Problems

5.1 A simple harmonic oscillator (in one dimension) is subjected to a perturbation

$$\lambda H_1 = bx,$$

where b is a real constant.

(a) Calculate the energy shift of the ground state to *lowest nonvanishing order*.

(b) Solve this problem *exactly* and compare with your result obtained in (a). You may assume without proof that

$$\langle u_{n'}|x|u_n\rangle = \sqrt{\frac{\hbar}{2m\omega}}(\sqrt{n+1}\delta_{n',n+1} + \sqrt{n}\delta_{n',n-1}).$$

5.2 In nondegenerate time-independent perturbation theory, what is the probability of finding in a perturbed energy eigenstate ($|k\rangle$) the corresponding unperturbed eigenstate ($|k^{(0)}\rangle$)? Solve this up to terms of order ~~g^2~~. λ^2

5.3 Consider a particle in a two-dimensional potential

$$V_0 = \begin{cases} 0, & \text{for } 0 \le x \le L, 0 \le y \le L \\ \infty, & \text{otherwise.} \end{cases}$$

Write the energy eigenfunctions for the ground state and the first excited state. We now add a time-independent perturbation of the form

$$V_1 = \begin{cases} \lambda xy, & \text{for } 0 \le x \le L, 0 \le y \le L \\ 0, & \text{otherwise.} \end{cases}$$

Obtain the zeroth-order energy eigenfunctions and the first-order energy shifts for the ground state and the first excited state.

5.4 Consider an isotropic harmonic oscillator in *two* dimensions. The Hamiltonian is given by

$$H_0 = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{m\omega^2}{2}(x^2 + y^2).$$

(a) What are the energies of the three lowest-lying states? Is there any degeneracy?

(b) We now apply a perturbation

$$V = \delta m\omega^2 xy,$$

where δ is a dimensionless real number much smaller than unity. Find the zeroth-order energy eigenket and the corresponding energy to first order [that is, the unperturbed energy obtained in (a) plus the first-order energy shift] for each of the three lowest-lying states.

(c) Solve the $H_0 + V$ problem *exactly*. Compare with the perturbation results obtained in (b). [You may use $\langle n'|x|n\rangle = \sqrt{\hbar/2m\omega}(\sqrt{n+1}\delta_{n',n+1} + \sqrt{n}\delta_{n',n-1})$.]

5.5 Establish (5.1.54) for the one-dimensional harmonic oscillator given by (5.1.50) with an additional perturbation $V = \frac{1}{2}\varepsilon m\omega^2 x^2$. Show that all other matrix elements V_{k0} vanish.

5.6 (From Merzbacher 1970.) A slightly anisotropic three-dimensional harmonic oscillator has $\omega_z \approx \omega_x = \omega_y$. A charged particle moves in the field of this oscillator and is at the same time exposed to a uniform magnetic field in the x-direction. Assuming that the Zeeman splitting is comparable to the splitting produced by the anisotropy, but small compared to $\hbar\omega$, calculate to first order the energies of the components of the first excited state. Discuss various limiting cases.

5.7 A one-electron atom whose ground state is nondegenerate is placed in a uniform electric field in the z-direction. Obtain an approximate expression for the induced electric dipole moment of the ground state by considering the expectation value of ez with respect to the perturbed-state vector computed to first order. Show that the same expression can also be obtained from the energy shift $\Delta = -\alpha|\mathbf{E}|^2/2$ of the ground state computed to second order. (*Note:* α stands for the polarizability.) Ignore spin.

5.8 Evaluate the matrix elements (or expectation values) given below. If any vanishes, explain why it vanishes using simple symmetry (or other) arguments.

(a) $\langle n=2, l=1, m=0|x|n=2, l=0, m=0\rangle$.

(b) $\langle n=2, l=1, m=0|p_z|n=2, l=0, m=0\rangle$.

[In (a) and (b), $|nlm\rangle$ stands for the energy eigenket of a nonrelativistic hydrogen atom with spin ignored.]

(c) $\langle L_z\rangle$ for an electron in a central field with $j=\frac{9}{2}, m=\frac{7}{2}, l=4$.

(d) $\langle \text{singlet}, m_s=0|S_z^{(e-)} - S_z^{(e+)}|\text{triplet}, m_s=0\rangle$ for an s-state positronium.

(e) $\langle \mathbf{S}^{(1)} \cdot \mathbf{S}^{(2)}\rangle$ for the ground state of a hydrogen *molecule*.

5.9 A p-orbital electron characterized by $|n, l=1, m=\pm 1, 0\rangle$ (ignore spin) is subjected to a potential

$$V = \lambda(x^2 - y^2) \quad (\lambda = \text{constant}).$$

(a) Obtain the "correct" zeroth-order energy eigenstates that diagonalize the perturbation. You need not evaluate the energy shifts in detail, but show that the original threefold degeneracy is now completely removed.

(b) Because V is invariant under time reversal and because there is no longer any degeneracy, we expect each of the energy eigenstates obtained in (a) to go into itself (up to a phase factor or sign) under time reversal. Check this point explicitly.

5.10 Consider a spinless particle in a two-dimensional infinite square well:

$$V = \begin{cases} 0, & \text{for } 0 \le x \le a, 0 \le y \le a \\ \infty, & \text{otherwise.} \end{cases}$$

(a) What are the energy eigenvalues for the three lowest states? Is there any degeneracy?

(b) We now add a potential

$$V_1 = \lambda xy, \quad 0 \le x \le a, 0 \le y \le a.$$

Taking this as a weak perturbation, answer the following:

(i) Is the energy shift due to the perturbation linear or quadratic in λ for each of the three states?

(ii) Obtain expressions for the energy shifts of the three lowest states accurate to order λ. (You need not evaluate integrals that may appear.)

(iii) Draw an energy diagram with and without the perturbation for the three energy states. Make sure to specify which unperturbed state is connected to which perturbed state.

5.11 The Hamiltonian matrix for a two-state system can be written as

$$\mathcal{H} = \begin{pmatrix} E_1^0 & \lambda\Delta \\ \lambda\Delta & E_2^0 \end{pmatrix}.$$

Clearly, the energy eigenfunctions for the unperturbed problems ($\lambda = 0$) are given by

$$\phi_1^{(0)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \phi_2^{(0)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

(a) Solve this problem *exactly* to find the energy eigenfunctions ψ_1 and ψ_2 and the energy eigenvalues E_1 and E_2.

(b) Assuming that $\lambda|\Delta| \ll |E_1^0 - E_2^0|$, solve the same problem using time-independent perturbation theory up to first order in the energy eigenfunctions and up to second order in the energy eigenvalues. Compare with the exact results obtained in (a).

(c) Suppose the two unperturbed energies are "almost degenerate"; that is,

$$|E_1^0 - E_2^0| \ll \lambda|\Delta|.$$

Show that the exact results obtained in (a) closely resemble what you would expect by applying *degenerate* perturbation theory to this problem with E_1^0 set exactly equal to E_2^0.

5.12 (This is a tricky problem because the degeneracy between the first state and the second state is not removed in first order. See also Gottfried 1966, p. 397, Problem 1.) This problem is from Schiff 1968, p. 295, Problem 4. A system that has three unperturbed states can be represented by the perturbed Hamiltonian matrix

$$\begin{pmatrix} E_1 & 0 & a \\ 0 & E_1 & b \\ a^* & b^* & E_2 \end{pmatrix},$$

where $E_2 > E_1$. The quantities a and b are to be regarded as perturbations that are of the same order and are small compared with $E_2 - E_1$. Use the second-order nondegenerate perturbation theory to calculate the perturbed eigenvalues. (Is this procedure correct?) Then diagonalize the matrix to find the exact eigenvalues. Finally, use the second-order degenerate perturbation theory. Compare the three results obtained.

5.13 Compute the Stark effect for the $2S_{1/2}$ and $2P_{1/2}$ levels of hydrogen for a field ε sufficiently weak that $e\varepsilon a_0$ is small compared to the fine structure, but take the Lamb shift δ ($\delta = 1{,}057$ MHz) into account (that is, ignore $2P_{3/2}$ in this calculation).

Show that for $e\varepsilon a_0 \ll \delta$, the energy shifts are quadratic in ε, whereas for $e\varepsilon a_0 \gg \delta$, they are linear in ε. (The radial integral you need is $\langle 2s|r|2p\rangle = 3\sqrt{3}a_0$.) Briefly discuss the consequences (if any) of time reversal for this problem. This problem is from Gottfried 1966, Problem 7-3.

5.14 Work out the Stark effect to lowest nonvanishing order for the $n = 3$ level of the hydrogen atom. Ignoring the spin-orbit force and relativistic correction (Lamb shift), obtain not only the energy shifts to lowest nonvanishing order but also the corresponding zeroth-order eigenket.

5.15 Suppose the electron had a very small intrinsic *electric* dipole moment analogous to the spin-magnetic moment (that is, $\boldsymbol{\mu}_{el}$ proportional to $\boldsymbol{\sigma}$). Treating the hypothetical $-\boldsymbol{\mu}_{el}\cdot\mathbf{E}$ interaction as a small perturbation, discuss qualitatively how the energy levels of the Na atom ($Z = 11$) would be altered in the absence of any external electromagnetic field. Are the level shifts first order or second order? Indicate explicitly which states get mixed with each other. Obtain an expression for the energy shift of the lowest level that is affected by the perturbation. Assume throughout that only the valence electron is subjected to the hypothetical interaction.

5.16 Consider a particle bound to a fixed center by a spherically symmetrical potential $V(r)$.

(a) Prove

$$|\psi(0)|^2 = \left(\frac{m}{2\pi\hbar^2}\right)\left\langle\frac{dV}{dr}\right\rangle$$

for all s-states, ground and excited.

(b) Check this relation for the ground state of a three-dimensional isotropic oscillator, the hydrogen atom, and so on. (*Note:* This relation has actually been found to be useful in guessing the form of the potential between a quark and an antiquark.)

5.17 **(a)** Suppose the Hamiltonian of a rigid rotator in a magnetic field perpendicular to the axis is of the form (Merzbacher 1970, Problem 17-1)

$$A\mathbf{L}^2 + BL_z + CL_y$$

if terms quadratic in the field are neglected. Assuming $B \gg C$, use perturbation theory to lowest nonvanishing order to get approximate energy eigenvalues.

(b) Consider the matrix elements

$$\langle n'l'm_l'm_s'|(3z^2 - r^2)|nlm_lm_s\rangle,$$

$$\langle n'l'm_l'm_s'|xy|nlm_lm_s\rangle$$

of a one-electron (for example, alkali) atom. Write the selection rules for Δl, Δm_l, and Δm_s. Justify your answer.

5.18 Work out the *quadratic* Zeeman effect for the ground-state hydrogen atom [$\langle\mathbf{x}|0\rangle = (1/\sqrt{\pi a_0^3})e^{-r/a_0}$] due to the usually neglected $e^2\mathbf{A}^2/2m_ec^2$-term in the Hamiltonian taken to first order. Write the energy shift as

$$\Delta = -\tfrac{1}{2}\chi\mathbf{B}^2$$

and obtain an expression for *diamagnetic susceptibility*, χ. The following definite integral may be useful:

$$\int_0^\infty e^{-\alpha r} r^n dr = \frac{n!}{\alpha^{n+1}}.$$

5.19 (Merzbacher 1970, p. 448, Problem 11.) For the He wave function, use

$$\psi(\mathbf{x}_1, \mathbf{x}_2) = (Z_{\text{eff}}^3/\pi a_0^3)\exp\left[\frac{-Z_{\text{eff}}(r_1+r_2)}{a_0}\right]$$

with $Z_{\text{eff}} = 2 - \frac{5}{16}$, as obtained by the variational method. The measured value of the diamagnetic susceptibility is $1.88 \times 10^{-6}\,\text{cm}^3/\text{mole}$.

Using the Hamiltonian for an atomic electron in a magnetic field, determine, for a state of zero angular momentum, the energy change to order B^2 if the system is in a uniform magnetic field represented by the vector potential $\mathbf{A} = \frac{1}{2}\mathbf{B} \times \mathbf{r}$.

Defining the atomic diamagnetic susceptibility χ by $E = -\frac{1}{2}\chi B^2$, calculate χ for a helium atom in the ground state and compare the result with the measured value.

5.20 Estimate the ground-state energy of a one-dimensional simple harmonic oscillator using

$$\langle x|\tilde{0}\rangle = e^{-\beta|x|}$$

as a trial function with β to be varied. You may use

$$\int_0^\infty e^{-\alpha x} x^n dx = \frac{n!}{\alpha^{n+1}}.$$

5.21 Estimate the lowest eigenvalue (λ) of the differential equation

$$\frac{d^2\psi}{dx^2} + (\lambda - |x|)\psi = 0, \quad \psi \to 0 \quad \text{for } |x| \to \infty$$

using the variational method with

$$\psi = \begin{cases} c(\alpha - |x|), & \text{for } |x| < \alpha \\ 0, & \text{for } |x| > \alpha \end{cases} \qquad (\alpha \text{ to be varied})$$

as a trial function. (*Caution:* $d\psi/dx$ is discontinuous at $x = 0$.) Numerical data that may be useful for this problem are

$$3^{1/3} = 1.442, \quad 5^{1/3} = 1.710, \quad 3^{2/3} = 2.080, \quad \pi^{2/3} = 2.145.$$

The *exact* value of the lowest eigenvalue can be shown to be 1.019.

5.22 Consider a one-dimensional simple harmonic oscillator whose classical angular frequency is ω_0. For $t < 0$ it is known to be in the ground state. For $t > 0$ there is also a time-dependent potential

$$V(t) = F_0 x \cos\omega t,$$

where F_0 is constant in both space and time. Obtain an expression for the expectation value $\langle x\rangle$ as a function of time using time-dependent perturbation theory to lowest nonvanishing order. Is this procedure valid for $\omega \simeq \omega_0$? [You may use $\langle n'|x|n\rangle = \sqrt{\hbar/2m\omega_0}(\sqrt{n+1}\delta_{n',n+1} + \sqrt{n}\delta_{n',n-1})$.]

5.23 A one-dimensional harmonic oscillator is in its ground state for $t < 0$. For $t \geq 0$ it is subjected to a time-dependent but spatially uniform *force* (not potential!) in the x-direction,

$$F(t) = F_0 e^{-t/\tau}.$$

(a) Using time-dependent perturbation theory to first order, obtain the probability of finding the oscillator in its first excited state for $t > 0$. Show that the $t \to \infty$ (τ finite) limit of your expression is independent of time. Is this reasonable or surprising?

(b) Can we find higher excited states? You may use

$$\langle n'|x|n\rangle = \sqrt{\hbar/2m\omega}(\sqrt{n}\delta_{n',n-1} + \sqrt{n+1}\delta_{n',n+1}).$$

5.24 Consider a particle bound in a simple harmonic-oscillator potential. Initially ($t < 0$), it is in the ground state. At $t = 0$ a perturbation of the form

$$H'(x,t) = Ax^2 e^{-t/\tau}$$

is switched on. Using time-dependent perturbation theory, calculate the probability that after a sufficiently long time ($t \gg \tau$), the system will have made a transition to a given excited state. Consider all final states.

5.25 The unperturbed Hamiltonian of a two-state system is represented by

$$H_0 = \begin{pmatrix} E_1^0 & 0 \\ 0 & E_2^0 \end{pmatrix}.$$

There is, in addition, a time-dependent perturbation

$$V(t) = \begin{pmatrix} 0 & \lambda\cos\omega t \\ \lambda\cos\omega t & 0 \end{pmatrix} \quad (\lambda \text{ real}).$$

(a) At $t = 0$ the system is known to be in the first state, represented by

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Using time-dependent perturbation theory and assuming that $E_1^0 - E_2^0$ is *not* close to $\pm\hbar\omega$, derive an expression for the probability that the system is found in the second state represented by

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

as a function of $t(t > 0)$.

(b) Why is this procedure not valid when $E_1^0 - E_2^0$ is close to $\pm\hbar\omega$?

5.26 A one-dimensional simple harmonic oscillator of angular frequency ω is acted upon by a spatially uniform but time-dependent force (*not* potential)

$$F(t) = \frac{(F_0\tau/\omega)}{(\tau^2 + t^2)}, \quad -\infty < t < \infty.$$

At $t = -\infty$, the oscillator is known to be in the ground state. Using the time-dependent perturbation theory to first order, calculate the probability that the oscillator is found in the first excited state at $t = +\infty$.

Challenge for experts: $F(t)$ is so normalized that the impulse

$$\int F(t)dt$$

imparted to the oscillator is always the same—that is, independent of τ; yet for $\tau \gg 1/\omega$, the probability for excitation is essentially negligible. Is this reasonable? [Matrix element of x: $\langle n'|x|n\rangle = (\hbar/2m\omega)^{1/2}(\sqrt{n}\delta_{n',n-1} + \sqrt{n+1}\delta_{n',n+1})$.]

5.27 Consider a particle in one dimension moving under the influence of some time-independent potential. The energy levels and the corresponding eigenfunctions for this problem are assumed to be known. We now subject the particle to a traveling pulse represented by a time-dependent potential,

$$V(t) = A\delta(x - ct).$$

(a) Suppose that at $t = -\infty$ the particle is known to be in the ground state whose energy eigenfunction is $\langle x|i\rangle = u_i(x)$. Obtain the probability for finding the system in some excited state with energy eigenfunction $\langle x|f\rangle = u_f(x)$ at $t = +\infty$.

(b) Interpret your result in (a) physically by regarding the δ-function pulse as a superposition of harmonic perturbations; recall

$$\delta(x - ct) = \frac{1}{2\pi c}\int_{-\infty}^{\infty} d\omega e^{i\omega[(x/c)-t]}.$$

Emphasize the role played by energy conservation, which holds even quantum-mechanically as long as the perturbation has been on for a very long time.

5.28 A hydrogen atom in its ground state $[(n,l,m) = (1,0,0)]$ is placed between the plates of a capacitor. A time-dependent but spatially uniform electric field (not potential!) is applied as follows:

$$\mathbf{E} = \begin{cases} 0, & \text{for } t < 0 \\ \mathbf{E}_0 e^{-t/\tau}, & \text{for } t > 0. \end{cases} \quad (\mathbf{E}_0 \text{ in the positive } z\text{-direction})$$

Using first-order time-dependent perturbation theory, compute the probability for the atom to be found at $t \gg \tau$ in each of the three $2p$ states: $(n,l,m) = (2,1,\pm 1 \text{ or } 0)$. Repeat the problem for the $2s$ state: $(n,l,m) = (2,0,0)$. You need not attempt to evaluate radial integrals, but perform all other integrations (with respect to angles and time).

5.29 Consider a composite system made up of two spin $\frac{1}{2}$ objects. For $t < 0$, the Hamiltonian does not depend on spin and can be taken to be zero by suitably adjusting the energy scale. For $t > 0$, the Hamiltonian is given by

$$H = \left(\frac{4\Delta}{\hbar^2}\right)\mathbf{S}_1 \cdot \mathbf{S}_2.$$

Suppose the system is in $|+-\rangle$ for $t \le 0$. Find, as a function of time, the probability for its being found in each of the following states $|++\rangle$, $|+-\rangle$, $|-+\rangle$, and $|--\rangle$

(a) By solving the problem exactly.

(b) By solving the problem assuming the validity of first-order time-dependent perturbation theory with H as a perturbation switched on at $t = 0$. Under what condition does (b) give the correct results?

5.30 Consider a two-level system with $E_1 < E_2$. There is a time-dependent potential that connects the two levels as follows:

$$V_{11} = V_{22} = 0, \quad V_{12} = \gamma e^{i\omega t}, \quad V_{21} = \gamma e^{-i\omega t} \quad (\gamma \text{ real}).$$

At $t = 0$, it is known that only the lower level is populated—that is, $c_1(0) = 1$, $c_2(0) = 0$.

(a) Find $|c_1(t)|^2$ and $|c_2(t)|^2$ for $t > 0$ by *exactly* solving the coupled differential equation

$$i\hbar \dot{c}_k = \sum_{n=1}^{2} V_{kn}(t) e^{i\omega_{kn} t} c_n, \quad (k = 1, 2).$$

(b) Do the same problem using time-dependent perturbation theory to lowest nonvanishing order. Compare the two approaches for small values of γ. Treat the following two cases separately: (i) ω very different from ω_{21} and (ii) ω close to ω_{21}.

Answer for (a): (Rabi's formula)

$$|c_2(t)|^2 = \frac{\gamma^2/\hbar^2}{\gamma^2/\hbar^2 + (\omega - \omega_{21})^2/4} \sin^2 \left\{ \left[\frac{\gamma^2}{\hbar^2} + \frac{(\omega - \omega_{21})^2}{4} \right]^{1/2} t \right\},$$

$$|c_1(t)|^2 = 1 - |c_2(t)|^2.$$

5.31 Show that the slow-turn-on of perturbation $V \to V e^{\eta t}$ (see Baym 1969, p. 257) can generate a contribution from the second term in (5.7.36).

5.32 **(a)** Consider the positronium problem you solved in Chapter 3, Problem 3.4. In the presence of a uniform and static magnetic field B along the z-axis, the Hamiltonian is given by

$$H = A\mathbf{S}_1 \cdot \mathbf{S}_2 + \left(\frac{eB}{m_e c} \right) (S_{1z} - S_{2z}).$$

Solve this problem to obtain the energy levels of all four states *using degenerate time-independent perturbation theory* (instead of diagonalizing the Hamiltonian matrix). Regard the first and second terms in the expression for H as H_0 and V, respectively. Compare your results with the exact expressions

$$E = -\left(\frac{\hbar^2 A}{4} \right) \left[1 \pm 2\sqrt{1 + 4\left(\frac{eB}{m_e c \hbar A} \right)^2} \right] \quad \text{for} \begin{cases} \text{singlet } m = 0 \\ \text{triplet } m = 0 \end{cases}$$

$$E = \frac{\hbar^2 A}{4} \quad \text{for triplet } m = \pm 1,$$

where *triplet* (*singlet*) $m = 0$ stands for the state that becomes a pure triplet (singlet) with $m = 0$ as $B \to 0$.

(b) We now attempt to cause transitions (via stimulated emission and absorption) between the two $m = 0$ states by introducing an oscillating magnetic field of the "right" frequency. Should we orient the magnetic field along the z-axis or along the x- (or y-) axis? Justify your choice. (The original static field is assumed to be along the z-axis throughout.)

(c) Calculate the eigenvectors to first order.

5.33 Repeat Problem 5.32, but with the atomic hydrogen Hamiltonian

$$H = A\mathbf{S}_1 \cdot \mathbf{S}_2 + \left(\frac{eB}{m_e c}\right)\mathbf{S}_1 \cdot \mathbf{B},$$

where in the hyperfine term, $A\mathbf{S}_1 \cdot \mathbf{S}_2$, $\mathbf{S}_1$ is the electron spin and $\mathbf{S}_2$ is the proton spin. [Note that the problem here has less symmetry than the positronium case].

5.34 Consider the spontaneous emission of a photon by an excited atom. The process is known to be an $E1$ transition. Suppose the magnetic quantum number of the atom decreases by one unit. What is the angular distribution of the emitted photon? Also discuss the polarization of the photon, with attention to angular-momentum conservation for the whole (atom plus photon) system.

5.35 Consider an atom made up of an electron and a singly charged ($Z = 1$) triton (^{3}H). Initially the system is in its ground state ($n = 1$, $l = 0$). Suppose the system undergoes beta decay, in which the nuclear charge *suddenly increases* by one unit (realistically by emitting an electron and an antineutrino). This means that the tritium nucleus (called a triton) turns into a helium ($Z = 2$) nucleus of mass 3 (^{3}He).

(a) Obtain the probability for the system to be found in the ground state of the resulting helium ion. The hydrogenic wave function is given by

$$\psi_{n=1,l=0}(\mathbf{x}) = \frac{1}{\sqrt{\pi}}\left(\frac{Z}{a_0}\right)^{3/2} e^{-Zr/a_0}.$$

(b) The available energy in tritium beta decay is about 18 keV, and the size of the ^{3}He atom is about 1Å. Check that the time scale T for the transformation satisfies the criterion of validity for the sudden approximation.

5.36 Show that $\mathbf{A}_n(\mathbf{R})$ defined in (5.6.23) is a purely real quantity.

5.37 Consider a neutron in a magnetic field, fixed at an angle θ with respect to the **z**-axis, but rotating slowly in the ϕ-direction. That is, the tip of the magnetic field traces out a circle on the surface of the sphere at "latitude" $\pi - \theta$. Explicitly calculate the Berry potential **A** for the spin-up state from (5.6.23), take its curl, and determine Berry's Phase γ_+. Thus, verify (5.6.42) for this particular example of a curve C. (For hints, see "The Adiabatic Theorem and Berry's Phase" by B. R. Holstein, *Am. J. Phys.* **57** (1989) 1079.)

5.38 The ground state of a hydrogen atom ($n = 1, l = 0$) is subjected to a time-dependent potential as follows:

$$V(\mathbf{x}, t) = V_0 \cos(kz - \omega t).$$

Using time-dependent perturbation theory, obtain an expression for the transition rate at which the electron is emitted with momentum **p**. Show, in particular, how you may compute the angular distribution of the ejected electron (in terms of θ and ϕ defined with respect to the z-axis). Discuss *briefly* the similarities and the differences between this problem and the (more realistic) photoelectric effect. (*Note:* For the initial wave function, see Problem 5.35. If you have a normalization problem, the final wave function may be taken to be

$$\psi_f(\mathbf{x}) = \left(\frac{1}{L^{3/2}}\right) e^{i\mathbf{p}\cdot\mathbf{x}/\hbar}$$

with L very large, but you should be able to show that the observable effects are independent of L.)

5.39 A particle of mass m constrained to move in one dimension is confined within $0 < x < L$ by an infinite-wall potential

$$\begin{aligned} V &= \infty \quad \text{for } x < 0, x > L, \\ V &= 0 \quad \text{for } 0 \le x \le L. \end{aligned}$$

Obtain an expression for the density of states (that is, the number of states per unit energy interval) for *high* energies as a function of E. (Check your dimension!)

5.40 Linearly polarized light of angular frequency ω is incident on a one-electron "atom" whose wave function can be approximated by the ground state of a three-dimensional isotropic harmonic oscillator of angular frequency ω_0. Show that the differential cross section for the ejection of a photoelectron is given by

$$\begin{aligned} \frac{d\sigma}{d\Omega} = \frac{4\alpha\hbar^2 k_f^3}{m^2\omega\omega_0}\sqrt{\frac{\pi\hbar}{m\omega_0}}\exp\left\{-\frac{\hbar}{m\omega_0}\left[k_f^2 + \left(\frac{\omega}{c}\right)^2\right]\right\} \\ \times \sin^2\theta\cos^2\phi\exp\left[\left(\frac{2\hbar k_f\omega}{m\omega_0 c}\right)\cos\theta\right], \end{aligned}$$

provided the ejected electron of momentum $\hbar k_f$ can be regarded as being in a plane-wave state. (The coordinate system used is shown in Figure 5.12.)

5.41 Find the probability $|\phi(\mathbf{p}')|^2 d^3p'$ of the particular momentum $\mathbf{p}'$ for the ground-state hydrogen atom. (This is a nice exercise in three-dimensional Fourier transforms. To perform the angular integration, choose the z-axis in the direction of $\mathbf{p}$.)

5.42 Obtain an expression for $\tau(2p \to 1s)$ for the hydrogen atom. Verify that it is equal to 1.6×10^{-9} s.