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Lembrando que:

$$\langle x', t'|x, t\rangle = \prod_{n=1}^{N-1} \int dx_n \langle x_n, t_n|x_{n-1}, t_{n-1}\rangle$$

Em que $x_N = x', x_0 = x$. Queremos avaliar $\langle x_n, t_n|x_{n-1}, t_{n-1}\rangle$:

$$\langle x_n, t_n|x_{n-1}, t_{n-1}\rangle = \langle x_n| e^{-\frac{i}{\hbar}\hat{H}t_n} e^{\frac{i}{\hbar}\hat{H}t_{n-1}} |x_{n-1}\rangle = \langle x_n| e^{-\frac{i}{\hbar}\hat{H}\Delta t} |x_{n-1}\rangle \qquad \Delta t = t_n - t_{n-1}$$

para $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{k\hat{x}^2}{2}$. Usando a identidade:

$$1 = \int \frac{dp_{n-1}}{2\pi\hbar} |p_{n-1}\rangle \langle p_{n-1}| \implies$$

$$\langle x_n| \left(\int \frac{dp_{n-1}}{2\pi\hbar} |p_{n-1}\rangle \langle p_{n-1}|\right) e^{-\frac{i}{\hbar}\hat{H}\Delta t} |x_{n-1}\rangle = \int \frac{dp_{n-1}}{2\pi\hbar} \langle x_n|p_{n-1}\rangle \langle p_{n-1}| e^{-\frac{i}{\hbar}\hat{H}\Delta t} |x_{n-1}\rangle$$

Usando:

$$\langle x_n|p_{n-1}\rangle = e^{\frac{i}{\hbar}x_n p_{n-1}}$$

e

$$\langle p_{n-1}| e^{-\frac{i}{\hbar}\hat{H}\Delta t} |x_{n-1}\rangle \approx \langle p_{n-1}| \left(1 - \frac{i}{\hbar}\hat{H}(\hat{p}, \hat{x})\Delta t\right) |x_{n-1}\rangle = \langle p_{n-1}|x_{n-1}\rangle \left(1 - \frac{i}{\hbar} H\left(p_{n-1}, \frac{x_n + x_{n-1}}{2}\right)\Delta t\right)$$
$$\approx \langle p_{n-1}|x_{n-1}\rangle e^{-\frac{i}{\hbar}H\left(p_{n-1}, \frac{x_n+x_{n-1}}{2}\right)\Delta t}$$

Usaremos:

$$H\left(p_{n-1}, \frac{x_n + x_{n-1}}{2}\right) = \frac{p_{n-1}^2}{2m} + \frac{k}{2}\left(\frac{x_n^2 + x_{n-1}^2}{2}\right) = \frac{p_{n-1}^2}{2m} + \frac{k\overline{x}_{n-1}^2}{2}$$

Computando a integral:

$$\int \frac{dp_{n-1}}{2\pi\hbar} \langle x_n|p_{n-1}\rangle \langle p_{n-1}|x_{n-1}\rangle e^{-\frac{i}{\hbar}H\left(p_{n-1}, \frac{x_n+x_{n-1}}{2}\right)\Delta t} =$$
$$\int \frac{dp_{n-1}}{2\pi\hbar} \exp\left[\frac{i}{\hbar}(x_n - x_{n-1})p_{n-1} - \frac{i}{\hbar}\Delta t\left(\frac{p_{n-1}^2}{2m} + \frac{k\overline{x}_{n-1}^2}{2}\right)\right]$$

Simplificando o argumento da exponencial:

$$\frac{i}{\hbar}(x_n - x_{n-1})p_{n-1} - \frac{i}{\hbar}\Delta t\left(\frac{p_{n-1}^2}{2m} + \frac{k\overline{x}_{n-1}^2}{2}\right) = -\frac{i\Delta t}{2\hbar m}\left(p_{n-1}^2 - 2m\left(\frac{x_n - x_{n-1}}{\Delta t}\right)p_{n-1}\right) - i\frac{k\Delta t}{2\hbar}\overline{x}_{n-1}^2$$
$$= -\frac{i\Delta t}{2\hbar m}\left(p_{n-1} - m\left(\frac{x_n - x_{n-1}}{\Delta t}\right)\right)^2 - i\frac{k\Delta t}{2\hbar}\overline{x}_{n-1}^2 + \frac{im}{2\hbar\Delta t}(x_n - x_{n-1})^2$$

Transladando a variável de integração $p_{n-1} \mapsto p_{n-1} + m\left(\frac{x_n - x_{n-1}}{\Delta t}\right)$ resulta em:

$$\langle x_n, t_n | x_{n-1}, t_{n-1}\rangle = \exp\left[-i\frac{k\Delta t}{2\hbar}\overline{x}_{n-1}^2 + \frac{im}{2\hbar\Delta t}(x_n - x_{n-1})^2\right]\int \frac{dp_{n-1}}{2\pi\hbar}\exp\left[-\frac{i\Delta t}{2\hbar m}p_{n-1}^2\right]$$

$$= \sqrt{\frac{2\hbar m\pi}{i\Delta t}}\frac{1}{2\pi\hbar}\exp\left[-i\frac{k\Delta t}{2\hbar}\overline{x}_{n-1}^2 + \frac{im}{2\hbar\Delta t}(x_n - x_{n-1})^2\right] = \sqrt{\frac{m}{2\pi\hbar i\Delta t}}\exp\left[\frac{-i}{2\hbar}\left(k\Delta t\overline{x}_{n-1}^2 - \frac{m}{\Delta t}(x_n - x_{n-1})^2\right)\right]$$

Rearranjando o argumento da exponencial, e substituindo $k = m\omega^2$:

$$\frac{-i}{2\hbar}\left(k\Delta t\overline{x}_{n-1}^2 - \frac{m}{\Delta t}(x_n - x_{n-1})^2\right) = \frac{-im}{2\hbar\Delta t}\left(\omega^2\Delta t^2\left(\frac{x_n^2 + x_{n-1}^2}{2}\right) - x_n^2 - x_{n-1}^2 + 2x_n x_{n-1}\right)$$

$$= \frac{im\omega}{2\hbar\omega\Delta t}\left((x_n^2 + x_{n-1}^2)\left(1 - \frac{\omega^2\Delta t^2}{2}\right) - 2x_n x_{n-1}\right) \approx \frac{im\omega}{2\hbar\sin(\omega\Delta t)}\left((x_n^2 + x_{n-1}^2)\cos(\omega\Delta t) - 2x_n x_{n-1}\right)$$

Da mesma forma para a constante:

$$\sqrt{\frac{m}{2\pi i\hbar\Delta t}} = \sqrt{\frac{m\omega}{2\pi i\hbar\omega\Delta t}} \approx \sqrt{\frac{m\omega}{2\pi i\hbar\sin(\omega\Delta t)}}$$

Portanto:

$$\langle x_n, t_n | x_{n-1}, t_{n-1}\rangle \approx \sqrt{\frac{m\omega}{2\pi i\hbar\sin(\omega\Delta t)}}\exp\left[\frac{im\omega}{2\hbar\sin(\omega\Delta t)}\left[(x_n^2 + x_{n-1}^2)\cos(\omega\Delta t) - 2x_n x_{n-1}\right]\right]$$

quando $\Delta t = t_n - t_{n-1} \mapsto 0$, que é a expressão do slide 13 quando $x_n = x''$ e $x_{n-1} = x'$.

Três maneiras diferentes de calcular o propagador para o oscilador harmônico: https://www.if.ufrj.br/~boschi/ensino/Pos/Barone_Boschi_Farina_AJP_2002.pdf