## Problems

**6.1** The Lippmann-Schwinger formalism can also be applied to a *one*-dimensional transmission-reflection problem with a finite-range potential, $V(x) \neq 0$ for $0 < |x| < a$ only.

**(a)** Suppose we have an incident wave coming from the left: $\langle x|\phi\rangle = e^{ikx}/\sqrt{2\pi}$. How must we handle the singular $1/(E - H_0)$ operator if we are to have a transmitted wave only for $x > a$ and a reflected wave and the original wave for $x < -a$? Is the $E \to E + i\varepsilon$ prescription still correct? Obtain an expression for the appropriate Green's function and write an integral equation for $\langle x|\psi^{(+)}\rangle$.

**(b)** Consider the special case of an attractive $\delta$-function potential

$$V = -\left(\frac{\gamma\hbar^2}{2m}\right)\delta(x) \quad (\gamma > 0).$$

Solve the integral equation to obtain the transmission and reflection amplitudes. Check your results with Gottfried 1966, p. 52.

**(c)** The one-dimensional $\delta$-function potential with $\gamma > 0$ admits one (and only one) bound state for any value of $\gamma$. Show that the transmission and reflection amplitudes you computed have bound-state poles at the expected positions when $k$ is regarded as a complex variable.

**6.2** Prove

$$\sigma_{\text{tot}} \simeq \frac{m^2}{\pi\hbar^4}\int d^3x \int d^3x' V(r)V(r')\frac{\sin^2 k|\mathbf{x}-\mathbf{x}'|}{k^2|\mathbf{x}-\mathbf{x}'|^2}$$

in each of the following ways.

**(a)** By integrating the differential cross section computed using the first-order Born approximation.

**(b)** By applying the optical theorem to the forward-scattering amplitude in the *second*-order Born approximation. [Note that $f(0)$ is real if the first-order Born approximation is used.]

**6.3** Estimate the radius of the $^{40}Ca$ nucleus from the data in Figure 6.6 and compare to that expected from the empirical value $\approx 1.4A^{1/3}$ fm, where $A$ is the nuclear mass number. Check the validity of using the first-order Born approximation for these data.

**6.4** Consider a potential

$$V = 0 \quad \text{for } r > R, \qquad V = V_0 = \text{constant} \quad \text{for } r < R,$$

where $V_0$ may be positive or negative. Using the method of partial waves, show that for $|V_0| \ll E = \hbar^2k^2/2m$ and $kR \ll 1$, the differential cross section is isotropic and that the total cross section is given by

$$\sigma_{\text{tot}} = \left(\frac{16\pi}{9}\right)\frac{m^2V_0^2R^6}{\hbar^4}.$$

Suppose the energy is raised slightly. Show that the angular distribution can then be written as

$$\frac{d\sigma}{d\Omega} = A + B\cos\theta.$$

Obtain an approximate expression for $B/A$.

**6.5** A spinless particle is scattered by a weak Yukawa potential

$$V = \frac{V_0 e^{-\mu r}}{\mu r},$$

where $\mu > 0$ but $V_0$ can be positive or negative. It was shown in the text that the first-order Born amplitude is given by

$$f^{(1)}(\theta) = -\frac{2mV_0}{\hbar^2\mu}\frac{1}{[2k^2(1-\cos\theta)+\mu^2]}.$$

**(a)** Using $f^{(1)}(\theta)$ and assuming $|\delta_l| \ll 1$, obtain an expression for $\delta_l$ in terms of a Legendre function of the second kind,

$$Q_l(\zeta) = \frac{1}{2}\int_{-1}^{1}\frac{P_l(\zeta')}{\zeta-\zeta'}d\zeta'.$$

**(b)** Use the expansion formula

$$Q_l(\zeta) = \frac{l!}{1\cdot 3\cdot 5\cdots(2l+1)} \times \left\{\frac{1}{\zeta^{l+1}} + \frac{(l+1)(l+2)}{2(2l+3)}\frac{1}{\zeta^{l+3}} + \frac{(l+1)(l+2)(l+3)(l+4)}{2\cdot 4\cdot(2l+3)(2l+5)}\frac{1}{\zeta^{l+5}} + \cdots\right\} \quad (|\zeta| > 1)$$

to prove each assertion.

**(i)** $\delta_l$ is negative (positive) when the potential is repulsive (attractive).

**(ii)** When the de Broglie wavelength is much longer than the range of the potential, $\delta_l$ is proportional to $k^{2l+1}$. Find the proportionality constant.

**6.6** Check explicitly the $x - p_x$ uncertainty relation for the ground state of a particle confined inside a hard sphere: $V = \infty$ for $r > a$, $V = 0$ for $r < a$. (*Hint:* Take advantage of spherical symmetry.)

**6.7** Consider the scattering of a particle by an impenetrable sphere

$$V(r) = \begin{cases} 0 & \text{for } r > a \\ \infty & \text{for } r < a. \end{cases}$$

**(a)** Derive an expression for the $s$-wave ($l = 0$) phase shift. (You need not know the detailed properties of the spherical Bessel functions to do this simple problem!)

**(b)** What is the total cross section $\sigma[\sigma = \int (d\sigma/d\Omega)d\Omega]$ in the extreme low-energy limit $k \to 0$? Compare your answer with the geometric cross section $\pi a^2$. You may assume without proof:

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2,$$

$$f(\theta) = \left(\frac{1}{k}\right)\sum_{l=0}^{\infty}(2l+1)e^{i\delta_l}\sin\delta_l P_l(\cos\theta).$$

**6.8** Use $\delta_l = \Delta(b)|_{b=l/k}$ to obtain the phase shift $\delta_l$ for scattering at high energies by (a) the Gaussian potential, $V = V_0\exp(-r^2/a^2)$, and (b) the Yukawa potential, $V = V_0\exp(-\mu r)/\mu r$. Verify the assertion that $\delta_l$ goes to zero very rapidly with increasing $l$ ($k$ fixed) for $l \gg kR$, where $R$ is the "range" of the potential. [The formula for $\Delta(b)$ is given in (6.5.14)].

**6.9** **(a)** Prove

$$\frac{\hbar^2}{2m}\langle\mathbf{x}|\frac{1}{E - H_0 + i\varepsilon}|\mathbf{x}'\rangle = -ik\sum_l\sum_m Y_l^m(\hat{\mathbf{r}})Y_l^{m*}(\hat{\mathbf{r}}')j_l(kr_<)h_l^{(1)}(kr_>),$$

where $r_<(r_>)$ stands for the smaller (larger) of $r$ and $r'$.

**(b)** For spherically symmetrical potentials, the Lippmann-Schwinger equation can be written for *spherical* waves:

$$|Elm(+)\rangle = |Elm\rangle + \frac{1}{E - H_0 + i\varepsilon}V|Elm(+)\rangle.$$

Using (a), show that this equation, written in the **x**-representation, leads to an equation for the radial function, $A_l(k; r)$, as follows:

$$A_l(k;r) = j_l(kr) - \frac{2mik}{\hbar^2} \times \int_0^\infty j_l(kr_<)h_l^{(1)}(kr_>)V(r')A_l(k;r')r'^2dr'.$$

By taking $r$ very large, also obtain

$$f_l(k) = e^{i\delta_l}\frac{\sin\delta_l}{k} = -\left(\frac{2m}{\hbar^2}\right)\int_0^\infty j_l(kr)A_l(k;r)V(r)r^2dr.$$

**6.10** Consider scattering by a repulsive $\delta$-shell potential:

$$\left(\frac{2m}{\hbar^2}\right)V(r) = \gamma\delta(r - R), \quad (\gamma > 0).$$

**(a)** Set up an equation that determines the $s$-wave phase shift $\delta_0$ as a function of $k(E = \hbar^2k^2/2m)$.

**(b)** Assume now that $\gamma$ is very large,

$$\gamma \gg \frac{1}{R}, k.$$

Show that if tan $kR$ is *not* close to zero, the $s$-wave phase shift resembles the hard-sphere result discussed in the text. Show also that for tan $kR$ close to (but not exactly equal to) zero, resonance behavior is possible; that is, $\cot\delta_0$ goes through zero from the positive side as $k$ increases. Determine approximately the positions of the resonances keeping terms of order $1/\gamma$; compare them with the bound-state energies for a particle confined *inside* a spherical wall of the same radius,

$$V = 0, \quad r < R; \qquad V = \infty, \quad r > R.$$

Also obtain an approximate expression for the resonance width $\Gamma$ defined by

$$\Gamma = \frac{-2}{[d(\cot\,\delta_0)/dE]|_{E=E_r}},$$

and notice, in particular, that the resonances become extremely sharp as $\gamma$ becomes large. (*Note:* For a different, more sophisticated approach to this problem, see Gottfried 1966, pp. 131–41, who discusses the analytic properties of the $D_l$-function defined by $A_l = j_l/D_l$.)

**6.11** A spinless particle is scattered by a time-dependent potential

$$\mathcal{V}(\mathbf{r},t) = V(\mathbf{r})\cos\omega t.$$

Show that if the potential is treated to first order in the transition amplitude, the energy of the scattered particle is increased or decreased by $\hbar\omega$. Obtain $d\sigma/d\Omega$. Discuss qualitatively what happens if the higher-order terms are taken into account.

**6.12** Show that the differential cross section for the elastic scattering of a fast electron by the ground state of the hydrogen atom is given by

$$\frac{d\sigma}{d\Omega} = \left(\frac{4m^2e^4}{\hbar^4q^4}\right)\left\{1 - \frac{16}{[4+(qa_0)^2]^2}\right\}^2.$$

(Ignore the effect of identity.)

**6.13** Let the energy of a particle moving in a central field be $E(J_1J_2J_3)$, where $(J_1, J_2, J_3)$ are the three action variables. How does the functional form of $E$ specialize for the Coulomb potential? Using the recipe of the action-angle method, compare the degeneracy of the central-field problem to that of the Coulomb problem, and relate it to the vector **A**.

If the Hamiltonian is

$$H = \frac{p^2}{2\mu} + V(r) + F(\mathbf{A}^2),$$

how are these statements changed?

Describe the corresponding degeneracies of the central-field and Coulomb problems in quantum theory in terms of the usual quantum numbers ($n$, $l$, $m$) and also in terms of the quantum numbers ($k$, $m$, $n$). Here the second set, ($k$, $m$, $n$), labels the wave functions $\mathcal{D}^k_{mn}(\alpha\beta\gamma)$.

How are the wave functions $\mathcal{D}^k_{mn}(\alpha\beta\gamma)$ related to Laguerre times spherical harmonics?