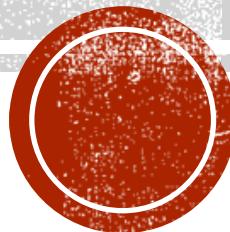


EQUAÇÃO DE DIRAC

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1°

$$[\Sigma \cdot \hat{p}, H] = 0$$

Hamiltoniano de
Dirac

$$H = \gamma^0 \gamma^i \cdot p + \gamma^0 m$$

$$H = \hat{\alpha} \cdot \hat{p} + \beta m$$

$$\hat{\alpha} = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix}$$

$$\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Matriz Identidade

$$\Sigma = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}$$



- Matrizes de Pauli (σ);
- Matrizes 4x4 em formato 2x2.
- $\mathbf{S} = \frac{\hbar}{2} \boldsymbol{\Sigma}$

$$[\beta, \Sigma \cdot \hat{p}] = 0 \longrightarrow \beta \text{ é Matriz diagonal} \longrightarrow \beta \Sigma = \Sigma \beta = \begin{pmatrix} \sigma \cdot \hat{p} & 0 \\ 0 & -\sigma \cdot \hat{p} \end{pmatrix}$$

$$[\Sigma \cdot \hat{p}, \hat{\alpha} \cdot \hat{p}] = (\Sigma \cdot \hat{p})(\hat{\alpha} \cdot \hat{p}) - (\hat{\alpha} \cdot \hat{p})(\Sigma \cdot \hat{p})$$

$$[\Sigma \cdot \hat{p}, \hat{\alpha} \cdot \hat{p}] = \begin{pmatrix} \sigma \cdot \hat{p} & 0 \\ 0 & \sigma \cdot \hat{p} \end{pmatrix} \begin{pmatrix} 0 & \sigma \cdot \hat{p} \\ \sigma \cdot \hat{p} & 0 \end{pmatrix} - \begin{pmatrix} 0 & \sigma \cdot \hat{p} \\ \sigma \cdot \hat{p} & 0 \end{pmatrix} \begin{pmatrix} \sigma \cdot \hat{p} & 0 \\ 0 & \sigma \cdot \hat{p} \end{pmatrix}$$

$$[\Sigma \cdot \hat{p}, \hat{\alpha} \cdot \hat{p}] = \begin{pmatrix} 0 & (\sigma \cdot \hat{p})^2 \\ (\sigma \cdot \hat{p})^2 & 0 \end{pmatrix} - \begin{pmatrix} 0 & (\sigma \cdot \hat{p})^2 \\ (\sigma \cdot \hat{p})^2 & 0 \end{pmatrix}$$

$$[\Sigma \cdot \hat{p}, \hat{\alpha} \cdot \hat{p}] = [\Sigma \cdot \hat{p}, H] = 0$$

$$\Sigma \cdot \hat{p} = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix} \cdot \hat{p}$$

$$\hat{\alpha} \cdot \hat{p} = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix} \cdot \hat{p}$$

Projeta as componentes na direção do momento.

!2°

$$[J, H] = 0$$

Momento angular total

$$J = L + \frac{\hbar}{2} \Sigma = \mathbf{L} + \mathbf{S}$$

$$\begin{aligned} [\alpha \cdot p + \beta m, L_i] &= [\alpha_l p_l, \epsilon_{ijk} x_j p_k] \\ &= \epsilon_{ijk} \alpha_l [p_l, x_j] p_k \\ &= \epsilon_{ijk} \alpha_l (-i \delta_{jl}) p_k \\ [\alpha \cdot p + \beta m, L_i] &= -i \epsilon_{ijk} \alpha_j p_k \end{aligned}$$

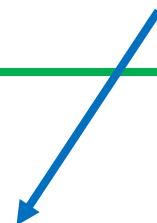


O hamiltoniano não comuta com o momento angular.

$$\begin{aligned} [\alpha \cdot p + \beta m, \Sigma_i] &= [\alpha_k p_k, \Sigma_i] = [\alpha_k, \Sigma_i] p_k \\ &= 2 i \epsilon_{kij} \sigma_j p_k \\ &= 2 i \epsilon_{ijk} \sigma_j p_k \end{aligned}$$



O hamiltoniano não comuta com o momento angular de Spin.



$$[\sigma_i, \Sigma_j] = \begin{pmatrix} \sigma_i \sigma_j - \sigma_j \sigma_i & 0 \\ 0 & \sigma_i \sigma_j - \sigma_j \sigma_i \end{pmatrix} = [\sigma_i, \sigma_j] \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$[\sigma_i, \Sigma_j] = 2 i \epsilon_{ijk} \sigma_k$$

$$[A, B + C] = [A, B] + [A, C]$$

$\hbar=1$

$$\begin{aligned} \left[\boldsymbol{\alpha} \cdot \boldsymbol{p}, L_i + \frac{\hbar}{2} \Sigma_i \right] &= [\boldsymbol{\alpha} \cdot \boldsymbol{p}, L_i] + \frac{1}{2} [\boldsymbol{\alpha} \cdot \boldsymbol{p}, \Sigma_i] \\ &= -i \varepsilon_{ijk} \alpha_j p_k + \frac{1}{2} (2 i \varepsilon_{ijk} \alpha_j p_k) \\ \left[\boldsymbol{\alpha} \cdot \boldsymbol{p}, L_i + \frac{\hbar}{2} \Sigma_i \right] &= 0 \end{aligned}$$

$$[J, H] = 0$$

3°

$$U_p = \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \beta^\dagger$$

$$\mathcal{P} = \pi U_p$$


Operador unitário e invariante
sob uma transformação de paridade

Operador Paridade Total

$$\begin{aligned} U_p \alpha U_p^\dagger &= -\alpha \\ U_p \beta U_p^\dagger &= \beta \\ U_p^2 &= 1 \end{aligned}$$

$$\alpha = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix}$$


$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -\sigma \\ \sigma & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\sigma \\ -\sigma & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\tilde{C}\tilde{C}^{-1} = 1 \quad \tilde{C}(\gamma^\mu)^*\tilde{C}^{-1} = -\gamma^\mu \quad \tilde{C} = i\gamma^2$$

Klein-Gordon

$$\psi_{partícula}(x, t) \equiv \psi_{E>0}(x, t)$$



Equação de Dirac com campo eletromagnético

$$(i\gamma^\mu \partial_\mu - e\gamma^\mu A_\mu - m) \psi(x, t) = 0$$

$$\psi_{antipartícula}(x, t) \equiv \psi_{E<0}^*(x, t)$$



$$(-i(\gamma^\mu)^*\partial_\mu - e(\gamma^\mu)^*A_\mu - m) \psi(x, t)^* = 0$$



$$\tilde{C}\tilde{C}^{-1} = 1 \quad \tilde{C}(\gamma^\mu)^*\tilde{C}^{-1} = -\gamma^\mu$$



$$(i\gamma^\mu \partial_\mu + e\gamma^\mu A_\mu - m) \tilde{C} \psi(x, t)^* = 0$$

Equação do Pósitron

$$i\gamma^2 = \begin{pmatrix} 0 & i\sigma_2 \\ -i\sigma_2 & \mathbf{0} \end{pmatrix} = \tilde{C}$$

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma_\mu \\ -\sigma_\mu & 0 \end{pmatrix}$$

$$\begin{aligned}\tilde{C}\tilde{C}^{-1} &= i \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & \mathbf{0} \end{pmatrix} i \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & \mathbf{0} \end{pmatrix} \\ &= i^2 \begin{pmatrix} -\sigma_2^2 & 0 \\ 0 & -\sigma_2^2 \end{pmatrix} \\ &= (-1) \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \\ \tilde{C}\tilde{C}^{-1} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\end{aligned}$$

$$\begin{aligned}\tilde{C}(\gamma^\mu)^*\tilde{C}^{-1} &= i \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & \mathbf{0} \end{pmatrix} \begin{pmatrix} 0 & -\sigma_\mu \\ \sigma_\mu & \mathbf{0} \end{pmatrix} i \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & \mathbf{0} \end{pmatrix} \\ &= (i)^2 \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} \begin{pmatrix} \sigma_\mu \sigma_2 & 0 \\ 0 & \sigma_\mu \sigma_2 \end{pmatrix} \\ &= (-1) \begin{pmatrix} 0 & \sigma_\mu \sigma_2^2 \\ -\sigma_\mu \sigma_2^2 & 0 \end{pmatrix} \\ \tilde{C}(\gamma^\mu)^*\tilde{C}^{-1} &= (-1) \begin{pmatrix} 0 & \sigma_\mu \\ -\sigma_\mu & 0 \end{pmatrix}\end{aligned}$$

Referências

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