A equação de Dirac: Solução para o potencial central

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Qual é o efeito de $(\boldsymbol{\sigma}\cdot\mathbf{p})$ em $\psi_1(\boldsymbol{x})$ e $\psi_2(\boldsymbol{x})$?

Solução:

Sabemos que

$$\boldsymbol{\sigma}\cdot\boldsymbol{p} = (\boldsymbol{\sigma}\cdot\hat{\boldsymbol{r}})\left[\hat{\boldsymbol{r}}\cdot\boldsymbol{p} + i\boldsymbol{\sigma}\cdot\frac{\boldsymbol{L}}{r}\right]$$

Também,

$$A: \ \psi_1(\boldsymbol{x}) = \mathcal{U}_A(r)\,\mathcal{Y}^{j,m}_{l=j-1/2}(\theta,\phi) \quad \text{e} \quad \psi_2(\boldsymbol{x}) = -i\mathcal{V}_A(r)\,\mathcal{Y}^{j,m}_{l=j+1/2}(\theta,\phi)$$

$$B: \ \psi_1(\boldsymbol{x}) = \mathcal{U}_B(r)\,\mathcal{Y}^{j,m}_{l=j+1/2}(\theta,\phi) \quad \text{e} \quad \psi_2(\boldsymbol{x}) = -i\mathcal{V}_B(r)\,\mathcal{Y}^{j,m}_{l=j-1/2}(\theta,\phi)$$

A equação de Dirac: solução para o potencial central

onde $\mathcal{Y}_l^{j,m}$ são funções de spin-angular que são da forma:

$$\mathcal{Y}_l^{j=l\pm 1/2,m} = \frac{1}{\sqrt{2l+1}} \begin{pmatrix} \pm\sqrt{l \pm m + 1/2}\, Y_l^{m-1/2}(\theta,\phi) \\ \sqrt{l \mp m + 1/2}\, Y_l^{m-1/2}(\theta,\phi) \end{pmatrix}$$

$$Y_l^m(\theta,\phi) = \frac{(-1)^l}{2^l l!} \sqrt{\frac{2l+1}{4\pi} \frac{(l+m)!}{(l-m)!}}\, e^{im\phi} \frac{1}{\sin^m\theta} \frac{d^{l-m}}{d(\cos\theta)^{l-m}} (\sin\theta)^{2l}$$

Por outro lado, o efeito nas funções de spin-angular são

$$\begin{aligned}(\boldsymbol{\sigma}\cdot\boldsymbol{L})\,\mathcal{Y}_l^{j,m}(\theta,\phi) =& \left[j(j+1) - l(l+1) - 3/4\right]\mathcal{Y}_l^{j,m}(\theta,\phi) \\ =& \mathcal{K}(j,l)\,\mathcal{Y}_l^{j,m}(\theta,\phi)\end{aligned}$$

A equação de Dirac: solução para o potencial central

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$$\begin{cases} \mathcal{K} = -(\lambda + 1) & \text{para} \quad l = j + \frac{1}{2} \\ \mathcal{K} = (\lambda - 1) & \text{para} \quad l = j - \frac{1}{2} \end{cases} \qquad \text{onde} \quad \lambda = j + \frac{1}{2}$$

também,

$$(\boldsymbol{\sigma} \cdot \hat{\boldsymbol{r}})\, \mathcal{Y}^{j,m}_{l=j\pm 1/2}(\theta, \phi) = -\mathcal{Y}^{j,m}_{l=j\mp 1/2}(\theta, \phi)$$

assim como

$$\hat{\boldsymbol{r}} \cdot \boldsymbol{p} \longrightarrow \hat{\boldsymbol{r}} \cdot (-i\nabla) = -i\frac{\partial}{\partial r}$$

A equação de Dirac: solução para o potencial central

Para A:

$$
\begin{aligned}
(\boldsymbol{\sigma}\cdot\boldsymbol{p})\,\psi_1(\boldsymbol{x}) &= (\boldsymbol{\sigma}\cdot\hat{\boldsymbol{r}})\left[\hat{\boldsymbol{r}}\cdot\boldsymbol{p}+i\boldsymbol{\sigma}\cdot\frac{\boldsymbol{L}}{r}\right]\psi_1(\boldsymbol{x}) \\
&= (\boldsymbol{\sigma}\cdot\hat{\boldsymbol{r}})\Big[(\hat{\boldsymbol{r}}\cdot\boldsymbol{p})\,\mathcal{U}_A(r)\,\mathcal{Y}^{j,m}_{l=j-1/2}(\theta,\phi) \\
&\quad +i\left(\boldsymbol{\sigma}\cdot\frac{\boldsymbol{L}}{r}\right)\mathcal{U}_A(r)\,\mathcal{Y}^{j,m}_{l=j-1/2}(\theta,\phi)\Big] \\
&= (\boldsymbol{\sigma}\cdot\hat{\boldsymbol{r}})\,\mathcal{Y}^{j,m}_{l=j-1/2}(\theta,\phi)\left[-i\frac{d}{dr}+\frac{i}{r}(\lambda-1)\right]\mathcal{U}_A(r) \\
&= -\mathcal{Y}^{j,m}_{l=j+1/2}(\theta,\phi)\left[-i\frac{d}{dr}+\frac{i}{r}(\lambda-1)\right]\mathcal{U}_A(r)
\end{aligned}
$$

A equação de Dirac: solução para o potencial central

Para A:

$$
\begin{aligned}
(\boldsymbol{\sigma}\cdot\boldsymbol{p})\,\psi_2(\boldsymbol{x}) &= (\boldsymbol{\sigma}\cdot\hat{\boldsymbol{r}})\left[\hat{\boldsymbol{r}}\cdot\boldsymbol{p}+i\boldsymbol{\sigma}\cdot\frac{\boldsymbol{L}}{r}\right]\psi_2(\boldsymbol{x})\\
&= (\boldsymbol{\sigma}\cdot\hat{\boldsymbol{r}})\Big[(\hat{\boldsymbol{r}}\cdot\boldsymbol{p})-i\mathcal{V}_A(r)\,\mathcal{Y}^{j,m}_{l=j+1/2}(\theta,\phi)\\
&\quad +i\left(\boldsymbol{\sigma}\cdot\frac{\boldsymbol{L}}{r}\right)-i\mathcal{V}_A(r)\,\mathcal{Y}^{j,m}_{l=j+1/2}(\theta,\phi)\Big]\\
&= (\boldsymbol{\sigma}\cdot\hat{\boldsymbol{r}})\,\mathcal{Y}^{j,m}_{l=j+1/2}(\theta,\phi)\left[-\frac{d}{dr}-\frac{1}{r}(\lambda+1)\right]\mathcal{V}_A(r)\\
&= -\mathcal{Y}^{j,m}_{l=j-1/2}(\theta,\phi)\left[-\frac{d}{dr}-\frac{1}{r}(\lambda+1)\right]\mathcal{V}_A(r)
\end{aligned}
$$

A equação de Dirac: solução para o potencial central

Para B:

$$(\boldsymbol{\sigma}\cdot\boldsymbol{p})\,\psi_1(\boldsymbol{x}) = -\mathcal{Y}^{j,m}_{l=j-1/2}(\theta,\phi)\left[-i\frac{d}{dr}-\frac{i}{r}(\lambda+1)\right]\mathcal{U}_B(r)$$

$$(\boldsymbol{\sigma}\cdot\boldsymbol{p})\,\psi_2(\boldsymbol{x}) = -\mathcal{Y}^{j,m}_{l=j+1/2}(\theta,\phi)\left[-\frac{d}{dr}+\frac{1}{r}(\lambda-1)\right]\mathcal{V}_B(r)$$

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As equações acopladas para os spinores $\psi_1(\boldsymbol{x})$ e $\psi_2(\boldsymbol{x})$ são

$$[E - m - V(r)]\,\psi_1(\boldsymbol{x}) - (\boldsymbol{\sigma}\cdot\boldsymbol{p})\,\psi_2(\boldsymbol{x}) = 0$$
$$[E + m - V(r)]\,\psi_2(\boldsymbol{x}) - (\boldsymbol{\sigma}\cdot\boldsymbol{p})\,\psi_1(\boldsymbol{x}) = 0$$

Para A:

$$[E - m - V(r)]\,\mathcal{U}_A(r) - \left[\frac{d}{dr} + \frac{\lambda+1}{r}\right]\mathcal{V}_A(r) = 0$$
$$[E + m - V(r)]\,\mathcal{V}_A(r) + \left[\frac{d}{dr} - \frac{\lambda-1}{r}\right]\mathcal{U}_A(r) = 0$$

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A equação de Dirac: solução para o potencial central

Para B:

$$\left[E - m - V\left(r\right)\right]\mathcal{U}_B\left(r\right) - \left[\frac{d}{dr} - \frac{\lambda - 1}{r}\right]\mathcal{V}_B\left(r\right) = 0$$

$$\left[E + m - V\left(r\right)\right]\mathcal{V}_B\left(r\right) + \left[\frac{d}{dr} + \frac{\lambda + 1}{r}\right]\mathcal{U}_B\left(r\right) = 0$$

A equação de Dirac: solução para o potencial central

$$[E - m - V(r)]\,\mathcal{U}_A(r) - \left[\frac{d}{dr} + \frac{\lambda+1}{r}\right]\mathcal{V}_A(r) = 0$$

$$[E + m - V(r)]\,\mathcal{V}_A(r) + \left[\frac{d}{dr} - \frac{\lambda-1}{r}\right]\mathcal{U}_A(r) = 0$$

$$[E - m - V(r)]\,\mathcal{U}_B(r) - \left[\frac{d}{dr} - \frac{\lambda-1}{r}\right]\mathcal{V}_B(r) = 0$$

$$[E + m - V(r)]\,\mathcal{V}_B(r) + \left[\frac{d}{dr} + \frac{\lambda+1}{r}\right]\mathcal{U}_B(r) = 0$$