ity to create and destroy particles is inconsistent with our "single-particle" approach to writing down dynamics in quantum mechanics. Instead, we would need to reexamine the Hamiltonian formalism, on which much of this book is based, in order to address these issues.

Quantum field theory is the correct framework for addressing relativistic quantum mechanics and mutli-particle quantum mechanics in general. There are essentially two ways to approach quantum field theory, neither of which is developed here. We mention them only for the reader interested in going on to the next steps.

One approach is through the method of "second quantization," wherein operators are introduced that create and destroy particles. These operators commute with each other if they have integer spin, and they anticommute for half-integer spin. Work needs to be done in order to build in relativistic covariance, but it is relatively straightforward. It is also, however, not necessary if the problem doesn't warrant it. This is the case, for example, in a vast number of fascinating problems in condensed-matter physics.

Second quantization is discussed in Section 7.5 of this book. For other examples, see *Quantum Mechanics*, 3rd ed., by Eugen Merzbacher, and *Quantum Theory of Many-Particle Systems*, by Alexander L. Fetter and John Dirk Walecka.

The second approach is through the path-integral approach to quantum mechanics, famously pioneered by Richard Feynman in his Ph.D. thesis. This conceptually appealing formalism is straightforward to extend from particle quantum mechanics to quantum fields. However, it is not straightforward to use this formalism for calculation of typical problems until one makes the connection to the "canonical" formalism that eventually becomes second quantization. Nevertheless, it is a worthwhile subject for students who would like to have a better understanding of the principles that lead to the quantum many-body problem.

Path integrals are not the basis for many books on quantum field theory, but they are beautifully exploited in *Quantum Field Theory in a Nutshell* by Anthony Zee.

Problems

8.11 These exercises are to give you some practice with natural units.

- **(a)** Express the proton mass $m_p = 1.67262158 \times 10^{27}$ kg in units of GeV.
- **(b)** Assume a particle with negligible mass is confined to a box the size of the proton, around 1 fm $= 10^{-15}$ m. Use the uncertainty principle estimate the energy of the confined particle. You might be interested to know that the mass, in natural units, of the pion, the lightest strongly interacting particle, is $m_\pi =$ 135 MeV.
- **(c)** String theory concerns the physics at a scale that combines gravity, relativity, and quantum mechanics. Use dimensional analysis to find the "Planck mass" M_P, which is formed from G, $\hbar$, and c, and express the result in GeV.

8.12 Show that a matrix $\eta^{\mu\nu}$ with the same elements as the metric tensor $\eta_{\mu\nu}$ used in this chapter has the property that $\eta^{\mu\lambda}\eta_{\lambda\nu} = \delta^\mu_\nu$, the identity matrix. Thus, show that the natural relationship $\eta^{\mu\nu} = \eta^{\mu\lambda}\eta^{\nu\sigma}\eta_{\lambda\sigma}$ in fact holds with this definition. Show also that $a^\mu b_\mu = a_\mu b^\mu$ for two four-vectors a^μ and b^μ.

8.13 Show that (8.1.11) is in fact a conserved current when $\Psi(\mathbf{x},t)$ satisfies the Klein-Gordon equation.

T6 **8.14** Show that (8.1.14) follows from (8.1.8).

8.15 Derive (8.1.16a), (8.1.16b), and (8.1.18).

8.16 Show that the free-particle energy eigenvalues of (8.1.18) are $E = \pm E_p$ and that the eigenfunctions are indeed given by (8.1.21), subject to the normalization that $\Upsilon^\dagger \tau_3 \Upsilon = \pm 1$ for $E = \pm E_p$.

8.17 This problem is taken from *Quantum Mechanics II: A Second Course in Quantum Theory*, 2nd ed., by Rubin H. Landau (1996). A spinless electron is bound by the Coulomb potential $V(r) = -Ze^2/r$ in a stationary state of total energy $E \leq m$. You can incorporate this interaction into the Klein-Gordon equation by using the covariant derivative with $V = -e\Phi$ and $\mathbf{A} = 0$. Work with the upper component of the wave function.

(a) Assume that the radial and angular parts of the equation separate and that the wave function can be written as $e^{-iEt}[u_l(kr)/r]Y_{lm}(\theta,\phi)$. Show that the radial equation becomes

$$\frac{d^2u}{d\rho^2} + \left[\frac{2EZ\alpha}{\gamma\rho} - \frac{1}{4} - \frac{l(l+1)-(Z\alpha)^2}{\rho^2}\right]u_l(\rho) = 0,$$

where $\alpha = e^2$, $\gamma^2 = 4(m^2 - E^2)$, and $\rho = kr$.

(b) Assume that this equation has a solution of the usual form of a power series times the $\rho \to \infty$ and $\rho \to 0$ solutions, that is,

$$u_l(\rho) = \rho^k(1 + c_1\rho + c_2\rho^2 + \cdots)e^{-\rho/2},$$

and show that

$$k = k_\pm = \frac{1}{2} \pm \sqrt{\left(l + \frac{1}{2}\right)^2 - (Z\alpha)^2}$$

and that only for k_+ is the expectation value of the kinetic energy finite and that this solution has a nonrelativistic limit that agrees with the solution found for the Schrödinger equation.

(c) Determine the recurrence relation among the c_i's for this to be a solution of the Klein-Gordon equation, and show that unless the power series terminates, the will function will have an incorrect asymptotic form.

(d) In the case where the series terminates, show that the energy eigenvalue for the k_+ solution is

$$E = \frac{m}{\left(1 + (Z\alpha)^2\left[n - l - \frac{1}{2} + \sqrt{\left(l + \frac{1}{2}\right)^2 - (Z\alpha)^2}\right]^{-2}\right)^{1/2}},$$

where n is the principal quantum number.

(e) Expand E in powers of $(Z\alpha)^2$ and show that the first-order term yields the Bohr formula. Connect the higher-order terms with relativistic corrections, and discuss the degree to which the degeneracy in l is removed.

Jenkins and Kunselman, in *Phys. Rev. Lett.* **17** (1966) 1148, report measurements of a large number of transition energies for π^- atoms in large-Z nuclei. Compare some of these to the calculated energies, and discuss the accuracy of the prediction. (For example, consider the $3d \to 2p$ transition in ^{59}Co, which emits a photon with energy 384.6 ± 1.0 keV.) You will probably need either to use a computer to carry out the energy differences with high enough precision, or else expand to higher powers of $(Z\alpha)^2$.

8.8 Prove that the traces of the γ^μ, $\boldsymbol{\alpha}$, and β are all zero.

8.9 **(a)** Derive the matrices γ^μ from (8.2.10) and show that they satisfy the Clifford algebra (8.2.4).

(b) Show that

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} = I \otimes \tau_3$$

and

$$\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} = \sigma^i \otimes i\tau_2,$$

where I is the 2×2 identity matrix, and σ^i and τ_i are the Pauli matrices. (The $\otimes$ notation is a formal way to write our 4×4 matrices as 2×2 matrices of 2×2 matrices.)

8.10 Prove the continuity equation (8.2.11) for the Dirac equation.

8.11 Find the eigenvalues for the free-particle Dirac equation (8.2.20).

8.12 Insert one of the four solutions $u^{(\pm)}_{R,L}(p)$ from (8.2.22) into the four-vector probability current (8.2.13) and interpret the result.

8.13 Make use of Problem 8.9 to show that U_T as defined by (8.3.28) is just $\sigma^2 \otimes I$, up to a phase factor.

8.14 Write down the positive-helicity, positive-energy free-particle Dirac spinor wave function $\Psi(\mathbf{x}, t)$.

P2

(a) Construct the spinors $\mathcal{P}\Psi$, $\mathcal{C}\Psi$, $\mathcal{T}\Psi$.

(b) Construct the spinor $\mathcal{CPT}\Psi$ and interpret it using the discussion of negative-energy solutions to the Dirac equation.

8.15 Show that (8.4.38) imply that $u(x)$ and $v(x)$ grow like exponentials if the series (8.4.32) and (8.4.33) do not terminate.

P2 **8.16** Expand the energy eigenvalues given by (8.4.43) in powers of $Z\alpha$, and show that the result is equivalent to including the relativistic correction to kinetic energy (5.3.10) and the spin-orbit interaction (5.3.31) to the nonrelativistic energy eigenvalues for the one-electron atom (8.4.44).

8.17 The National Institute of Standards and Technology (NIST) maintains a web site with up-to-date high-precision data on the atomic energy levels of hydrogen and deuterium:

http://physics.nist.gov/PhysRefData/HDEL/data.html

The accompanying table of data was obtained from that web site. It gives the energies of transitions between the $(n,l,j) = (1,0,1/2)$ energy level and the energy level indicated by the columns on the left.

n	l	j	$[E(n,l,j) - E(1,0,1/2)]/hc$ (cm^{-1})
2	0	1/2	82 258.954 399 2832(15)
2	1	1/2	82 258.919 113 406(80)
2	1	3/2	82 259.285 001 249(80)
3	0	1/2	97 492.221 724 658(46)
3	1	1/2	97 492.211 221 463(24)
3	1	3/2	97 492.319 632 775(24)
3	2	3/2	97 492.319 454 928(23)
3	2	5/2	97 492.355 591 167(23)
4	0	1/2	102 823.853 020 867(68)
4	1	1/2	102 823.848 581 881(58)
4	1	3/2	102 823.894 317 849(58)
4	2	3/2	102 823.894 241 542(58)
4	2	5/2	102 823.909 486 535(58)
4	3	5/2	102 823.909 459 541(58)
4	3	7/2	102 823.917 081 991(58)

(The number in parentheses is the numerical value of the standard uncertainty referred to the last figures of the quoted value.) Compare these values to those predicted by (8.4.43). (You may want to make use of Problem 8.16.) In particular:

(a) Compare fine-structure splitting between the $n = 2, j = 1/2$ and $n = 2, j = 3/2$ states to (8.4.43).

(b) Compare fine-structure splitting between the $n = 4, j = 5/2$ and $n = 4, j = 7/2$ states to (8.4.43).

(c) Compare the $1S \to 2S$ transition energy to the first line in the table. Use as many significant figures as necessary in the values of the fundamental constants, to compare the results within standard uncertainty.

(d) How many examples of the Lamb shift are demonstrated in this table? Identify one example near the top and another near the bottom of the table, and compare their values.