B makes no measurement? Solve the same problem for $s_{1x} = \hbar/2$.

(b) Observer B determines the spin of particle 2 to be in the $s_{2z} = \hbar/2$ state with certainty. What can we then conclude about the outcome of observer A's measurement (i) if A measures s_{1z}; (ii) if A measures s_{1x}? Justify your answer.

3.29 Consider a spherical tensor of rank 1 (that is, a vector)

$$V^{(1)}_{\pm 1} = \mp \frac{V_x \pm iV_y}{\sqrt{2}}, \quad V^{(1)}_0 = V_z.$$

Using the expression for $d^{(j=1)}$ given in Problem 3.26, evaluate

$$\sum_{q'} d^{(1)}_{qq'}(\beta) V^{(1)}_{q'}$$

and show that your results are just what you expect from the transformation properties of $V_{x,y,z}$ under rotations about the y-axis.

3.30 **(a)** Construct a spherical tensor of rank 1 out of two different vectors $\mathbf{U} = (U_x, U_y, U_z)$ and $\mathbf{V} = (V_x, V_y, V_z)$. Explicitly write $T^{(1)}_{\pm 1,0}$ in terms of $U_{x,y,z}$ and $V_{x,y,z}$.

(b) Construct a spherical tensor of rank 2 out of two different vectors $\mathbf{U}$ and $\mathbf{V}$. Write down explicitly $T^{(2)}_{\pm 2,\pm 1,0}$ in terms of $U_{x,y,z}$ and $V_{x,y,z}$.

3.31 Consider a spinless particle bound to a fixed center by a central force potential.

(a) Relate, as much as possible, the matrix elements

$$\langle n',l',m'|\mp\frac{1}{\sqrt{2}}(x \pm iy)|n,l,m\rangle \quad \text{and} \quad \langle n',l',m'|z|n,l,m\rangle$$

using *only* the Wigner-Eckart theorem. Make sure to state under what conditions the matrix elements are nonvanishing.

(b) Do the same problem using wave functions $\psi(\mathbf{x}) = R_{nl}(r)Y_l^m(\theta,\phi)$.

3.32 **(a)** Write xy, xz, and $(x^2 - y^2)$ as components of a spherical (irreducible) tensor of rank 2.

(b) The expectation value

$$Q \equiv e\langle \alpha, j, m = j|(3z^2 - r^2)|\alpha, j, m = j\rangle$$

is known as the *quadrupole moment*. Evaluate

$$e\langle \alpha, j, m'|(x^2 - y^2)|\alpha, j, m = j\rangle,$$

where $m' = j, j-1, j-2, \ldots$, in terms of Q and appropriate Clebsch-Gordan coefficients.

3.33 A spin $\frac{3}{2}$ nucleus situated at the origin is subjected to an external inhomogeneous electric field. The basic electric quadrupole interaction may by taken to be

$$H_{\text{int}} = \frac{eQ}{2s(s-1)\hbar^2}\left[\left(\frac{\partial^2\phi}{\partial x^2}\right)_0 S_x^2 + \left(\frac{\partial^2\phi}{\partial y^2}\right)_0 S_y^2 + \left(\frac{\partial^2\phi}{\partial z^2}\right)_0 S_z^2\right],$$

where ϕ is the electrostatic potential satisfying Laplace's equation, and the coordinate axes are chosen such that

$$\left(\frac{\partial^2\phi}{\partial x \partial y}\right)_0 = \left(\frac{\partial^2\phi}{\partial y \partial z}\right)_0 = \left(\frac{\partial^2\phi}{\partial x \partial z}\right)_0 = 0.$$

Show that the interaction energy can be written as

$$A(3S_z^2 - \mathbf{S}^2) + B(S_+^2 + S_-^2),$$

and express A and B in terms of $(\partial^2\phi/\partial x^2)_0$ and so on. Determine the energy eigenkets (in terms of $|m\rangle$, where $m = \pm\frac{3}{2}, \pm\frac{1}{2}$) and the corresponding energy eigenvalues. Is there any degeneracy?