

# Student understanding of quantum mechanics

Chandralekha Singh<sup>a)</sup>

Department of Physics, University of Pittsburgh, Pittsburgh, Pennsylvania 15260

(Received 27 July 2000; accepted 18 January 2001)

We investigate the difficulties of advanced undergraduate students toward the end of a full year upper-level quantum mechanics course with concepts related to quantum measurements and time development. Our analysis is based upon a test administered to 89 students from six universities and interviews with 9 students. Strikingly, most students shared the same difficulties despite variations in background, teaching styles, and textbooks. Concepts related to stationary states, eigenstates, and time dependence of expectation values were found to be particularly difficult. An analysis of written tests and interviews suggests that widespread misconceptions originate from an inability to discriminate between related concepts and a tendency to overgeneralize. © 2001 American Association of Physics Teachers.

[DOI: 10.1119/1.1365404]

## I. INTRODUCTION

Quantum mechanics is a technically difficult and abstract subject. The subject matter makes instruction quite challenging, and able students constantly struggle to master the basic concepts. In this study, we investigate the difficulties students have with concepts related to quantum measurements and time development. Our analysis is based upon a test that was designed and administered to 89 students from six universities<sup>1</sup> and interviews with 9 students.

The goal of this study is to identify common difficulties that students have about quantum measurements and time development, and determine whether they are correlated with teaching style, place of study, or textbook. We are also interested in comparing difficulties and misconceptions in the upper-level courses with those in the lower-level courses. There is a vast literature<sup>2,3</sup> detailing misconceptions in introductory courses, showing that misconceptions are pervasive and often arise from an incorrect “world-view.” The realm of quantum physics deals with phenomena not directly observable in everyday experience; we want to explore whether the ways in which misconceptions arise in upper-level courses are similar to those in introductory courses, or whether fundamentally different processes are involved.

## II. TEST DESIGN

The students who participated in this study were advanced undergraduates nearing the end of a full year upper-level quantum mechanics course.<sup>4</sup> There have been earlier studies investigating student difficulties with waves<sup>5</sup> and quantum mechanics.<sup>6</sup> Those quantum mechanics investigations concentrated on difficulties related to material covered in a *modern physics* sequence, courses taken as a prerequisite to quantum mechanics or in place of it.<sup>6</sup> Common misconceptions regarding quantum mechanics have also been documented.<sup>7</sup> The present study focuses on quantum measurement and time development, advanced topics covered only in upper-level quantum mechanics courses. Quantum measurement theory is particularly difficult because of the statistical nature of the measurement outcome. Although questions about the foundations of the theory of quantum measurement are still being debated and investigated, at present the Copenhagen interpretation<sup>8</sup> is widely accepted and universally taught to students.

To aid in the design of test questions, three University of Pittsburgh (Pitt) faculty members were consulted, each of whom had recently taught a full year quantum mechanics course. Each faculty member was asked about what he or she considered to be the fundamental concepts in quantum measurements and time development that advanced undergraduate students should know. Many test questions were selected and modified from those used as homework and exam questions that had helped diagnose difficulties.

During the design phase, we went through several iterations of the test with the three Pitt faculty members and two physics postdocs. A preliminary version was administered to students enrolled in quantum mechanics at Duquesne University. After administering the test, there was an extensive discussion in the class, followed by individual discussions with student volunteers. Based upon these discussions, the test was modified before being administered to the students in this study. Appendix A shows the final version of the test,<sup>9</sup> which is slightly improved and revised from the test actually presented to students. Half of the test questions deal with measurements and the other half deal with time development. The test is designed to be administered in one class period (50 min).

The test explores student understanding of a number of important concepts related to quantum measurements and time development: the basic formalism, the special role of energy eigenstates or “stationary states,” the significance of eigenstates of an operator, the calculation of expectation values, and the conditions under which expectation values will be time independent. The test also probes student understanding of how prior measurements affect future measurements, and how the time dependence of spin angular momentum operators compares with operators such as position and linear momentum.

## III. ANALYSIS OF WRITTEN TEST RESULTS AND INTERVIEWS

An analysis of students’ written tests and student interviews shows that most students share a number of common difficulties and misconceptions, despite variations in their backgrounds and the abstract nature of the subject matter.

Table I lists the names of the participating universities, the number of students from each university who took the test, and the textbooks used. Since the number of students from

Table I. The names of participating universities, the number of participating students (most students enrolled in quantum mechanics took the test since it counted toward the course grade in all universities), and authors of the quantum mechanics textbooks used (in some cases, instructors supplemented the text with additional notes).

Name of university	Number of students	Author of quantum textbook
University of Pittsburgh	11	Liboff
Carnegie Mellon University	7	Shankar
Univ. of Illinois, Urbana Champaign	17	Goswami
Boston University	13	Griffiths
Univ. of California, Santa Barbara	34	Griffiths
University of Colorado, Boulder	7	Griffiths

each university is different, we calculate a weighted average of scores (in percent) and standard deviations for each question on the test (see Table II). The concepts that are probed in the test were covered in all of the classes that participated in the study. Students were told in advance that they would take a test on quantum measurements on a set date but were not told about the exact nature of the test. In all of the participating universities, students were given 50 min to take the test and were informed that it counted for one homework grade.

We also conducted audiotaped interviews with nine paid student volunteers from Pitt and analyzed the transcripts for a better understanding of the reasoning involved in answering the questions. We believe their verbal responses echo those of students from the other universities; the written responses clearly reflect the universal nature of the difficulties. Each interview lasted approximately 1 h. The students interviewed were not given the written test earlier because we wanted them to discuss the test without having seen it before.

During the interviews, we provided students with a pen and paper and asked them to “think aloud”<sup>10</sup> while answering the questions. Students first read the questions on their own and answered them without interruptions (except that we prompted them to think aloud if they were quiet for a long time). After students had finished answering a particular question to the best of their ability, we asked them to further clarify and elaborate issues which they had not clearly addressed earlier. This process was repeated for every question on the test. After the interviews, we carefully re-analyzed the written responses and the reasoning provided by the 89 students. Many of the written responses were more easily interpreted after the interviews. The interviews also helped us to gauge the general confidence level of students while responding to a particular question. Often, students seemed

unsure about their responses, had difficulty in discriminating between concepts, and provided conflicting justifications. Some admitted that some of their answers were based upon “gut feeling” or “educated guess.” However, when asked to justify their responses, students sometimes pursued incorrect justifications quite far.

Below, we list each test question, the correct answer, and then students’ written and interviewed responses to that question. We will use the masculine pronoun to refer to all students regardless of their gender. As indicated in Appendix A, for all questions, we refer to a generic observable  $Q$  and its corresponding quantum mechanical operator  $\hat{Q}$ . *It is also noted in Appendix A that for all questions, the Hamiltonian  $\hat{H}$  and operators  $\hat{Q}$  do not depend upon time explicitly.*

Table III lists several common misconceptions evident in student responses to questions (2), (4), and (5). For ease in referring to them, we label the misconceptions (M1)–(M7).

### A. Basic formalism of quantum mechanics

*Question (1):* The eigenvalue equation for an operator  $\hat{Q}$  is given by  $\hat{Q}|\psi_i\rangle = \lambda_i|\psi_i\rangle$ ,  $i = 1, \dots, N$ . Using this information, write a mathematical expression for  $\langle\phi|\hat{Q}|\phi\rangle$ , where  $|\phi\rangle$  is a general state.<sup>11</sup>

*Answer (1):*  $\langle\phi|\hat{Q}|\phi\rangle = \sum_i \langle\phi|\psi_i\rangle^2 \lambda_i$ , or simply  $\sum_i |C_i|^2 \lambda_i$ , where  $C_i = \langle\phi|\psi_i\rangle$ .

Only 43% of students provided the correct response. Some had difficulty with the principle of linear superposition and could not expand a general state in terms of the complete set of eigenstates of an operator. The common mistakes include the following types of answers:

Let

$$|\phi\rangle = \sum_i |\psi_i\rangle,$$

then

$$\langle\phi|\hat{Q}|\phi\rangle = \sum_i \lambda_i, \quad (1)$$

$$\langle\phi|\hat{Q}\left|\sum_i C_i\psi_i\right\rangle = \sum_i \lambda_i C_i \langle\phi|\psi_i\rangle = \sum_i \lambda_i C_i, \quad (2)$$

$$\langle\phi|\hat{Q}|\phi\rangle = \langle\phi|\hat{Q}|\psi\rangle = \langle\phi|\lambda|\psi\rangle = \lambda\langle\phi|\psi\rangle. \quad (3)$$

Let

$$|\phi\rangle = |\psi\rangle,$$

Table II. The percent of correct responses,  $\bar{S}$ , and a weighted standard deviation,  $\sigma$ , for each question on the test. These are defined as  $\bar{S} = \sum_i n_i S_i / N$ , and  $\sigma = \sqrt{\sum_i n_i (\bar{S} - S_i)^2 / N}$ , where the sum  $i$  runs over all the six universities,  $n_i$  is the number of students from the  $i$ th university,  $S_i$  is the average percent score of students from  $i$ th university,  $N = 89$  is the total number of participating students. For questions 5e–5h, the number in parentheses refer to students who wrote “yes” or “no” correctly but either did not justify their answer or gave wrong justification.

Question	1	2	3	4a	4b	5a	5b	5c	5d	5e	5f	5g	5h
$\bar{S}\%$	43	76	83	11	17	95	75	73	73	22	22	13	25
										(29)	(32)	(32)	(55)
$\sigma\%$	14	8	7	8	10	4	12	12	11	12	16	7	12
										(18)	(13)	(13)	(9)

Table III. Common misconceptions of students, the symbols used for ease in referring to them, and the questions to which they relate.

Symbol	Misconception	Question Nos.
M1	If the system is initially in an eigenstate of any operator $\hat{Q}$ , then the expectation value of another operator $\hat{Q}'$ will be time independent if $[\hat{Q}, \hat{Q}'] = 0$ .	(4) and (5)
M2	If the system is initially in an eigenstate of an operator $\hat{Q}$ , then the expectation value of that operator is time independent.	(4) and (5)
M3.1	An eigenstate of any operator is a stationary state.	(2), (4), and (5)
M3.2	If the system is in an eigenstate of any operator $\hat{Q}$ , then it remains in the eigenstate of $\hat{Q}$ forever unless an external perturbation is applied.	
M3.3	The statement “the time dependent exponential factors cancel out in the expectation value” is synonymous with the statement “the system does not evolve in an eigenstate.”	
M4	The expectation value of an operator in an energy eigenstate may depend upon time.	(4) and (5)
M5	If the expectation value of an operator $\hat{Q}$ is zero in some initial state, the expectation value cannot have any time dependence.	(5)
M6	Individual terms ( $\hat{H}_0, \hat{H}_1, \dots$ ) in a time-independent Hamiltonian $\hat{H} = \hat{H}_0 + \hat{H}_1 + \dots$ can cause transitions from one eigenstate of $\hat{H}$ to another.	(5)
M7	Time evolution of an arbitrary state cannot change the probability of obtaining a particular outcome when any observable is measured because the time evolution operator is of the form $\exp(-i\hat{H}t/\hbar)$ .	(5)

Then

$$\langle \psi | \hat{Q} | \psi \rangle = \lambda, \tag{4}$$

$$\langle \phi | \hat{Q} | \phi \rangle = \lambda_i \langle \phi | \phi \rangle = \lambda_i. \tag{5}$$

Six percent of students based their answers on Eq. (1). Nine percent initially expanded the wave function correctly but ended up with an incorrect answer. Six percent of the students did not realize that  $\langle \phi | \psi_i \rangle$  is not unity and  $C_i = \langle \psi_i | \phi \rangle$  and provided a response similar to Eq. (2). Fourteen percent of students wrote  $\lambda$  without any subscript in their final answer similar in spirit to Eqs. (3) and (4). Eq. (3)–Eq. (5) show that when presented with an operator  $\hat{Q}$  and a state  $|\phi\rangle$ , many students assume that the state is an eigenstate of the operator, i.e.,  $\hat{Q}|\phi\rangle = \lambda|\phi\rangle$ , whether it is justified or not. Some students also made mistakes with summation indices. The above responses suggest that many advanced students are uncomfortable with the Dirac formalism and notation, even though it was used in all of the classes in this study.

In the interview, in response to question (1), one student said that “the eigenvalue gives the probability of getting a particular eigenstate” and expanded the state as “ $|\phi\rangle = \sum_i \lambda_i |\psi_i\rangle$ .” Then, he made another mistake by writing the expectation value as “ $\langle \phi | \hat{Q} \sum_i \lambda_i |\psi_i\rangle = \langle \phi | \sum_i \lambda_i^2 |\psi_i\rangle = \sum \lambda_i^2$ .” When asked to explain the final step, he said “ $\sum_i \lambda_i^2$  gets pulled out and this bra and ket states (pointing to the bra and ket explicitly) will give 1.” Another student made the same mistake and contracted different bra and ket vectors to obtain 1. He wrote “ $\langle \phi | \hat{Q} | \sum_n C_n \psi_n \rangle = \sum_n C_n \langle \phi | \hat{Q} | \psi_n \rangle = \sum_n C_n \langle \phi | \lambda_n | \psi_n \rangle = \sum_n C_n \lambda_n \langle \phi | \psi_n \rangle = \sum_n C_n \lambda_n$ .” When asked to explain the final step, he said “ $\psi_n$  will pick out the  $n$ th state from  $\phi$  and give 1 assuming that the states are normalized.” The fact that many students in the written test and interview could retrieve from memory

that a general state  $|\phi\rangle$  can be expanded as  $\sum_n C_n |\psi_n\rangle$ , but did not realize that  $\langle \phi | \psi_n \rangle$  is not unity, shows that students lack a clear understanding of what the expansion  $|\phi\rangle = \sum_n C_n |\psi_n\rangle$  means and that  $C_n = \langle \psi_n | \phi \rangle$  (which implies  $\langle \phi | \psi_n \rangle = C_n^*$ ).

## B. Effect of prior measurements on future measurement and measurements on identically prepared systems

*Question (2):* If you make measurements of a physical observable  $Q$  on a system in rapid succession, do you expect the outcome to be the same every time? Justify your answer.<sup>11</sup>

*Answer (2):* Yes. The first measurement collapses the wave function into an eigenstate of the operator corresponding to the observable being measured. If successive measurements are rapid so that the state of the system does not have the time to evolve, the outcomes will be the same every time.

*Question (3):* If you make measurements of a physical observable  $Q$  on an ensemble of identically prepared systems which are not in an eigenstate of  $\hat{Q}$ , do you expect the outcome to be the same every time? Justify your answer.

*Answer (3):* No, a measurement on a system in a definite state could yield a multitude of results. Therefore, an ensemble of particles prepared in identical states  $|\phi\rangle$  may collapse into different eigenstates  $|\psi_i\rangle$  of  $\hat{Q}$ , yielding different eigenvalues  $\lambda_i$  with probability  $|\langle \psi_i | \phi \rangle|^2$ .

In questions (2) and (3), students might have misunderstood the technical terms “rapid succession” and “identically prepared.” Therefore, regardless of their answer, we considered their response correct if they justified it and showed correct understanding. Students performed relatively well on both questions (2) and (3) with weighted average scores of 76% and 83%, respectively. Therefore, it appears that most advanced students have some idea that the measurement of an observable collapses the wave function into

an eigenstate of the corresponding operator, that a measurement on a system in a definite state could yield a multitude of results, and that prior measurements affect future measurements. Asking additional questions similar to (2) and (3) but for specific systems would provide further insight into the depth of student understanding. In response to question (2), one student wrote “No, for example, from the uncertainty principle in the book there is a 50-50 chance of measuring up and down spins everytime you measure  $S_z$ .” This student seems oblivious of the collapse of the wave function upon measurement of an operator.

In the interview, in response to question (2), one student began with a correct statement: “if you measure  $Q$ , the system will collapse into an eigenstate of that operator. Then, if you wait for a while the measurement will be different.” But then he added incorrectly: “if  $Q$  has a continuous spectrum then the system would gently evolve and the next measurement won’t be very different from the first one. But if the spectrum of eigenvalues is discrete then you will get very different answers even if you did the next measurement after a very short time.” When the student was asked to elaborate, he said: “For example, imagine measuring the position of an electron. It is a continuous function so the time dependence is gentle and after a few seconds you can only go from A to its neighboring point. [Pointing to an  $x$  vs  $t$  graph that he sketches on the paper]...you cannot go from this place to this without going through this intermediate space.” When asked to elaborate on the discrete spectrum case, he said: “...think of discrete variables like spin...they can give you very different values in a short time because the system must flip from up to down. I find it a little strange that such [large] changes can happen almost instantaneously. But that’s what quantum mechanics predicts...” This student had the misconception that successive measurements of continuous variables, e.g., position, produce “somewhat” deterministic outcomes whereas successive measurements of discrete variables, e.g., spin, can produce very different outcomes. This type of response may also be due to the difficulty in reconciling classical and quantum mechanical ideas; in classical mechanics the position of a particle is deterministic and can be unambiguously predicted for all times from the knowledge of the initial conditions and potential.

In response to question (2), one student who had earlier claimed that the system is stuck in an eigenstate unless you apply an external perturbation said, “Yes, once the first measurement is made...the wave function collapses to an eigenstate where it will stay for all times.” Such a response reflects misconception (M3.2) (see Table III) that if the system is in an eigenstate of any operator  $\hat{Q}$ , then it remains in that eigenstate.

In response to question (3), student S1 (we call him student S1 for ease in referring to him later), who appeared not to remember that the wave function collapses into an eigenstate of the operator that is measured, said “If  $Q$  is not in an eigenstate then  $Q|\psi\rangle = \lambda|\psi\rangle$  is not true...so if you measure  $Q$  you won’t be able to get  $\lambda$  and your results will be different every time.” Another student claimed that identically prepared systems should give the same measured value of  $\hat{Q}$ . Even when explicitly told by the interviewer to compare individual measurements made on identically prepared systems with each other and not the expectation values, his answer was unchanged. He said “Barring external influence, if I measure say...the position in two identical systems...I should

get the same position.” The student had apparently forgotten that identically prepared systems can yield different outcomes with probabilities that depend upon the wave function.

### C. Significance of eigenstates, expectation values, and their time dependence

The responses to questions (4) and (5) (below) suggest that only a handful of students have given careful thought to the time dependence of operator expectation values. Very few students appeared to understand that the Hamiltonian plays a crucial role in the time evolution of a state. The general formula for the time dependence of the expectation value of an operator  $\hat{Q}$  (with no explicit time dependence of any operator) is

$$\frac{d}{dt}\langle\phi|\hat{Q}|\phi\rangle = \frac{i}{\hbar}\langle\phi|[\hat{H},\hat{Q}]|\phi\rangle, \quad (6)$$

where  $[\hat{H},\hat{Q}]$  is the commutator of the Hamiltonian  $\hat{H}$  and  $\hat{Q}$ . Two major results can be deduced from Eq. (6). We label them (C1) and (C2) for ease in referring to them later:

- The expectation value of an operator that commutes with the Hamiltonian is time independent regardless of the initial state. (C1)
- If the system is initially in an energy eigenstate, the expectation value of any operator  $\hat{Q}$  will be time independent. (C2)

Students can also deduce (C1) and (C2) by explicitly writing the expectation value of an operator  $\hat{Q}$  at time  $t$  for a system which is initially ( $t=0$ ) in a state  $|\phi_0\rangle$ :

$$\langle\phi(t)|\hat{Q}|\phi(t)\rangle = \langle\phi_0|\exp(i\hat{H}t/\hbar)\hat{Q}\exp(-i\hat{H}t/\hbar)|\phi_0\rangle. \quad (7)$$

From Eq. (7), if  $\hat{Q}$  and  $\hat{H}$  commute, the time evolution operators cancel leading to (C1). Similarly, if  $|\phi_0\rangle$  is an energy eigenstate, Eq. (7) reduces to  $\langle\phi_0|\exp(iE_0t/\hbar)\hat{Q}\times\exp(-iE_0t/\hbar)|\phi_0\rangle = \langle\phi_0|\hat{Q}|\phi_0\rangle$  ( $\hat{H}$  is a Hermitian operator) which implies (C2).

*Question (4):* A particle is in a one-dimensional harmonic oscillator potential. Under what conditions will the expectation value of an operator  $\hat{Q}$  depend on time if

- the particle is initially in a momentum eigenstate.
- the particle is initially in an energy eigenstate.

Justify your answer in both cases.

*Answer (4):* (a) Always, except when  $[\hat{H},\hat{Q}]=0$ , (b) Never (no explicit time dependence of any operator).

This question was the most difficult of all, with only 11% and 17% of the students providing correct responses to questions (4a) and (4b), respectively. Table IV lists statistics pertaining to question (4) in addition to those given in Table II. It shows that only 3% of the students answered both (4a) and (4b) correctly. In question (4a), there is nothing special about the time evolution of the system if the particle is initially in a momentum eigenstate of a one-dimensional harmonic oscillator potential. Therefore, the expectation value of any operator  $\hat{Q}$  will be time independent only if it commutes with the Hamiltonian (i.e., when  $\hat{Q}$  is a conserved quantity). In

Table IV. Additional statistics related to question (4) on the written test.

Student response to question (4)	% of students
Correctly answered both (4a) and (4b)	3
Correctly answered <i>only</i> (4a)	8
Correctly answered <i>only</i> (4b)	13
Condition for time independence of expectation value of $\hat{Q}$	
$[\hat{Q}, \hat{P}] = 0$ in (4a), and $[\hat{Q}, \hat{H}] = 0$ in (4b)	28 <sup>a</sup>
$\hat{Q} = \hat{P}$ (or $f(P)$ ), and $\hat{Q} = \hat{H}$ (or $f(H)$ )	10
$[\hat{H}, \hat{Q}] = 0$ for both (4a) and (4b)	8
$[\hat{H}, \hat{P}] = 0$ for (4a)	3
Expectation value of $\hat{Q}$ does not depend on time in both (4a) and (4b)	
With no reasoning provided	8
With reasoning of the type “always independent of time unless perturbation is added”	9
Explicitly wrote Eq. (6)	
Explicitly wrote Eq. (6) with $\hat{H}$ replaced by $\hat{P}$ for (4a)	7 <sup>b</sup>
	3

<sup>a</sup>Except in 3% cases where students realized that energy eigenstates are stationary, students wrote  $[\hat{Q}, \hat{P}] = 0$  in (4a), and  $[\hat{Q}, \hat{H}] = 0$  in (4b) simultaneously.

<sup>b</sup>Only 1% answered (4b) correctly.

question (4b), the particle is initially in an energy eigenstate of a one-dimensional harmonic oscillator potential. This is a stationary state and the expectation value of any operator  $\hat{Q}$  will be time independent.

Tables III and IV show that 28% of the students held misconception (M1), that if the system is initially in an eigenstate of any operator  $\hat{Q}$ , then the expectation value of a second operator  $\hat{Q}'$  will be time independent if  $[\hat{Q}, \hat{Q}'] = 0$ . Based upon (M1), many students wrote in response to question (4a) that if the system is initially in a momentum eigenstate, any operator  $\hat{Q}$  will have a time independent expectation value if it commutes with the momentum operator  $\hat{P}$ , i.e., if  $[\hat{Q}, \hat{P}] = 0$  [except for 3% students, others also wrote  $[\hat{Q}, \hat{H}] = 0$  in (4b) by similar reasoning]. This belief was so deep-rooted that 3% students wrote explicit formulas similar to Eq. (6) with  $\hat{H}$  replaced by  $\hat{P}$ .

A particularly interesting fact to note is that although the test makes no explicit reference to commutation relations anywhere, 42% of students based their answers on such relations (mostly incorrectly, see Table IV). Appendix B lists incorrect sample student responses to question (4) in which responses 1–4 invoke commutation relations in determining the condition for the time dependence of expectation value of an operator. 7% students (Table IV) explicitly wrote Eq. (6) but only 1% answered (4b) correctly. The rest were focused on the commutation relation, and did not pay attention to the state. They wrote  $[\hat{H}, \hat{Q}] = 0$  as the condition for time independence of expectation value in both (4a) and (4b), despite writing Eq. (6) correctly. Some students had difficulty recalling whether the Hamiltonian should commute with  $\hat{Q}$  or the operator whose eigenstate the system is initially in.

Table IV shows that at least 38% of students held misconception (M2), and believed that if the system is in an eigenstate of an operator  $\hat{Q}$ , then the expectation value of that operator is time independent [note that (M2) is also a special case of (M1)]. Based upon (M2), many wrote in response to question (4a) that when the system is initially in a momentum eigenstate, only the momentum operator can have a time-independent expectation value [by similar faulty reasoning,  $\hat{Q}$  must necessarily be the Hamiltonian in (4b)].

For many students, (M2) was related to (M3), which actually can be divided into related misconceptions (M3.1), (M3.2), and (M3.3). Misconception (M3.1) addresses the difficulty many students have in distinguishing between energy eigenstates and eigenstates of other operators. They believed that an eigenstate of *any* operator is a stationary state. A related misconception (M3.2) is summarized well in this remark by a student: “If the system is in an eigenstate of  $\hat{Q}$ , then the system remains in the eigenstate of  $\hat{Q}$  forever unless an external perturbation is applied.” Written reasonings and interviews both show that many students also believe that the statements “time-dependent phase factors cancel out from the expectation value” and “the system does not evolve in an eigenstate” are equivalent [misconception (M3.3)]. Seventeen percent of the students wrote that there is never any time dependence in either (4a) or (4b) and 9% explicitly provided misconception (M3) as their reasoning (others provided no reasoning, see Table IV). One student exclaimed with surprise when the test was discussed afterwards: “Oh, so only the energy eigenstates are stationary states. I thought that stationary states refer to eigenstates of any operator because the system does not evolve in an eigenstate.” Misconceptions (M3.1) and (M3.2) are both evident in this statement.

Seventeen percent of students answered (4b) correctly, and another 17% answered “No” to both (4a) and (4b) [see (M3) above and Table IV]. The rest held misconception (M4) and believed that the expectation value of an operator in an energy eigenstate may depend on time. They treated an energy eigenstate as a general state while determining the condition for the time dependence of an operator’s expectation value. They apparently forgot the meaning of “stationary states,” a concept that is usually introduced earlier in the course. They did not remember that energy eigenstates evolve in time via a simple phase factor, and that the expectation value of an operator  $\hat{Q}$  in such a state is *time independent regardless of*  $[\hat{H}, \hat{Q}]$ .

In the interview, two students said that the expectation values in question (4) will be time independent only if  $\hat{Q} = \hat{P}$  for part (a), and  $\hat{Q} = \hat{H}$  for part (b) [misconception (M2)]. When asked to elaborate, one said: “...if the system is in an eigenstate...it is stuck there...and if successive measurement of that operator is performed...it obviously will yield the same result independent of time.” This kind of reasoning is consistent with the written responses. The student did not understand that only *sufficiently rapid* measurements of the same operator yield the same result unless energy is measured repeatedly in the energy eigenstate because the system in an eigenstate of a general operator indeed evolves in time in a nontrivial manner.

*Question (5):* Questions (a)–(b) refer to the following system. An electron is at rest in an external magnetic field  $B$  which is pointing in the  $z$  direction. The Hamiltonian for this

system is given by  $\hat{H} = -\gamma B \hat{S}_z$ , where  $\gamma$  is the gyromagnetic ratio and  $\hat{S}_z$  is the  $z$  component of the spin angular momentum operator.<sup>11</sup>

Notation:  $\hat{S}_z|\uparrow\rangle = \hbar/2|\uparrow\rangle$ , and  $\hat{S}_z|\downarrow\rangle = -\hbar/2|\downarrow\rangle$

For reference, the unnormalized eigenstates of  $\hat{S}_x$  and  $\hat{S}_y$  are given by

$$\hat{S}_x(|\uparrow\rangle \pm |\downarrow\rangle) = \pm \hbar/2(|\uparrow\rangle \pm |\downarrow\rangle),$$

$$\hat{S}_y(|\uparrow\rangle \pm i|\downarrow\rangle) = \pm \hbar/2(|\uparrow\rangle \pm i|\downarrow\rangle).$$

- (a) If you measure  $S_z$  in a state  $|\chi\rangle = (|\uparrow\rangle + |\downarrow\rangle)/\sqrt{2}$ , what are the possible results, and what are their respective probabilities?
- (b) If the result of the first measurement of  $S_z$  was  $\hbar/2$ , and you immediately measure  $S_z$  again, what are the possible results, and what are their respective probabilities?
- (c) If the result of the first measurement of  $S_z$  was  $\hbar/2$ , and you immediately measure  $S_x$ , what are the possible results, and what are their respective probabilities?
- (d) What is the expectation value  $\langle \hat{S}_z \rangle$  in the state  $|\chi\rangle = (|\uparrow\rangle + |\downarrow\rangle)/\sqrt{2}$ ?
- (e) If the electron is initially in an eigenstate of  $\hat{S}_x$ , does the expectation value of  $\hat{S}_y$  depend on time? Justify your answer.
- (f) If the electron is initially in an eigenstate of  $\hat{S}_x$ , does the expectation value of  $\hat{S}_z$  depend on time? Justify your answer.
- (g) If the electron is initially in an eigenstate of  $\hat{S}_z$ , does the expectation value of  $\hat{S}_x$  depend on time? Justify your answer.
- (h) If the electron is initially in an eigenstate of  $\hat{S}_z$ , does the expectation value of  $\hat{S}_z$  depend on time? Justify your answer.

*Answer (5):* (a)  $\hbar/2$  with probability 1/2, and  $-\hbar/2$  with probability 1/2, (b)  $\hbar/2$  with probability 1, (c)  $\hbar/2$  with probability 1/2, and  $-\hbar/2$  with probability 1/2, (d) Zero, (e) Yes, because  $[\hat{H}, \hat{S}_y] \neq 0$ , (f) No, because  $[\hat{H}, \hat{S}_z] = 0$ , (g) and (h) No, because eigenstates of  $\hat{S}_z$  are stationary states.

Students performed well on questions (5a)–(5d), with weighted average scores of 95%, 75%, 73%, and 73%, respectively. In response to question (5b), the most common mistake was “ $\hbar/2$  and  $-\hbar/2$  each with a probability 1/2” while in question (5c), the most common mistake was “ $\hbar/2$  with a probability 1.” Incorrect responses to question (5d) about the expectation value  $\langle \hat{S}_z \rangle$  include  $\hbar/2$ ,  $\hbar^2/2$ ,  $\hbar^2/4$ ,  $\sin(\omega t)$ .

Both written responses and student interviews suggest that some students did not understand the difference between the expectation value and an individual measurement of  $\hat{S}_z$  and thought that questions (5a) and (5d) are the same. Also, many students who incorrectly answered question (5b) also answered question (2) incorrectly since they are similar. The written justifications in question (2) made it clear that they did not understand that prior measurements affect future measurements.

Table V. Additional statistics related to questions (5e)–(5h) on the written test.

Student response to questions (5e)–(5h)	% of students
Correctly answered all of (5e)–(5h)	7
Condition for time independence of expectation value of $\hat{Q}$	
If initial state is eigenstate of $\hat{Q}'$ , $[\hat{Q}, \hat{Q}'] = 0$ , e.g., $[\hat{S}_x, \hat{S}_y] = 0$ in (5e)	20 <sup>a</sup>
If initial state is eigenstate of $\hat{Q}'$ , $[\hat{H}, \hat{Q}'] = 0$ , e.g., $[\hat{H}, \hat{S}_x] = 0$ in (5e)	3 <sup>a</sup>
At least for parts of (5e)–(5h) invoked Larmor precession of spins	10
All four parts correct	2
Suggested that the static magnetic field, $B$ , will cause transitions between energy levels [Typically answered “yes” in (5h) since $B$ is in $z$ direction and often in (5f) and (5g)]	10
Answered “No” to all of (5e)–(5h)	
Reasoning	
Eigenstates do not evolve	6
$\langle \phi_{S_i}   \hat{S}_j   \Phi_{S_i} \rangle = 0$ if $i \neq j$ , and constant if $i = j$ where $ \phi_{S_i}\rangle$ is an eigenstate of $ \hat{S}_i\rangle$	6
If expectation value is zero in initial state, it cannot depend on time.	
Spin operators do not depend on time	3
None or other reasoning	3
Comparison of questions (4) and (5e)–(5h)	
Answered “No” to both (4a)–(4b) but not to <i>all</i> of (5e)–(5h)	6
Answered “No” to both (4a)–(4b) and also to <i>all</i> of (5e)–(5h)	6

<sup>a</sup>For at least some of questions (5e)–(5h), this reasoning was invoked.

The weighted average scores on questions (5e)–(5h) were between 15% and 25%. Table V shows the statistics related to these questions in addition to those in Table II. Only 7% of students answered all of these questions correctly. Appendix C shows examples of the incorrect reasoning used in the written test for these questions. A majority of the incorrect responses to questions (5e)–(5h) were similar to those for question (4) (see Tables III and V, and Appendix C) and were based upon misconceptions (M1)–(M4). However, there were additional misconceptions related to (5e)–(5h). Twenty percent of the students explicitly displayed misconception (M1), while 18% answered “No” to all of (5e)–(5h) (see Table V and Appendix C). The reason for the latter varied widely. Some attributed it to “eigenstates do not evolve” [misconception (M3)]. Others said that the expectation values of spin operators cannot depend on time (they confused the spin operators’ position and linear momentum independence with their time independence). Others noted that  $\langle \phi_{S_i} | \hat{S}_j | \phi_{S_i} \rangle = 0$  if  $i \neq j$ , and constant if  $i = j$ , where  $|\phi_{S_i}\rangle$  is an eigenstate of  $\hat{S}_i$ . They argued that if the expectation value is zero in the initial state, it cannot depend on time [misconception (M5)].

As noted in Table V and Appendix C (see responses 8 and 9), 10% of the students explicitly displayed misconception (M6). They believed that individual terms ( $\hat{H}_0, \hat{H}_1, \dots$ ) in a time-independent Hamiltonian  $\hat{H} = \hat{H}_0 + \hat{H}_1 + \dots$  can cause

transitions from one eigenstate of  $\hat{H}$  to another. In particular, they thought that the presence of magnetic field in  $\hat{H}$  will lead to transitions between different eigenstates of  $\hat{H}$ . Also, 10% of students based their answers on Larmor precession of spin (see Table V and responses 6 and 7 in Appendix C). However, only 2% students out of that could answer all of (5e)–(5h) correctly.

In the interview, in response to questions (4) and (5), three students at some point or other explicitly displayed misconception (M1) and cited  $[\hat{Q}, \hat{Q}'] = 0$  as the condition for time independence. When asked to elaborate one student said “...it has to do with the uncertainty principle...there is no uncertainty in measuring two operators that commute...so eigenstate of one is also eigenstate of another.” Another student said something similar: “...if two operators commute...they can have simultaneous eigenfunctions.” When asked by the interviewer to explain what that has to do with the time dependence of the expectation value of  $\hat{Q}$ ; the student thought for a moment and responded, pointing to question (4a): “since eigenstates do not change...the expectation value of  $P$  is time independent...if  $P$  and  $Q$  commute then expectation value of both will be time independent.” This pattern of reasoning is consistent with the written responses. One of the students later expressed concern about this claim but remained unsure about what needed to be done.

Transcribed below are excerpts from interviews (by interviewer I) with two students (S2 and S3) who were asked to elaborate on their incorrect claim that the expectation value of any operator is always time independent if the system is initially in any eigenstate [misconception (M3)]:

S2: ...tell me how can it [the system] get out of an eigenstate on its own? ...how can there be time dependence in this situation?

I: Can you show that by writing down the expectation value explicitly?

S2: (does not write anything but thinks for a moment)...I remember that in an eigenstate the expectation value has  $\psi^2$  and there is no time dependence.

Note the confidence and insistence of student S3 below:

S3: ...in an eigenstate the system cannot evolve...so when you measure something you get the same thing...I mean the expectation value of any operator will be time independent.

I: Can you show that by writing down the expectation value explicitly?

S3: (writes down)  $\langle \phi | Q | \phi \rangle$

I: Where is the time dependence?

S3: there isn't...because  $\phi$  is an eigenstate.

I: Can you write down the time dependence for a general state?

S3: it is something like...(writes down)  $\exp(-iHt)|\phi\rangle$

I: Why does this not apply here?

S3: (thinks)...you can write it like this if you want...but if you substitute the exponential in the expectation value...it will cancel out and you will get what I told you earlier.

Thus, student S3 was so confident that the time evolution operators cancel in the expectation value, that he did not feel the need to insert it in the expression for the expectation

value, even when explicitly questioned. Also, S3 initially stated that an eigenstate does not evolve. Later, when explicitly asked to show the time dependence of a general state, he wrote an almost correct time evolution operator. Yet, he claimed that he had been correct all along because the expectation value is unaffected by it. Similar to this claim, written responses and other interviews show misconception (M3.3), the fact that many students cannot make a distinction between “the time-dependent exponential factors cancel out in the expectation value” versus “the system does not evolve in an eigenstate.” They invoked one or the other indiscriminately often in the same answer and when asked for clarification claimed that both statements are equivalent. Many relied on their memorized knowledge about energy eigenstates. Most students who claimed that the time-dependent phase factors cancel were as confident as S3 about the validity of their claim. They may be confusing eigenstates of a general operator with energy eigenstates and using the term “the system does not evolve” because the absolute value of the wave function is time independent in an energy eigenstate. From the type of response provided by S3 it also appears that some students may not remember that  $\hat{H}$  is not a number but an operator and it may not commute with  $\hat{Q}$ .

Except for one interviewed student, none could discriminate between eigenstates of the Hamiltonian  $\hat{H}$  and eigenstates of other operators [misconception (M3)]. Two of the interviewed students used the word “stationary state” explicitly to refer to eigenstates of any operator. One student who thought that any eigenstate is stationary said: “...I remember from class that there are two conditions for a state to be a stationary state...One is that all eigenstates are stationary and...at the moment I forget the other [condition].” A discussion with this student after the interview revealed that his instructor had discussed that there were two general conditions for the expectation value of an operator to be time independent [(C1) and (C2)]. The student was mistaking them as two conditions for a state to be stationary.

On the written test we had noted that some students treated questions (4) and (5) very differently [misconception (M6), see Tables III and V]. In the interview, one student clearly applied a similar differential treatment for the two. When asked to elaborate on the difference between questions (4) and (5), he said: “These two are very different systems...harmonic oscillator is an isolated system so if the system is in an eigenstate it stays there...[pointing to question (5)] this is a dynamic problem because there is an external perturbation...harmonic oscillator is not [dynamic]...generally speaking there will be transitions from one eigenstate to another [in question (5)] and the expectation value of  $Q$  will depend on time.” The student was incorrectly attributing the time dependence of the expectation value in Eq. (6) to electrons making transitions from one energy eigenstate to another due to a “static” magnetic field in the Hamiltonian [misconception (M6)]. Written responses are consistent with this student’s assertion.

In questions (5e)–(5h), one interviewed student argued that the expectation value is zero when the initial state is not an eigenstate of the operator whose time dependence of expectation value is desired. His argument was along the lines that, in question (5e) for example, all eigenstates of  $\hat{S}_x$  are orthogonal to all eigenstates of  $\hat{S}_y$  (which is actually not true although  $\langle \hat{S}_y \rangle$  is zero in the initial state which is an eigen-

state of  $\hat{S}_x$ ). The interviewer reminded him that the eigenstate of  $\hat{S}_x$  is only the initial state and he had to find the time dependence of the expectation value of  $\hat{S}_y$  when the initial state was an eigenstate of  $\hat{S}_x$ . The student immediately responded: *“I understand that...[but] since the expectation value is zero in the initial state...so is its time dependence.”* This type of justification is reminiscent of a common misconception in introductory physics that if the velocity of a particle is zero, so is its acceleration.

A misconception that was somewhat difficult to detect from the written responses to question (5) alone (although we find evidence for it after the interviews) is that the time evolution of an arbitrary state cannot change the probability of obtaining a particular outcome when any observable is measured because the time evolution operator is of the form  $\exp(-i\hat{H}t/\hbar)$  [misconception (M7)]. In response to questions (5e)–(5g), one student said that if the initial state is not an eigenstate of the operator whose expectation value we want to calculate, the expectation value is zero. When asked to elaborate, the student correctly noted that if the electron is initially in an eigenstate of  $\hat{S}_x$ , the expectation value of  $\hat{S}_y$  is zero because when we write eigenstates of  $\hat{S}_x$  in terms of eigenstates of  $\hat{S}_y$ , the probabilities of finding an up spin and a down spin are the same. However, when he was reminded that the eigenstate of  $\hat{S}_x$  is only an initial state, the student insisted *“...but the time dependent factors are just exponentials...they cannot change the probability of getting up and down spins.”* The student waited for a moment and continued, *“hmmm...it does look a bit strange though that these expectation values will always be zero. Now I am confused whether only  $\pm\hbar/2$  are allowed [eigenvalues for electron spin] or the whole spectrum between  $\pm\hbar/2$  is allowed. (waits)...Maybe the whole spectrum is allowed...then the expectation value will be non-zero.”* The student appeared so certain that the probability of obtaining  $\pm\hbar/2$  is unaffected by the time evolution of the system that he incorrectly started to speculate that every value between  $\pm\hbar/2$  is allowed when a spin component is measured. It is true that the time evolution will not change the probability of collapsing a general state into different energy eigenstates. However, it will change the probability of collapsing a general state into eigenstates of a general operator. For example, if the system is initially in an eigenstate of  $\hat{S}_x$ , the time evolution will change the probability of collapsing into different eigenstates of  $\hat{S}_y$  in problem (5).

In response to question (5e), one interviewed student who we referred to as S1 earlier said: *“ $S_y$  will evolve in time because it cannot get into an eigenstate...because  $S_x$  is in an eigenstate and the uncertainty principle disallows both being known exactly...”* He continued: *“same is true for (5f) and (5g) but since there is no contradiction with uncertainty principle in (5h), it won’t evolve.”* We asked him if the operators could evolve in time in the Schrödinger formalism. He pointed to (5e) and said *“Oh...No...I meant the wave function is not evolving because it is in an eigenstate of  $S_x$ ...so it shouldn’t matter what you measure... $S_x$  or  $S_y$  or  $S_z$ ...since the eigenstate will be unchanged...[therefore]  $S_y$  and  $S_z$  will be time-dependent...as for  $S_x$ ...it will be a constant since wave function is in its eigenstate.”* This student

appears to have forgotten about the collapse of a wave function into an eigenstate of  $\hat{Q}$  when  $\hat{Q}$  is measured.

As noted earlier,<sup>9</sup> question (5i) in Appendix A was not administered in the written test. However, we included it in the interviews because it directly addresses misconception (M2). In the written test, 80% had answered “No” in response to question (5h), which appears to be correct on the surface, but 55% did not provide the correct reasoning (see responses 15–18 in Appendix C). The latter displayed misconceptions (M1)–(M4), and did not base their answer on the fact that the eigenstates of  $\hat{S}_z$  are stationary states. In the interview, in response to question (5i) (for which the correct response is “Yes,” because  $[\hat{H}, \hat{S}_x] \neq 0$ ) seven out of nine interviewed students answered “No.” As we had suspected, most of them incorrectly claimed that (5h) and (5i) were very similar based upon misconceptions (M1) (citing  $[\hat{S}_x, \hat{S}_x] = 0$  as the condition for time independence) and (M2).

#### IV. DISCUSSION

Although students in advanced quantum mechanics courses may have learned to solve the Schrödinger equation with complicated potentials and boundary conditions, many have difficulties with conceptual understanding of quantum measurements and time development.<sup>12</sup> The written test and interviews reveal a number of common misconceptions (M1)–(M7) (see Tables III–V). Students often used conflicting justifications throughout the test, and there was a lack of discrimination between related concepts. Most students did not remember that energy eigenstates or stationary states play a special role in quantum mechanics, and thought that eigenstates of any operator have the same properties as energy eigenstates.

Students performed much worse on the time development aspects of the test (7 of the 14 test questions) than on the pure measurement aspects. Our investigation shows that most students’ knowledge about time development is fragmented. Most students did not understand that the Hamiltonian plays a crucial role in the time development of the system. A majority did not remember the significance of operators that commute with the Hamiltonian. Many students could only remember that there was a commutation relation involved in the time dependence of expectation value. The few who remembered Eq. (6) did not realize that for energy eigenstates, every operator has a time-independent expectation value. Few textbooks explicitly discuss that if  $|\phi\rangle$  is an energy eigenstate, the time independence of the expectation value of any operator  $\hat{Q}$  follows from Eq. (6) or Eq. (7). Before the written test was administered, some instructors noted that questions (5e)–(5h) should be relatively easy for their students because they were either worked out in class while discussing Larmor precession of spins or they were assigned as homework. However, they turned out not to be easy for students.

The detected knowledge deficiencies can be broadly divided into three levels with increased difficulty in overcoming them:<sup>13</sup> (I) lack of knowledge related to a particular concept, (II) knowledge that is retrieved from memory but cannot be interpreted correctly, (III) knowledge that is retrieved and interpreted at the basic level but cannot be used to draw inferences in specific situations. Our investigations show evidence that for advanced students, the difficulties with quantum measurement and time development concepts



were due to the deficiencies spread across all three levels. Since many of the questions required students to predict the outcomes of experiments, they necessitated a transition from the mathematical representation to a concrete case. This task requires that students interpret and draw qualitative inferences from quantitative tools [e.g., in question (5), does  $\langle \hat{S}_y \rangle$  depend on time if the system is initially in an eigenstate of  $\hat{S}_x$  for an electron in a static magnetic field  $B_z$ ?]. Therefore, deficiencies at levels II and III were frequently observed. Students in general had difficulty in differentiating between related concepts. For example, some claimed that there is no time dependence in all of questions (5e)–(5h) because the expectation value of  $\hat{Q}$  is zero in the initial state (similar inability to distinguish between velocity and acceleration is common in introductory courses). In question (1), some students were unable to expand a general state in terms of the eigenstates of an operator (deficiency level I). They either lacked the knowledge including knowledge about Dirac formalism or could not retrieve it from memory (i.e., they might have recognized the correct relationship if shown explicitly). Others could not interpret the information retrieved from memory at a later stage (deficiency levels II and III). For example, some students who correctly invoked  $|\phi\rangle = \sum_n C_n |\psi_n\rangle$  did not realize that  $C_n = \langle \psi_n | \phi \rangle$  and at a later step thought that  $\langle \phi | \psi_n \rangle$  is unity. This type of lack of consistency is reminiscent of the response of introductory students (e.g., when asked about the acceleration of a baseball after it has left the bat, most students correctly retrieve from memory that the acceleration of a projectile is  $9.8 \text{ m/s}^2$  vertically downward, but when asked about the force on the baseball after it has left the bat, many believe that the initial force by the bat should be added to the gravitational force to obtain the net force<sup>14</sup>).

A significant finding is that most students had *common* difficulties and misconceptions about quantum measurements and time development, despite their varied background and the abstract nature of the subject matter. This finding is strikingly similar to the “universal” nature of misconceptions documented for introductory physics courses.<sup>2,3</sup> In introductory courses, many misconceptions are really deep-rooted *preconceptions*, fitting in well with students’ overall views about the world. It is therefore very difficult to correct them with traditional instruction; and studies have shown that they reappear within a short time.<sup>2,3</sup> The misconceptions about quantum measurements cannot properly be called preconceptions because students are introduced to novel concepts during the course, and they do not explicitly encounter relevant phenomena in everyday experience. Some misconceptions, e.g., that successive measurements on continuous variables produce somewhat deterministic outcomes, may be due to the difficulty in reconciling the classical concepts learned earlier with the quantum concepts. *The analysis of test results suggests that the widespread misconceptions originate largely from students’ inability to discriminate between related concepts and a tendency to overgeneralize.* A preconception, e.g., that motion implies force, can often be viewed as an overgeneralization, e.g., of the observation that motion implies force for an object initially at rest in a reference frame. The contrasting feature is that in introductory courses, overgeneralizations are often inappropriate extrapolations of everyday experiences, whereas in the context of

quantum mechanics they are inappropriate extrapolations of concepts learned in one context during the course (or previous courses) to another.

## ACKNOWLEDGMENTS

We are very grateful to Professor M. Crommie, Professor J. Fetkovich, Professor C. Y. Fong, Professor L. Greene, Professor E. Gwinn, Professor J. Levy, and Professor S. Pollock for administering the test to their students. We also gratefully acknowledge helpful discussions with Professor F. Reif, Professor J. Levy, Professor M. Vincent, Professor A. Janis, Professor P. Shepard, Professor Y. Goldschmidt, Professor E. Newman, Professor J. Boudreau, M. Niedermaier, and R. Tate. The work is supported in part by the National Science Foundation.

## APPENDIX A: QUANTUM MEASUREMENT TEST

For the questions below, we refer to a generic observable  $Q$  and its corresponding quantum mechanical operator  $\hat{Q}$ . For all of the questions, the Hamiltonian and operators  $\hat{Q}$  do not depend upon time explicitly.

(1) The eigenvalue equation for an operator  $\hat{Q}$  is given by  $\hat{Q}|\psi_i\rangle = \lambda_i |\psi_i\rangle$ ,  $i = 1, \dots, N$ . Write an expression for  $\langle \phi | \hat{Q} | \phi \rangle$ , where  $|\phi\rangle$  is a general state, in terms of the projections  $\langle \phi | \psi_i \rangle$ .

(2) If you make measurements of a physical observable  $Q$  on a system in immediate succession, do you expect the outcome to be the same every time? Justify your answer.

(3) If you make measurements of a physical observable  $Q$  on an ensemble of identically prepared systems which are not in an eigenstate of  $\hat{Q}$ , do you expect the outcome to be the same every time? Justify your answer.

(4) A particle is in a one-dimensional harmonic oscillator potential. Under what conditions will the expectation value of an operator  $\hat{Q}$  depend on time if

- (a) the particle is initially in a momentum eigenstate.
- (b) the particle is initially in an energy eigenstate.

Justify your answer in both cases.

(5) Questions (a)–(i) refer to the following system. An electron is in a uniform magnetic field  $B$  which is pointing in the  $z$  direction. The Hamiltonian for the spin degree of freedom for this system is given by  $\hat{H} = -\gamma B \hat{S}_z$  where  $\gamma$  is the gyromagnetic ratio and  $\hat{S}_z$  is the  $z$  component of the spin angular momentum operator.

Notation:  $\hat{S}_z |\uparrow\rangle = \hbar/2 |\uparrow\rangle$ , and  $\hat{S}_z |\downarrow\rangle = -\hbar/2 |\downarrow\rangle$ .

For reference, the unnormalized eigenstates of  $\hat{S}_x$  and  $\hat{S}_y$  are given by

$$\hat{S}_x (|\uparrow\rangle \pm |\downarrow\rangle) = \pm \hbar/2 (|\uparrow\rangle \pm |\downarrow\rangle),$$

$$\hat{S}_y (|\uparrow\rangle \pm i|\downarrow\rangle) = \pm \hbar/2 (|\uparrow\rangle \pm i|\downarrow\rangle).$$

- (a) If you measure  $S_z$  in a state  $|\chi\rangle = (|\uparrow\rangle + |\downarrow\rangle)/\sqrt{2}$ , what are the possible results, and what are their respective probabilities?

- (b) If the result of the first measurement of  $S_z$  was  $\hbar/2$ , and you immediately measure  $S_z$  again, what are the possible results, and what are their respective probabilities?
- (c) If the result of the first measurement of  $S_z$  was  $\hbar/2$ , and you immediately measure  $S_x$ , what are the possible results, and what are their respective probabilities?
- (d) What is the expectation value  $\langle \hat{S}_z \rangle$  in the state  $|\chi\rangle = (|\uparrow\rangle + |\downarrow\rangle)/\sqrt{2}$ ?
- (e) If the electron is initially in an eigenstate of  $\hat{S}_x$ , does the expectation value of  $\hat{S}_y$  depend on time? Justify your answer.
- (f) If the electron is initially in an eigenstate of  $\hat{S}_x$ , does the expectation value of  $\hat{S}_z$  depend on time? Justify your answer.
- (g) If the electron is initially in an eigenstate of  $\hat{S}_z$ , does the expectation value of  $\hat{S}_x$  depend on time? Justify your answer.
- (h) If the electron is initially in an eigenstate of  $\hat{S}_z$ , does the expectation value of  $\hat{S}_z$  depend on time? Justify your answer.
- (i) If the electron is initially in an eigenstate of  $\hat{S}_x$ , does the expectation value of  $\hat{S}_x$  depend on time? Justify your answer.

#### APPENDIX B: EXAMPLES OF INCORRECT WRITTEN RESPONSES TO QUESTION (4)

1. (4a) Always, unless  $[Q,P]=0$ , i.e., measurement of a commuting operator is conducted (fundamental theorem of QM) [analogous reasoning with  $[Q,H]=0$  for (4b)].
2. (4a) If  $[Q,P]=0$  no spreading of wave function in time because particle has well-defined momentum [analogous reasoning with  $[Q,H]=0$  for (4b)].
3. (4a) Unless  $[Q,P]=0$ , i.e.,  $Q$  measures something whose eigenstate is not shared with momentum [analogous reasoning with energy for (4b)].
4. (4a)–(4b) if  $[H,Q]=0$  → constant of motion.
5. (4a) If  $Q$  is not a momentum operator [analogous reasoning with Hamiltonian for (4b)].
6. (4a) Measuring anything but momentum or velocity [analogous reasoning with energy for (4b)].
7. (4a)–(4b) Never, it's in an eigenstate, expectation values are constant.
8. (4a)–(4b) If the wave function is time dependent.
9. (4a)–(4b) If spontaneous transition to other energy levels occur through time.
10. (4a)–(4b) Only if the potential is changing with time.
11. (4a)–(4b) Unless  $Q$  is a constant.
12. (4a)–(4b) If  $Q$  has more than one eigenstate.

#### APPENDIX C: EXAMPLES OF INCORRECT WRITTEN RESPONSES TO QUESTION (5e)–(5h)

1. (5e) Yes,  $[S_x, S_y] \neq 0$  since no common eigenstates [Yes for (5f) and (5g) and No for (5h) by analogous reasoning].
2. (5e)–(5g) Yes, if electron is initially in an eigenstate of  $S_x$ ,  $\langle S_y \rangle$  will be time dependent because  $S_y$  is in a different spin basis than  $S_x$  [(5h) No, by analogous reasoning].

3. (5e) Yes,  $S_x$  eigenstates are superposition of  $S_y$  eigenstates → we expect  $\langle S_y \rangle$  to be time dependent [(5f) and (5g) Yes, (5h) No, by analogous reasoning].

4. (5e)–(5h) No, each eigenstate of  $S_i$  is an equal superposition of eigenstates of another operator  $S_j$  so the expectation value won't change with time. However, other observables, e.g., momentum, position, etc., would.

5. (5e)  $\langle \chi^x | S_y | \chi^x \rangle = 0$  → always time independent [analogous reasoning for (5f) and (5g)].

6. (5e) Since  $\psi$  has collapsed onto eigenstate of  $S_x$ ,  $\langle S_y \rangle$  cannot depend on time [analogous reasoning for (5f)–(5h)].

7. (5e)–(5h) No, it won't get out of eigenstate.

8. (5e)–(5h) All questions time independent because expectation value cannot change unless state is changing (here eigenstate). Note: Even if the wave function is time dependent, it would cancel out with its complex conjugate and time dependence would go away.

9. (5e)–(5g) Yes, must precess. (5h) No, no need to precess.

10. (5g) Yes, the magnetic moment precesses about the field so although system is in an eigenstate of  $S_z$ , the amount of  $\langle S_x \rangle$  and  $\langle S_y \rangle$  will change with time.

11. (5e) No, magnetic field is only in the  $z$  direction so electron is not influenced in  $x$  direction by it and stay in the eigenstate. (5f)–(5h) Yes, Magnetic field will serve to align the spin of electrons in  $z$  direction.

12. (5f) Yes,  $B$  field in the  $Z$  direction will bring system out of initial eigenstate → cause transition and make  $d\langle S_z \rangle / dt \neq 0$  [No for (5e) and Yes for (5g) and (5h) by analogous reasoning].

13. (5f) No, I got this wrong in the last exam and I am still not sure.

14. (5g)  $[H, S_x] \propto [S_z, S_x] \neq 0$  →  $\langle S_x \rangle$  time dependent.

15. (5h) No, this is obvious since  $[S_z, S_z] = 0$ .

16. (5h) No, once in an eigenstate of an operator, future measurements of that operator won't change the state.

17. (5h) No, it has already been observed to be in  $S_z$  eigenstate so it will stay there.

18. (5h) No, by definition eigenstate is a state of an operator which does not change in time. So the eigenvalues and expectation values do not change in time.

<sup>a</sup>Electronic mail: singh@bondi.phyast.pitt.edu

<sup>1</sup>The test was actually administered in seven universities but we do not report the data from the University of California Davis, where it was given to graduate students.

<sup>2</sup>A. B. Arons, *A Guide to Introductory Physics Teaching* (Wiley, New York, 1990).

<sup>3</sup>L. C. McDermott, "Millikan Lecture 1990: What we teach and what is learned—Closing the gap," *Am. J. Phys.* **59**, 301–315 (1991); F. Reif, "Millikan Lecture 1994: Understanding and teaching important scientific thought processes," *ibid.* **63**, 17–32 (1995).

<sup>4</sup>Students took the test at the end of a two-semester sequence in quantum mechanics except those from the University of California Santa Barbara who took it at the end of the second quarter in a three-quarter sequence.

<sup>5</sup>S. Vokos, P. S. Shaffer, B. S. Ambrose, and L. C. McDermott, "Student understanding of the wave nature of matter: Diffraction and interference of particles," *Phys. Ed. Res. Suppl. to Am. J. Phys.* **68**, S42–S51 (2000).

<sup>6</sup>Research on teaching and learning quantum mechanics (Papers presented at the annual meeting of National Association for Research in Science Teaching, D. Zollman editor, March 1999), available <http://www.phys.ksu.edu/perg/papers/narst/>; G. J. Aubrecht II, D. B. May, T. J. Kassebaum, and J. Stith, "Revising a questionnaire about quantization and the photon," *Summer AAPT Announcer* **29**, 93 (1999); M. C. Wittman, R. Steinberg, B. Lei, and E. F. Redish, "Curriculum development to address student difficulties with models of conductivity," *ibid.* **29**, 94 (1999); M.

Thoresen and D. Zollman, "Tutorials for introductory quantum mechanics," *ibid.* **29**, 84 (1999).

<sup>7</sup>D. F. Styer, "Common misconceptions regarding quantum mechanics," *Am. J. Phys.* **64**, 31–34 (1996); **64**, 1202 (1996).

<sup>8</sup>N. D. Mermin, "Is the moon there when nobody looks? Reality and the quantum theory," *Phys. Today* **38**, 38–47 (April 1985); J. Bell, *Speakable and Unspeakable in Quantum Mechanics* (Cambridge U.P., Cambridge, 1987).

<sup>9</sup>The order of questions in the written test administered to students was (1), (4), (5), (2), (3). The last line of question (1) was "Using this information, write a mathematical expression for  $\langle \phi | \hat{Q} | \phi \rangle$ , where  $|\phi\rangle$  is a general state" instead of the present wording in Appendix A. In question (2) the word "rapid" has been changed to "immediate" and in question (5) the wording "An electron at rest..." has been changed to "The Hamiltonian for the spin degree of freedom" in the third line. The test did not have question (5i). It had an additional question about the mathematical origin of the uncertainty relation in quantum measurements. Students approached

this question in several ways, and many alternatives are valid in their own right. We do not include this question in our analysis here.

<sup>10</sup>M. T. H. Chi, "Thinking Aloud," in *The Think Aloud Method*, edited by M. W. van Someren, Y. F. Barnard, and J. A. C. Sandberg (Academic, London, 1994), Chap. 1.

<sup>11</sup>This is the wording that was actually used and is slightly different from the revised wording in Appendix A.

<sup>12</sup>D. F. Styer, "The motion of wavepackets through their expectation values and uncertainties," *Am. J. Phys.* **58**, 742–744 (1990); Instructive simulations by D. F. Styer related to time development can be found in "Quantum mechanical time development" published as part of *Quantum Mechanics Simulations*, by J. Hiller, I. Johnston, and D. Styer (Wiley, New York, 1994).

<sup>13</sup>F. Reif (private communications).

<sup>14</sup>D. Hestenes, M. Wells, and G. Swackhamer, "Force concept inventory," *Phys. Teach.* **30**, 141–151 (1992).