3.26 **(a)** Consider a system with $j = 1$. Explicitly write

$$\langle j = 1, m' | J_y | j = 1, m \rangle$$

in 3×3 matrix form.

(b) Show that for $j = 1$ only, it is legitimate to replace $e^{-iJ_y\beta/\hbar}$ by

$$1 - i\left(\frac{J_y}{\hbar}\right)\sin\beta - \left(\frac{J_y}{\hbar}\right)^2(1 - \cos\beta).$$

(c) Using (b), prove

$$d^{(j=1)}(\beta) = \begin{pmatrix} \left(\frac{1}{2}\right)(1+\cos\beta) & -\left(\frac{1}{\sqrt{2}}\right)\sin\beta & \left(\frac{1}{2}\right)(1-\cos\beta) \\ \left(\frac{1}{\sqrt{2}}\right)\sin\beta & \cos\beta & -\left(\frac{1}{\sqrt{2}}\right)\sin\beta \\ \left(\frac{1}{2}\right)(1-\cos\beta) & \left(\frac{1}{\sqrt{2}}\right)\sin\beta & \left(\frac{1}{2}\right)(1+\cos\beta) \end{pmatrix}.$$

a)Primeiramente temos que:

$$J_y = \frac{J_+ - J_-}{2i},$$

logo:

$$\langle j=1, m'|J_y|j=1, m\rangle = \frac{\langle 1m'|J_+|1m\rangle}{2i} - \frac{\langle 1m'|J_-|1m\rangle}{2i}.$$

Tem-se também que:

$$\langle j', m'|J_\pm|j, m\rangle = \sqrt{(j \mp m)(j \pm m + 1)}\hbar\delta_{j',j}\delta_{m',m\pm1},$$

logo:

$$\langle 1m'|J_y|1m\rangle = \frac{\hbar}{2i}\left[\sqrt{(1-m)(2+m)}\delta_{m',m+1} - \sqrt{(1+m)(2-m)}\delta_{m',m-1}\right],$$

que em forma matricial fica:

$$\langle 1m'|J_y|1m\rangle = \frac{\hbar}{2}\begin{pmatrix} 0 & -\sqrt{2}i & 0 \\ \sqrt{2}i & 0 & -\sqrt{2}i \\ 0 & \sqrt{2}i & 0 \end{pmatrix}.$$

b) Primeiramente tem-se que, para $j = 1$:

$$J_y^3 = J_y \frac{\hbar}{2}\begin{pmatrix} 0 & -\sqrt{2}i & 0 \\ \sqrt{2}i & 0 & -\sqrt{2}i \\ 0 & \sqrt{2}i & 0 \end{pmatrix}\frac{\hbar}{2}\begin{pmatrix} 0 & -\sqrt{2}i & 0 \\ \sqrt{2}i & 0 & -\sqrt{2}i \\ 0 & \sqrt{2}i & 0 \end{pmatrix}$$

$$J_y^3 = \frac{\hbar}{2}\begin{pmatrix} 0 & -\sqrt{2}i & 0 \\ \sqrt{2}i & 0 & -\sqrt{2}i \\ 0 & \sqrt{2}i & 0 \end{pmatrix}\frac{\hbar^2}{4}\begin{pmatrix} 2 & 0 & -2 \\ 0 & 4 & 0 \\ -2 & 0 & 2 \end{pmatrix}$$

$$J_y^3 = \frac{\hbar^3}{8}\begin{pmatrix} 0 & -\sqrt{2}i & 0 \\ \sqrt{2}i & 0 & -\sqrt{2}i \\ 0 & \sqrt{2}i & 0 \end{pmatrix}$$

$$J_y^3 = \hbar^3 \frac{J_y}{\hbar} \Rightarrow \left(\frac{J_y}{\hbar}\right)^3 = \frac{J_y}{\hbar},$$

então o operador de rotação é:

$$e^{\frac{-iJ_y\beta}{\hbar}} = 1 - i\frac{J_y}{\hbar}\beta - \left(\frac{J_y}{\hbar}\right)^2 \frac{\beta^2}{2!} + i\left(\frac{J_y}{\hbar}\right)^3 \frac{\beta^3}{3!} + \left(\frac{J_y}{\hbar}\right)^4 \frac{\beta^4}{4!} - i\left(\frac{J_y}{\hbar}\right)^5 \frac{\beta^5}{5!} + \cdots$$

$$e^{\frac{-iJ_y\beta}{\hbar}} = 1 - i\frac{J_y}{\hbar}\beta - \left(\frac{J_y}{\hbar}\right)^2 \frac{\beta^2}{2!} + i\frac{J_y}{\hbar}\frac{\beta^3}{3!} + \left(\frac{J_y}{\hbar}\right)^2 \frac{\beta^4}{4!} - i\frac{J_y}{\hbar}\frac{\beta^5}{5!} + \cdots$$

$$e^{\frac{-iJ_y\beta}{\hbar}} = 1 - i\frac{J_y}{\hbar}\left(\beta - \frac{\beta^3}{3!} + \frac{\beta^5}{5!} + \cdots\right) - \left(\frac{J_y}{\hbar}\right)^2\left(1 - 1 + \frac{\beta^2}{2!} - \frac{\beta^4}{4!} + \cdots\right)$$

$$e^{\frac{-iJ_y\beta}{\hbar}} = 1 - i\frac{J_y}{\hbar}\sin\beta - \left(\frac{J_y}{\hbar}\right)^2(1 - \cos\beta).$$

c) Primeiramente tem-se que:

$$J_y^2 = \hbar^2\begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix},$$

então, como $d^{(1)}_{m1,m}(\beta) = \left\langle 1m' \middle| e^{\frac{-iJ_y\beta}{\hbar}} \middle| 1m \right\rangle$, tem-se:

$$d^{(1)}_{m1,m}(\beta) = 1 - i\begin{pmatrix} 0 & -\frac{i}{\sqrt{2}} & 0 \\ \frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ 0 & \frac{i}{\sqrt{2}} & 0 \end{pmatrix}\sin\beta - \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}(1-\cos\beta)$$

$$d^{(1)}_{m1,m}(\beta) = \begin{pmatrix} 1 & -\frac{\sin\beta}{\sqrt{2}} & 0 \\ \frac{\sin\beta}{\sqrt{2}} & 1 & -\frac{\sin\beta}{\sqrt{2}} \\ 0 & \frac{\sin\beta}{\sqrt{2}} & 1 \end{pmatrix} + \begin{pmatrix} \frac{\cos\beta-1}{2} & 0 & \frac{-\cos\beta+1}{2} \\ 0 & \cos\beta-1 & 0 \\ \frac{-\cos\beta+1}{2} & 0 & \frac{\cos\beta-1}{2} \end{pmatrix}$$

$$d^{(1)}_{m1,m}(\beta) = \begin{pmatrix} \frac{1+\cos\beta}{2} & -\frac{\sin\beta}{\sqrt{2}} & \frac{1-\cos\beta}{2} \\ \frac{\sin\beta}{\sqrt{2}} & \cos\beta & -\frac{\sin\beta}{\sqrt{2}} \\ \frac{1-\cos\beta}{2} & \frac{\sin\beta}{\sqrt{2}} & \frac{1+\cos\beta}{2} \end{pmatrix}$$