As a trivial example, for a spin $\frac{1}{2}$ system the spin-up state $|+\rangle$ and its time-reversed state $|-\rangle$ no longer have the same energy in the presence of an external magnetic field. In general, Kramers degeneracy in a system containing an odd number of electrons can be lifted by applying an external magnetic field.

Notice that when we treat **B** as external, we do not change **B** under time reversal; this is because the atomic electron is viewed as a closed quantum-mechanical system to which we apply the time-reversal operator. This should not be confused with our earlier remarks concerning the invariance of the Maxwell equations (4.4.2) and the Lorentz force equation under $t \to -t$ and (4.4.3). There we were to apply time reversal to the *whole world*, for example, even to the currents in the wire that produces the **B** field!

Problems

Calculate the *three lowest* energy levels, together with their degeneracies, for the following systems (assume equal-mass *distinguishable* particles).

- (a) Three noninteracting spin $\frac{1}{2}$ particles in a box of length L.
- (b) Four noninteracting spin $\frac{1}{2}$ particles in a box of length L.
- Let $\mathcal{T}_{\mathbf{d}}$ denote the translation operator (displacement vector \mathbf{d}); let $\mathcal{D}(\hat{\mathbf{n}}, \phi)$ denote the rotation operator ($\hat{\mathbf{n}}$ and ϕ are the axis and angle of rotation, respectively); and let π denote the parity operator. Which, if any, of the following pairs commute? Why?
 - (a) \mathcal{T}_d and $\mathcal{T}_{d'}$ (d and d' in different directions).
 - **(b)** $\mathcal{D}(\hat{\mathbf{n}}, \phi)$ and $\mathcal{D}(\hat{\mathbf{n}}', \phi')$ ($\hat{\mathbf{n}}$ and $\hat{\mathbf{n}}'$ in different directions).
 - (c) \mathcal{T}_d and π .
 - (d) $\mathcal{D}(\hat{\mathbf{n}}, \phi)$ and π .

A quantum-mechanical state Ψ is known to be a simultaneous eigenstate of two Hermitian operators A and B that anticommute:

$$AB + BA = 0.$$

What can you say about the eigenvalues of A and B for state Ψ ? Illustrate your point using the parity operator (which can be chosen to satisfy $\pi = \pi^{-1} = \pi^{\dagger}$) and the momentum operator.

- **4.4** A spin $\frac{1}{2}$ particle is bound to a fixed center by a spherically symmetrical potential.
 - (a) Write down the spin-angular function $\mathcal{Y}_{l=0}^{j=1/2,m=1/2}$.
 - **(b)** Express $(\sigma \cdot \mathbf{x}) \mathcal{Y}_{l=0}^{j=1/2,m=1/2}$ in terms of some other $\mathcal{Y}_{l}^{j,m}$.
 - (c) Show that your result in (b) is understandable in view of the transformation properties of the operator $\mathbf{S} \cdot \mathbf{x}$ under rotations and under space inversion (parity).
- **4.5** Because of weak (neutral-current) interactions, there is a parity-violating potential between the atomic electron and the nucleus as follows:

$$V = \lambda [\delta^{(3)}(\mathbf{x})\mathbf{S} \cdot \mathbf{p} + \mathbf{S} \cdot \mathbf{p}\delta^{(3)}(\mathbf{x})],$$

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where **S** and **p** are the spin and momentum operators of the electron, and the nucleus is assumed to be situated at the origin. As a result, the ground state of an alkali atom, usually characterized by $|n,l,j,m\rangle$, actually contains very tiny contributions from other eigenstates as follows:

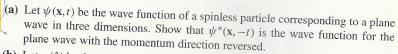
$$|n,l,j,m\rangle \rightarrow |n,l,j,m\rangle + \sum_{n'l'j'm'} \, C_{n'l'j'm'} |n',l',j',m'\rangle. \label{eq:local_local_problem}$$

On the basis of symmetry considerations *alone*, what can you say about (n', l', j', m'), which give rise to nonvanishing contributions? Suppose the radial wave functions and the energy levels are all known. Indicate how you may calculate $C_{n'l'j'm'}$. Do we get further restrictions on (n', l', j', m')?

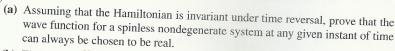
4.6 Consider a symmetric rectangular double-well potential:

$$V = \begin{cases} \infty & \text{for } |x| > a + b; \\ 0 & \text{for } a < |x| < a + b; \\ V_0 > 0 & \text{for } |x| < a. \end{cases}$$

Assuming that V_0 is very high compared to the quantized energies of low-lying states, obtain an approximate expression for the energy splitting between the two lowest-lying states.



(b) Let $\chi(\hat{\mathbf{n}})$ be the two-component eigenspinor of $\sigma \cdot \hat{\mathbf{n}}$ with eigenvalue +1. Using the explicit form of $\chi(\hat{\mathbf{n}})$ (in terms of the polar and azimuthal angles β and γ that characterize $\hat{\mathbf{n}}$), verify that $-i\sigma_2\chi^*(\hat{\mathbf{n}})$ is the two-component eigenspinor with the spin direction reversed.



(b) The wave function for a plane-wave state at t = 0 is given by a complex function $e^{i\mathbf{p}\cdot\mathbf{x}/\hbar}$. Why does this not violate time-reversal invariance?

Let $\phi(\mathbf{p}')$ be the momentum-space wave function for state $|\alpha\rangle$ —that is, $\phi(\mathbf{p}') = \langle \mathbf{p}' | \alpha \rangle$. Is the momentum-space wave function for the time-reversed state $\theta | \alpha \rangle$ given by $\phi(\mathbf{p}')$, by $\phi(-\mathbf{p}')$, by $\phi^*(\mathbf{p}')$, or by $\phi^*(-\mathbf{p}')$? Justify your answer.

4.10 (a) What is the time-reversed state corresponding to $\mathcal{D}(R)|j,m\rangle$?

(b) Using the properties of time reversal and rotations, prove

$$\mathcal{D}_{m'm}^{(j)*}(R) = (-1)^{m-m'} \mathcal{D}_{-m',-m}^{(j)}(R).$$

(c) Prove $\theta|j,m\rangle = i^{2m}|j,-m\rangle$.

4.11 Suppose a spinless particle is bound to a fixed center by a potential $V(\mathbf{x})$ so asymmetrical that no energy level is degenerate. Using time-reversal invariance, prove

$$\langle \mathbf{L} \rangle = 0$$

for any energy eigenstate. (This is known as **quenching** of orbital angular momentum.) If the wave function of such a nondegenerate eigenstate is expanded as

$$\sum_{l}\sum_{m}F_{lm}(r)Y_{l}^{m}(\theta,\phi),$$

what kind of phase restrictions do we obtain on $F_{lm}(r)$?

12 The Hamiltonian for a spin 1 system is given by

$$H = AS_z^2 + B(S_x^2 - S_y^2).$$

Solve this problem *exactly* to find the normalized energy eigenstates and eigenvalues. (A spin-dependent Hamiltonian of this kind actually appears in crystal physics.) Is this Hamiltonian invariant under time reversal? How do the normalized eigenstates you obtained transform under time reversal?