As a trivial example, for a spin $\frac{1}{2}$ system the spin-up state $|+\rangle$ and its time-reversed state $|-\rangle$ no longer have the same energy in the presence of an external magnetic field. In general, Kramers degeneracy in a system containing an odd number of electrons can be lifted by applying an external magnetic field.

Notice that when we treat **B** as external, we do not change **B** under time reversal; this is because the atomic electron is viewed as a closed quantum-mechanical system to which we apply the time-reversal operator. This should not be confused with our earlier remarks concerning the invariance of the Maxwell equations (4.4.2) and the Lorentz force equation under $t \to -t$ and (4.4.3). There we were to apply time reversal to the *whole world*, for example, even to the currents in the wire that produces the **B** field!

Problems

4.1 Calculate the *three lowest* energy levels, together with their degeneracies, for the following systems (assume equal-mass *distinguishable* particles).

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(a) Three noninteracting spin $\frac{1}{2}$ particles in a box of length L.

(b) Four noninteracting spin $\frac{1}{2}$ particles in a box of length L.

4.2 Let $\mathcal{T}_{\mathbf{d}}$ denote the translation operator (displacement vector $\mathbf{d}$); let $\mathcal{D}(\hat{\mathbf{n}}, \phi)$ denote the rotation operator ($\hat{\mathbf{n}}$ and ϕ are the axis and angle of rotation, respectively); and let π denote the parity operator. Which, if any, of the following pairs commute? Why?

(a) $\mathcal{T}_{\mathbf{d}}$ and $\mathcal{T}_{\mathbf{d}'}$ ($\mathbf{d}$ and $\mathbf{d}'$ in different directions).

(b) $\mathcal{D}(\hat{\mathbf{n}}, \phi)$ and $\mathcal{D}(\hat{\mathbf{n}}', \phi')$ ($\hat{\mathbf{n}}$ and $\hat{\mathbf{n}}'$ in different directions).

(c) $\mathcal{T}_{\mathbf{d}}$ and π.

(d) $\mathcal{D}(\hat{\mathbf{n}}, \phi)$ and π.

4.3 A quantum-mechanical state Ψ is known to be a simultaneous eigenstate of two Hermitian operators A and B that *anticommute*:

$$AB + BA = 0.$$

What can you say about the eigenvalues of A and B for state Ψ? Illustrate your point using the parity operator (which can be chosen to satisfy $\pi = \pi^{-1} = \pi^{\dagger}$) and the momentum operator.

4.4 A spin $\frac{1}{2}$ particle is bound to a fixed center by a spherically symmetrical potential.

(a) Write down the spin-angular function $\mathcal{Y}_{l=0}^{j=1/2, m=1/2}$.

(b) Express $(\boldsymbol{\sigma} \cdot \mathbf{x})\, \mathcal{Y}_{l=0}^{j=1/2, m=1/2}$ in terms of some other $\mathcal{Y}_{l}^{j,m}$.

(c) Show that your result in (b) is understandable in view of the transformation properties of the operator $\mathbf{S} \cdot \mathbf{x}$ under rotations and under space inversion (parity).

4.5 Because of weak (neutral-current) interactions, there is a parity-violating potential between the atomic electron and the nucleus as follows:

$$V = \lambda[\delta^{(3)}(\mathbf{x})\mathbf{S} \cdot \mathbf{p} + \mathbf{S} \cdot \mathbf{p}\delta^{(3)}(\mathbf{x})],$$

where $\mathbf{S}$ and $\mathbf{p}$ are the spin and momentum operators of the electron, and the nucleus is assumed to be situated at the origin. As a result, the ground state of an alkali atom, usually characterized by $|n,l,j,m\rangle$, actually contains very tiny contributions from other eigenstates as follows:

$$|n,l,j,m\rangle \to |n,l,j,m\rangle + \sum_{n'l'j'm'} C_{n'l'j'm'}|n',l',j',m'\rangle.$$

On the basis of symmetry considerations *alone*, what can you say about (n',l',j',m'), which give rise to nonvanishing contributions? Suppose the radial wave functions and the energy levels are all known. Indicate how you may calculate $C_{n'l'j'm'}$. Do we get further restrictions on (n',l',j',m')?

4.6 Consider a symmetric rectangular double-well potential:

$$V = \begin{cases} \infty & \text{for } |x| > a+b; \\ 0 & \text{for } a < |x| < a+b; \\ V_0 > 0 & \text{for } |x| < a. \end{cases}$$

Assuming that V_0 is very high compared to the quantized energies of low-lying states, obtain an approximate expression for the energy splitting between the two lowest-lying states.

4.7 **(a)** Let $\psi(\mathbf{x},t)$ be the wave function of a spinless particle corresponding to a plane wave in three dimensions. Show that $\psi^*(\mathbf{x},-t)$ is the wave function for the plane wave with the momentum direction reversed.

(b) Let $\chi(\hat{\mathbf{n}})$ be the two-component eigenspinor of $\boldsymbol{\sigma}\cdot\hat{\mathbf{n}}$ with eigenvalue $+1$. *Using the explicit form of* $\chi(\hat{\mathbf{n}})$ (in terms of the polar and azimuthal angles β and γ that characterize $\hat{\mathbf{n}}$), verify that $-i\sigma_2\chi^*(\hat{\mathbf{n}})$ is the two-component eigenspinor with the spin direction reversed.

4.8 **(a)** Assuming that the Hamiltonian is invariant under time reversal, prove that the wave function for a spinless nondegenerate system at any given instant of time can always be chosen to be real.

(b) The wave function for a plane-wave state at $t=0$ is given by a complex function $e^{i\mathbf{p}\cdot\mathbf{x}/\hbar}$. Why does this not violate time-reversal invariance?

4.9 Let $\phi(\mathbf{p}')$ be the momentum-space wave function for state $|\alpha\rangle$—that is, $\phi(\mathbf{p}') = \langle\mathbf{p}'|\alpha\rangle$. Is the momentum-space wave function for the time-reversed state $\theta|\alpha\rangle$ given by $\phi(\mathbf{p}')$, by $\phi(-\mathbf{p}')$, by $\phi^*(\mathbf{p}')$, or by $\phi^*(-\mathbf{p}')$? Justify your answer.

4.10 **(a)** What is the time-reversed state corresponding to $\mathcal{D}(R)|j,m\rangle$?

(b) Using the properties of time reversal and rotations, prove

$$\mathcal{D}^{(j)*}_{m'm}(R) = (-1)^{m-m'}\mathcal{D}^{(j)}_{-m',-m}(R).$$

(c) Prove $\theta|j,m\rangle = i^{2m}|j,-m\rangle$.

4.11 Suppose a spinless particle is bound to a fixed center by a potential $V(\mathbf{x})$ so asymmetrical that no energy level is degenerate. Using time-reversal invariance, prove

$$\langle\mathbf{L}\rangle = 0$$

for any energy eigenstate. (This is known as **quenching** of orbital angular momentum.) If the wave function of such a nondegenerate eigenstate is expanded as

$$\sum_l \sum_m F_{lm}(r) Y_l^m(\theta, \phi),$$

what kind of phase restrictions do we obtain on $F_{lm}(r)$?

4.12 The Hamiltonian for a spin 1 system is given by

$$H = AS_z^2 + B(S_x^2 - S_y^2).$$

Solve this problem *exactly* to find the normalized energy eigenstates and eigenvalues. (A spin-dependent Hamiltonian of this kind actually appears in crystal physics.) Is this Hamiltonian invariant under time reversal? How do the normalized eigenstates you obtained transform under time reversal?