Momento angular orbital

Gustavo Matheus Gauy

FI-001 - Mecânica quântica 1

1 de Junho, 2020

- Queremos mostrar que

$$\left\langle \boldsymbol{r}' \left| L_x \right| \alpha \right\rangle = i\hbar \left(\sin\phi \frac{\partial}{\partial\theta} + \mathrm{cotg}\theta\cos\phi \frac{\partial}{\partial\phi} \right) \left\langle \boldsymbol{r}' | \alpha \right\rangle$$

e

$$\left\langle \boldsymbol{r}' \left| L_y \right| \alpha \right\rangle = i\hbar \left(-\cos\phi \frac{\partial}{\partial\theta} + \mathrm{cotg}\theta\sin\phi \frac{\partial}{\partial\phi} \right) \left\langle \boldsymbol{r}' | \alpha \right\rangle$$

onde $|\alpha\rangle$ é um ket qualquer e (r, θ, ϕ) são as coordenadas esféricas.

- Definição do operador de momento angular orbital:

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$$

- Definição do operador de momento angular orbital:

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p} \Rightarrow L_k = \sum_{ij} \epsilon_{ijk} x_i p_j$$

- Definição do operador de momento angular orbital:

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p} \Rightarrow L_k = \sum_{ij} \epsilon_{ijk} x_i p_j$$

então

$$\left\langle \boldsymbol{r}' \right| L_k = \left\langle \boldsymbol{r}' \right| \sum_{ij} \epsilon_{ijk} x_i p_j$$

- Definição do operador de momento angular orbital:

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p} \Rightarrow L_k = \sum_{ij} \epsilon_{ijk} x_i p_j$$

então

$$\langle \boldsymbol{r}' | \, L_k = \sum_{ij} \epsilon_{ijk} \langle \boldsymbol{r}' | \, x_i p_j$$

- Definição do operador de momento angular orbital:

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p} \Rightarrow L_k = \sum_{ij} \epsilon_{ijk} x_i p_j$$

então

$$\left\langle \boldsymbol{r}' \right| L_k = \sum_{ij} \epsilon_{ijk} \left\langle \boldsymbol{r}' \right| x_i' p_j$$

- Definição do operador de momento angular orbital:

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p} \Rightarrow L_k = \sum_{ij} \epsilon_{ijk} x_i p_j$$

então

$$\langle \boldsymbol{r}' | \, L_k = \sum_{ij} \epsilon_{ijk} x_i' \langle \boldsymbol{r}' | \, p_j$$

- Definição do operador de momento angular orbital:

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p} \Rightarrow L_k = \sum_{ij} \epsilon_{ijk} x_i p_j$$

então

$$\langle \boldsymbol{r}' | \, L_k = \sum_{ij} \epsilon_{ijk} x'_i \langle \boldsymbol{r}' | \left(-i\hbar \frac{\partial}{\partial x'_j} \right)$$

- Definição do operador de momento angular orbital:

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p} \Rightarrow L_k = \sum_{ij} \epsilon_{ijk} x_i p_j$$

então

$$\langle \boldsymbol{r}' | \, L_k = -i\hbar \sum_{ij} \epsilon_{ijk} x_i' \frac{\partial}{\partial x_j'} \langle \boldsymbol{r}' |$$

- Definição do operador de momento angular orbital:

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p} \Rightarrow L_k = \sum_{ij} \epsilon_{ijk} x_i p_j$$

então

$$\langle \boldsymbol{r}' | \, L_k = \, -i\hbar \sum_{ij} \epsilon_{ijk} x'_i \frac{\partial}{\partial x'_j} \langle \boldsymbol{r}' |$$

- Portanto

$$\langle \boldsymbol{r}' \, | L_x | \, \alpha \rangle = -i\hbar \sum_{ij} \epsilon_{ij1} x'_i \frac{\partial}{\partial x'_j} \langle \boldsymbol{r}' | \alpha \rangle$$

- Definição do operador de momento angular orbital:

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p} \Rightarrow L_k = \sum_{ij} \epsilon_{ijk} x_i p_j$$

então

$$\langle \boldsymbol{r}' | \, L_k = \, -i\hbar \sum_{ij} \epsilon_{ijk} x_i' \frac{\partial}{\partial x_j'} \langle \boldsymbol{r}' |$$

- Portanto

$$\langle \boldsymbol{r}' \, | L_x | \, \alpha \rangle = -i\hbar \sum_{ij} \epsilon_{ij1} x_i' \frac{\partial}{\partial x_j'} \langle \boldsymbol{r}' | \alpha \rangle \, = -i\hbar \left(y' \frac{\partial}{\partial z'} - z' \frac{\partial}{\partial y'} \right) \langle \boldsymbol{r}' | \alpha \rangle$$

- Definição do operador de momento angular orbital:

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p} \Rightarrow L_k = \sum_{ij} \epsilon_{ijk} x_i p_j$$

então

$$\langle \boldsymbol{r}' | \, L_k = \; - i\hbar \sum_{ij} \epsilon_{ijk} x_i' \frac{\partial}{\partial x_j'} \langle \boldsymbol{r}' |$$

- Portanto

$$\langle \boldsymbol{r}' \, | L_x | \, \alpha \rangle = -i\hbar \sum_{ij} \epsilon_{ij1} x_i' \frac{\partial}{\partial x_j'} \langle \boldsymbol{r}' | \alpha \rangle = -i\hbar \left(y' \frac{\partial}{\partial z'} - z' \frac{\partial}{\partial y'} \right) \langle \boldsymbol{r}' | \alpha \rangle$$

e

$$\langle \boldsymbol{r}' \, | L_y | \, \alpha \rangle = -i\hbar \sum_{ij} \epsilon_{ij2} x_i' \frac{\partial}{\partial x_j'} \langle \boldsymbol{r}' | \alpha \rangle$$

- Definição do operador de momento angular orbital:

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p} \Rightarrow L_k = \sum_{ij} \epsilon_{ijk} x_i p_j$$

então

$$\langle \boldsymbol{r}' | \, L_k = \; - i\hbar \sum_{ij} \epsilon_{ijk} x'_i \frac{\partial}{\partial x'_j} \langle \boldsymbol{r}' |$$

- Portanto

$$\langle \boldsymbol{r}' \, | L_x | \, \alpha \rangle = -i\hbar \sum_{ij} \epsilon_{ij1} x'_i \frac{\partial}{\partial x'_j} \langle \boldsymbol{r}' | \alpha \rangle \, = -i\hbar \left(y' \frac{\partial}{\partial z'} - z' \frac{\partial}{\partial y'} \right) \langle \boldsymbol{r}' | \alpha \rangle$$

e

$$\langle \boldsymbol{r}' \, | L_y | \, \alpha \rangle = -i\hbar \sum_{ij} \epsilon_{ij2} x'_i \frac{\partial}{\partial x'_j} \langle \boldsymbol{r}' | \alpha \rangle \, = -i\hbar \left(z' \frac{\partial}{\partial x'} - x' \frac{\partial}{\partial z'} \right) \langle \boldsymbol{r}' | \alpha \rangle$$

- Queremos o resultado anterior em coordenadas esféricas

$$x = r\sin\theta\cos\phi$$

$$y = r\sin\theta\sin\phi$$

$$z = r\cos\theta$$

- Queremos o resultado anterior em coordenadas esféricas

$$x = r\sin\theta\cos\phi$$

$$y = r\sin\theta\sin\phi$$

$$z = r\cos\theta$$

- Sabemos que dada uma transformação de coordenadas

$$x_i = x_i(y)$$

- Queremos o resultado anterior em coordenadas esféricas

$$x = r\sin\theta\cos\phi$$

$$y = r\sin\theta\sin\phi$$

$$z = r\cos\theta$$

- Sabemos que dada uma transformação de coordenadas

$$x_i = x_i(y)$$

temos

$$\frac{\partial}{\partial x_i} = \sum_j \frac{\partial y_j}{\partial x_i}\frac{\partial}{\partial y_j}$$

- Queremos o resultado anterior em coordenadas esféricas

$$x = r\sin\theta\cos\phi$$

$$y = r\sin\theta\sin\phi$$

$$z = r\cos\theta$$

- Sabemos que dada uma transformação de coordenadas

$$x_i = x_i(y)$$

temos

$$\frac{\partial}{\partial x_i} = \sum_j \frac{\partial y_j}{\partial x_i}\frac{\partial}{\partial y_j}$$

para o x

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial x}\frac{\partial}{\partial \phi}$$

- Queremos o resultado anterior em coordenadas esféricas

$$x = r\sin\theta\cos\phi$$

$$y = r\sin\theta\sin\phi$$

$$z = r\cos\theta$$

- Sabemos que dada uma transformação de coordenadas

$$x_i = x_i(y)$$

temos

$$\frac{\partial}{\partial x_i} = \sum_j \frac{\partial y_j}{\partial x_i}\frac{\partial}{\partial y_j}$$

para o y

$$\frac{\partial}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial y}\frac{\partial}{\partial \phi}$$

- Queremos o resultado anterior em coordenadas esféricas

$$x = r\sin\theta\cos\phi$$

$$y = r\sin\theta\sin\phi$$

$$z = r\cos\theta$$

- Sabemos que dada uma transformação de coordenadas

$$x_i = x_i(y)$$

temos

$$\frac{\partial}{\partial x_i} = \sum_j \frac{\partial y_j}{\partial x_i}\frac{\partial}{\partial y_j}$$

para o z

$$\frac{\partial}{\partial z} = \frac{\partial r}{\partial z}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial z}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial z}\frac{\partial}{\partial \phi}$$

- Usando a definição de coordenadas esféricas temos

$$r = \sqrt{x^2 + y^2 + z^2}, \quad \cos\theta = \frac{z}{r}, \quad \mathrm{tg}\phi = \frac{y}{x}$$

- Usando a definição de coordenadas esféricas temos

$$r = \sqrt{x^2 + y^2 + z^2}, \quad \cos\theta = \frac{z}{r}, \quad \mathrm{tg}\phi = \frac{y}{x}$$

- Com isso, as componentes do Jacobiano são

$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

- Usando a definição de coordenadas esféricas temos

$$r = \sqrt{x^2 + y^2 + z^2}, \quad \cos\theta = \frac{z}{r}, \quad \mathrm{tg}\phi = \frac{y}{x}$$

- Com isso, as componentes do Jacobiano são

$$\frac{\partial r}{\partial x} = \sin\theta\cos\phi$$

- Usando a definição de coordenadas esféricas temos

$$r = \sqrt{x^2 + y^2 + z^2}, \quad \cos\theta = \frac{z}{r}, \quad \mathrm{tg}\phi = \frac{y}{x}$$

- Com isso, as componentes do Jacobiano são

$$\frac{\partial r}{\partial x} = \sin\theta\cos\phi, \quad \frac{\partial r}{\partial y} = \frac{y}{r}$$

- Usando a definição de coordenadas esféricas temos

$$r = \sqrt{x^2 + y^2 + z^2}, \quad \cos\theta = \frac{z}{r}, \quad \mathrm{tg}\phi = \frac{y}{x}$$

- Com isso, as componentes do Jacobiano são

$$\frac{\partial r}{\partial x} = \sin\theta\cos\phi, \quad \frac{\partial r}{\partial y} = \sin\theta\sin\phi$$

- Usando a definição de coordenadas esféricas temos

$$r = \sqrt{x^2 + y^2 + z^2}, \quad \cos\theta = \frac{z}{r}, \quad \mathrm{tg}\phi = \frac{y}{x}$$

- Com isso, as componentes do Jacobiano são

$$\frac{\partial r}{\partial x} = \sin\theta\cos\phi, \quad \frac{\partial r}{\partial y} = \sin\theta\sin\phi, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

- Usando a definição de coordenadas esféricas temos

$$r = \sqrt{x^2 + y^2 + z^2}, \quad \cos\theta = \frac{z}{r}, \quad \text{tg}\phi = \frac{y}{x}$$

- Com isso, as componentes do Jacobiano são

$$\frac{\partial r}{\partial x} = \sin\theta\cos\phi, \quad \frac{\partial r}{\partial y} = \sin\theta\sin\phi, \quad \frac{\partial r}{\partial z} = \cos\theta$$

- Usando a definição de coordenadas esféricas temos

$$r = \sqrt{x^2 + y^2 + z^2}, \quad \cos\theta = \frac{z}{r}, \quad \mathrm{tg}\phi = \frac{y}{x}$$

- Com isso, as componentes do Jacobiano são

$$\frac{\partial r}{\partial x} = \sin\theta\cos\phi, \quad \frac{\partial r}{\partial y} = \sin\theta\sin\phi, \quad \frac{\partial r}{\partial z} = \cos\theta;$$

$$\frac{\partial \cos\theta}{\partial x} = \frac{\partial}{\partial x}\left(\frac{z}{r}\right)$$

- Usando a definição de coordenadas esféricas temos

$$r = \sqrt{x^2 + y^2 + z^2}, \quad \cos\theta = \frac{z}{r}, \quad \mathrm{tg}\phi = \frac{y}{x}$$

- Com isso, as componentes do Jacobiano são

$$\frac{\partial r}{\partial x} = \sin\theta\cos\phi, \quad \frac{\partial r}{\partial y} = \sin\theta\sin\phi, \quad \frac{\partial r}{\partial z} = \cos\theta;$$

$$-\sin\theta\frac{\partial\theta}{\partial x} = \frac{\partial}{\partial x}\left(\frac{z}{r}\right)$$

- Usando a definição de coordenadas esféricas temos

$$r = \sqrt{x^2 + y^2 + z^2}, \quad \cos\theta = \frac{z}{r}, \quad \mathrm{tg}\phi = \frac{y}{x}$$

- Com isso, as componentes do Jacobiano são

$$\frac{\partial r}{\partial x} = \sin\theta\cos\phi, \quad \frac{\partial r}{\partial y} = \sin\theta\sin\phi, \quad \frac{\partial r}{\partial z} = \cos\theta;$$

$$-\sin\theta\frac{\partial\theta}{\partial x} = -\frac{z}{r^2}\frac{\partial r}{\partial x}$$

- Usando a definição de coordenadas esféricas temos

$$r = \sqrt{x^2 + y^2 + z^2}, \quad \cos\theta = \frac{z}{r}, \quad \mathrm{tg}\phi = \frac{y}{x}$$

- Com isso, as componentes do Jacobiano são

$$\frac{\partial r}{\partial x} = \sin\theta\cos\phi, \quad \frac{\partial r}{\partial y} = \sin\theta\sin\phi, \quad \frac{\partial r}{\partial z} = \cos\theta;$$

$$\frac{\partial\theta}{\partial x} = \frac{\cos\phi\cos\theta}{r}$$

- Usando a definição de coordenadas esféricas temos

$$r = \sqrt{x^2 + y^2 + z^2}, \quad \cos\theta = \frac{z}{r}, \quad \mathrm{tg}\phi = \frac{y}{x}$$

- Com isso, as componentes do Jacobiano são

$$\frac{\partial r}{\partial x} = \sin\theta\cos\phi, \quad \frac{\partial r}{\partial y} = \sin\theta\sin\phi, \quad \frac{\partial r}{\partial z} = \cos\theta;$$

$$\frac{\partial\theta}{\partial x} = \frac{\cos\phi\cos\theta}{r}, \quad \frac{\partial\cos\theta}{\partial y} = \frac{\partial}{\partial y}\left(\frac{z}{r}\right)$$

- Usando a definição de coordenadas esféricas temos

$$r = \sqrt{x^2 + y^2 + z^2}, \quad \cos\theta = \frac{z}{r}, \quad \mathrm{tg}\phi = \frac{y}{x}$$

- Com isso, as componentes do Jacobiano são

$$\frac{\partial r}{\partial x} = \sin\theta\cos\phi, \quad \frac{\partial r}{\partial y} = \sin\theta\sin\phi, \quad \frac{\partial r}{\partial z} = \cos\theta;$$

$$\frac{\partial \theta}{\partial x} = \frac{\cos\phi\cos\theta}{r}, \quad \frac{\partial \theta}{\partial y} = \frac{\sin\phi\cos\theta}{r}$$

- Usando a definição de coordenadas esféricas temos

$$r = \sqrt{x^2 + y^2 + z^2}, \quad \cos\theta = \frac{z}{r}, \quad \mathrm{tg}\phi = \frac{y}{x}$$

- Com isso, as componentes do Jacobiano são

$$\frac{\partial r}{\partial x} = \sin\theta\cos\phi, \quad \frac{\partial r}{\partial y} = \sin\theta\sin\phi, \quad \frac{\partial r}{\partial z} = \cos\theta;$$

$$\frac{\partial \theta}{\partial x} = \frac{\cos\phi\cos\theta}{r}, \quad \frac{\partial \theta}{\partial y} = \frac{\sin\phi\cos\theta}{r}, \quad \frac{\partial \cos\theta}{\partial z} = \frac{\partial}{\partial z}\left(\frac{z}{r}\right)$$

- Usando a definição de coordenadas esféricas temos

$$r = \sqrt{x^2 + y^2 + z^2}, \quad \cos\theta = \frac{z}{r}, \quad \mathrm{tg}\phi = \frac{y}{x}$$

- Com isso, as componentes do Jacobiano são

$$\frac{\partial r}{\partial x} = \sin\theta\cos\phi, \quad \frac{\partial r}{\partial y} = \sin\theta\sin\phi, \quad \frac{\partial r}{\partial z} = \cos\theta;$$

$$\frac{\partial \theta}{\partial x} = \frac{\cos\phi\cos\theta}{r}, \quad \frac{\partial \theta}{\partial y} = \frac{\sin\phi\cos\theta}{r}, \quad \frac{\partial \theta}{\partial z} = -\frac{\sin\theta}{r}$$

- Usando a definição de coordenadas esféricas temos

$$r = \sqrt{x^2 + y^2 + z^2}, \quad \cos\theta = \frac{z}{r}, \quad \mathrm{tg}\phi = \frac{y}{x}$$

- Com isso, as componentes do Jacobiano são

$$\frac{\partial r}{\partial x} = \sin\theta\cos\phi, \quad \frac{\partial r}{\partial y} = \sin\theta\sin\phi, \quad \frac{\partial r}{\partial z} = \cos\theta;$$

$$\frac{\partial \theta}{\partial x} = \frac{\cos\phi\cos\theta}{r}, \quad \frac{\partial \theta}{\partial y} = \frac{\sin\phi\cos\theta}{r}, \quad \frac{\partial \theta}{\partial z} = -\frac{\sin\theta}{r};$$

$$\frac{\partial \mathrm{tg}\phi}{\partial x} = \frac{\partial}{\partial x}\left(\frac{y}{x}\right)$$

- Usando a definição de coordenadas esféricas temos

$$r = \sqrt{x^2 + y^2 + z^2}, \quad \cos\theta = \frac{z}{r}, \quad \mathrm{tg}\phi = \frac{y}{x}$$

- Com isso, as componentes do Jacobiano são

$$\frac{\partial r}{\partial x} = \sin\theta\cos\phi, \quad \frac{\partial r}{\partial y} = \sin\theta\sin\phi, \quad \frac{\partial r}{\partial z} = \cos\theta;$$

$$\frac{\partial \theta}{\partial x} = \frac{\cos\phi\cos\theta}{r}, \quad \frac{\partial \theta}{\partial y} = \frac{\sin\phi\cos\theta}{r}, \quad \frac{\partial \theta}{\partial z} = -\frac{\sin\theta}{r};$$

$$\sec^2\phi\frac{\partial \phi}{\partial x} = \frac{\partial}{\partial x}\left(\frac{y}{x}\right)$$

- Usando a definição de coordenadas esféricas temos

$$r = \sqrt{x^2 + y^2 + z^2}, \quad \cos\theta = \frac{z}{r}, \quad \mathrm{tg}\phi = \frac{y}{x}$$

- Com isso, as componentes do Jacobiano são

$$\frac{\partial r}{\partial x} = \sin\theta\cos\phi, \quad \frac{\partial r}{\partial y} = \sin\theta\sin\phi, \quad \frac{\partial r}{\partial z} = \cos\theta;$$

$$\frac{\partial \theta}{\partial x} = \frac{\cos\phi\cos\theta}{r}, \quad \frac{\partial \theta}{\partial y} = \frac{\sin\phi\cos\theta}{r}, \quad \frac{\partial \theta}{\partial z} = -\frac{\sin\theta}{r};$$

$$\sec^2\phi\frac{\partial \phi}{\partial x} = -\frac{y}{x^2}$$

- Usando a definição de coordenadas esféricas temos

$$r = \sqrt{x^2 + y^2 + z^2}, \quad \cos\theta = \frac{z}{r}, \quad \mathrm{tg}\phi = \frac{y}{x}$$

- Com isso, as componentes do Jacobiano são

$$\frac{\partial r}{\partial x} = \sin\theta\cos\phi, \quad \frac{\partial r}{\partial y} = \sin\theta\sin\phi, \quad \frac{\partial r}{\partial z} = \cos\theta;$$

$$\frac{\partial \theta}{\partial x} = \frac{\cos\phi\cos\theta}{r}, \quad \frac{\partial \theta}{\partial y} = \frac{\sin\phi\cos\theta}{r}, \quad \frac{\partial \theta}{\partial z} = -\frac{\sin\theta}{r};$$

$$\frac{\partial \phi}{\partial x} = -\frac{\sin\phi}{r\sin\theta}$$

- Usando a definição de coordenadas esféricas temos

$$r = \sqrt{x^2 + y^2 + z^2}, \quad \cos\theta = \frac{z}{r}, \quad \mathrm{tg}\phi = \frac{y}{x}$$

- Com isso, as componentes do Jacobiano são

$$\frac{\partial r}{\partial x} = \sin\theta\cos\phi, \quad \frac{\partial r}{\partial y} = \sin\theta\sin\phi, \quad \frac{\partial r}{\partial z} = \cos\theta;$$

$$\frac{\partial \theta}{\partial x} = \frac{\cos\phi\cos\theta}{r}, \quad \frac{\partial \theta}{\partial y} = \frac{\sin\phi\cos\theta}{r}, \quad \frac{\partial \theta}{\partial z} = -\frac{\sin\theta}{r};$$

$$\frac{\partial \phi}{\partial x} = -\frac{\sin\phi}{r\sin\theta}, \quad \frac{\partial \mathrm{tg}\phi}{\partial y} = \frac{\partial}{\partial y}\left(\frac{y}{x}\right)$$

- Usando a definição de coordenadas esféricas temos

$$r = \sqrt{x^2 + y^2 + z^2}, \quad \cos\theta = \frac{z}{r}, \quad \mathrm{tg}\phi = \frac{y}{x}$$

- Com isso, as componentes do Jacobiano são

$$\frac{\partial r}{\partial x} = \sin\theta\cos\phi, \quad \frac{\partial r}{\partial y} = \sin\theta\sin\phi, \quad \frac{\partial r}{\partial z} = \cos\theta;$$

$$\frac{\partial \theta}{\partial x} = \frac{\cos\phi\cos\theta}{r}, \quad \frac{\partial \theta}{\partial y} = \frac{\sin\phi\cos\theta}{r}, \quad \frac{\partial \theta}{\partial z} = -\frac{\sin\theta}{r};$$

$$\frac{\partial \phi}{\partial x} = -\frac{\sin\phi}{r\sin\theta}, \quad \frac{\partial \phi}{\partial y} = \frac{\cos\phi}{r\sin\theta}$$

- Usando a definição de coordenadas esféricas temos

$$r = \sqrt{x^2 + y^2 + z^2}, \quad \cos\theta = \frac{z}{r}, \quad \mathrm{tg}\phi = \frac{y}{x}$$

- Com isso, as componentes do Jacobiano são

$$\frac{\partial r}{\partial x} = \sin\theta\cos\phi, \quad \frac{\partial r}{\partial y} = \sin\theta\sin\phi, \quad \frac{\partial r}{\partial z} = \cos\theta;$$

$$\frac{\partial \theta}{\partial x} = \frac{\cos\phi\cos\theta}{r}, \quad \frac{\partial \theta}{\partial y} = \frac{\sin\phi\cos\theta}{r}, \quad \frac{\partial \theta}{\partial z} = -\frac{\sin\theta}{r};$$

$$\frac{\partial \phi}{\partial x} = -\frac{\sin\phi}{r\sin\theta}, \quad \frac{\partial \phi}{\partial y} = \frac{\cos\phi}{r\sin\theta}, \quad \frac{\partial \mathrm{tg}\phi}{\partial z} = \frac{\partial}{\partial z}\left(\frac{y}{x}\right)$$

- Usando a definição de coordenadas esféricas temos

$$r = \sqrt{x^2 + y^2 + z^2}, \quad \cos\theta = \frac{z}{r}, \quad \mathrm{tg}\phi = \frac{y}{x}$$

- Com isso, as componentes do Jacobiano são

$$\frac{\partial r}{\partial x} = \sin\theta\cos\phi, \quad \frac{\partial r}{\partial y} = \sin\theta\sin\phi, \quad \frac{\partial r}{\partial z} = \cos\theta;$$

$$\frac{\partial \theta}{\partial x} = \frac{\cos\phi\cos\theta}{r}, \quad \frac{\partial \theta}{\partial y} = \frac{\sin\phi\cos\theta}{r}, \quad \frac{\partial \theta}{\partial z} = -\frac{\sin\theta}{r};$$

$$\frac{\partial \phi}{\partial x} = -\frac{\sin\phi}{r\sin\theta}, \quad \frac{\partial \phi}{\partial y} = \frac{\cos\phi}{r\sin\theta}, \quad \frac{\partial \phi}{\partial z} = 0.$$

- Portanto

$$\langle \mathbf{r}' | L_x | \alpha \rangle = -i\hbar \left(y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y} \right) \langle \mathbf{r}' | \alpha \rangle$$

- Portanto

$$\langle \mathbf{r}' | L_x | \alpha \rangle = -i\hbar \left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} \right) \langle \mathbf{r}' | \alpha \rangle$$

$$= -i\hbar \left[r\sin\theta\sin\phi \left(\frac{\partial r}{\partial z}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial z}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial z}\frac{\partial}{\partial \phi} \right) - r\cos\theta \left(\frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial y}\frac{\partial}{\partial \phi} \right) \right] \langle \mathbf{r}' | \alpha \rangle$$

- Portanto

$$\langle \mathbf{r}' | L_x | \alpha \rangle = -i\hbar \left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} \right) \langle \mathbf{r}' | \alpha \rangle$$

$$= -i\hbar \left[r\sin\theta\sin\phi \left(\frac{\partial r}{\partial z}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial z}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial z}\frac{\partial}{\partial \phi} \right) - r\cos\theta \left(\frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial y}\frac{\partial}{\partial \phi} \right) \right] \langle \mathbf{r}' | \alpha \rangle$$

$$\Leftrightarrow \langle \mathbf{r}' | L_x | \alpha \rangle = i\hbar \left(\sin\phi\frac{\partial}{\partial \theta} + \cos\phi \mathrm{cotg}\theta\frac{\partial}{\partial \phi} \right) \langle \mathbf{r}' | \alpha \rangle$$

- Portanto

$$\langle \mathbf{r}' | L_x | \alpha \rangle = -i\hbar \left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} \right) \langle \mathbf{r}' | \alpha \rangle$$

$$= -i\hbar \left[r\sin\theta\sin\phi \left(\frac{\partial r}{\partial z}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial z}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial z}\frac{\partial}{\partial \phi} \right) - r\cos\theta \left(\frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial y}\frac{\partial}{\partial \phi} \right) \right] \langle \mathbf{r}' | \alpha \rangle$$

$$\Leftrightarrow \langle \mathbf{r}' | L_x | \alpha \rangle = i\hbar \left(\sin\phi \frac{\partial}{\partial \theta} + \cos\phi \, \mathrm{cotg}\theta \frac{\partial}{\partial \phi} \right) \langle \mathbf{r}' | \alpha \rangle$$

- De forma análoga para o y:

$$\langle \mathbf{r}' | L_y | \alpha \rangle = i\hbar \left(-\cos\phi \frac{\partial}{\partial \theta} + \sin\phi \, \mathrm{cotg}\theta \frac{\partial}{\partial \phi} \right) \langle \mathbf{r}' | \alpha \rangle$$