

# An algebraic method for solving central force problems

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A simple algebraic method, which is as easy to use as the angular momentum algebra, is demonstrated as a pedagogical way to solve certain central force problems exactly. Solutions for the hydrogen atom and the three-dimensional isotropic harmonic oscillator are presented together with a discussion of the limits of the method. © 2002 American Association of Physics Teachers.

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## I. INTRODUCTION

The central force problem,

$$H|Elm\rangle = E|Elm\rangle, \quad (1)$$

where

$$H = \frac{p^2}{2m} + \frac{l(l+1)\hbar^2}{2mr^2} + V(r), \quad (2)$$

and  $p = p_r$  is a familiar one in quantum physics. It covers the Coulomb potential (Fig. 1), which describes the hydrogen atom, as well as the three-dimensional isotropic harmonic oscillator potential (Fig. 2), which is used as an approximation to the strong force independent-particle mean field in nuclei. Finding exact solutions to these problems by differential methods is tedious. We outline an algebraic way<sup>1</sup> of solving for the energy eigenvalues and eigenfunctions that is more elegant than the traditional differential methods, and also explore the limits of this method.

## II. BUILDING THE ALGEBRA

A well-known example of an algebraic solution to a standard quantum mechanical problem is the solution to the angular momentum problem,

$$L^2|\lambda\mu\rangle = \lambda|\lambda\mu\rangle, \quad (3)$$

$$L_z|\lambda\mu\rangle = \mu|\lambda\mu\rangle, \quad (4)$$

which uses the commutator brackets

$$[L_x, L_y] = i\hbar L_z, \quad (5)$$

$$[L_y, L_z] = i\hbar L_x, \quad (6)$$

$$[L_z, L_x] = i\hbar L_y, \quad (7)$$

and yields  $\lambda = l(l+1)\hbar^2$ ,  $\mu = m\hbar$  ( $m = l, l-1, \dots, 1, 0, -1, \dots, -l$ ). These commutator brackets, or Lie products, define the Lie algebra<sup>2</sup>  $so(3)$ .

The algebraic solution to (1) can be achieved using the commutator bracket,

$$[r, p^m] = i\hbar m p^{m-1}, \quad (8)$$

and the Lie products that result for the operators  $r^a$ ,  $r^b p$ , and  $r^c p^2$ . [We only consider powers of  $p$  up to  $p^2$  because that is the highest power of  $p$  in Eq. (1).] From Eq. (8) we obtain

$$[r^a, r^b p] = r^b [r^a, p] = r^b (i\hbar a r^{a-1}) = i\hbar a r^{a+b-1}. \quad (9)$$

Just as  $L_x$ ,  $L_y$ , and  $L_z$  form the closed algebra of Eqs. (5)–(7), we want the Lie products of  $r^a$ ,  $r^b p$ , and  $r^c p^2$  to form a

closed algebra. Thus,  $r^{a+b-1}$  must be equal to  $r^a$ , whence  $b = 1$  and  $[r^a, r p] = i a \hbar r^a$ . Similarly, we obtain for the other commutator brackets,

$$[r^a, r^{2-a} p^2] = a(a-1)\hbar^2 + 2ia\hbar r p, \quad (10)$$

$$[r p, r^{2-a} p^2] = i\hbar a r^{2-a} p^2, \quad (11)$$

where  $c = 2 - a$  is chosen so that the algebra closes. Note that these brackets do not depend in any way on the choice of a potential. Indeed, we choose the remaining degree of freedom,  $a$ , to fit our algebra to the potentials we solve.

We introduce the change of variables, viz.,

$$V_1 = r^a, \quad (12)$$

$$V_2 = \frac{1}{a} [r p - \frac{1}{2} i (a-1) \hbar], \quad (13)$$

$$V_3 = \frac{1}{a^2} r^{2-a} p^2. \quad (14)$$

The commutator brackets, Eqs. (9), (10), and (11), can now be written as

$$[V_1, V_2] = i\hbar V_1, \quad (15)$$

$$[V_2, V_3] = i\hbar V_3, \quad (16)$$

$$[V_3, V_1] = -2i\hbar V_2. \quad (17)$$

A subtle and key extension of this algebraic structure is realized<sup>3</sup> by noting that  $a$  can take on both positive and negative values in Eq. (12), and therefore that  $r^{-a} = V_1^{-1}$  yields, from Eqs. (9) and (13),

$$[V_2, V_1^{-1}] = i\hbar V_1^{-1}. \quad (18)$$

Therefore, from Eqs. (16) and (18) we obtain

$$[V_2, (V_3 + \tau V_1^{-1})] = i\hbar (V_3 + \tau V_1^{-1}), \quad (19)$$

where  $\tau$  is a constant or any operator that commutes with  $V_1$ ,  $V_2$ , and  $V_3$ . That is, the algebra of Eqs. (15)–(17) is unchanged by the replacement of  $V_3$  with  $V_3 + \tau V_1^{-1}$ . With one last linear combination (extension) of the algebra, viz.,

$$T_1 = \frac{1}{2} (V_3 + \tau V_1^{-1} - V_1), \quad (20)$$

$$T_2 = V_2, \quad (21)$$

$$T_3 = \frac{1}{2} (V_3 + \tau V_1^{-1} + V_1), \quad (22)$$

we obtain the commutator algebra,

$$[T_1, T_2] = -i\hbar T_3, \quad (23)$$

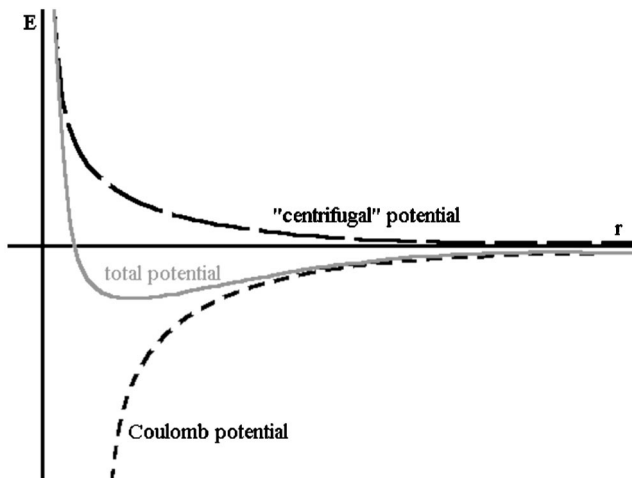


Fig. 1. The hydrogen atom—the Coulomb potential is the dotted line, the “centrifugal” potential is the dashed line, and the total potential is the solid line.

$$[T_2, T_3] = i\hbar T_1, \quad (24)$$

$$[T_3, T_1] = i\hbar T_2. \quad (25)$$

These are reminiscent of the angular momentum commutator brackets mentioned in Eqs. (5)–(7). Equations (23)–(25) are identical to Eqs. (5)–(7) except for the sign difference between Eqs. (5) and (23). The Lie algebra described by  $L_x$ ,  $L_y$ , and  $L_z$  is  $so(3)$ , whereas the algebra described by  $T_1$ ,  $T_2$ , and  $T_3$  is  $so(2,1)$ . We learn a great deal about this algebra by comparing it with our knowledge of angular momentum.

### III. A COMPARISON OF $so(3)$ AND $so(2,1)$

To compare the algebras  $so(2,1)$  and  $so(3)$ , we write them in the condensed form

$$[T_1, T_2] = i\gamma\hbar T_3, \quad (26)$$

$$[T_2, T_3] = i\hbar T_1, \quad (27)$$

$$[T_3, T_1] = i\hbar T_2, \quad (28)$$

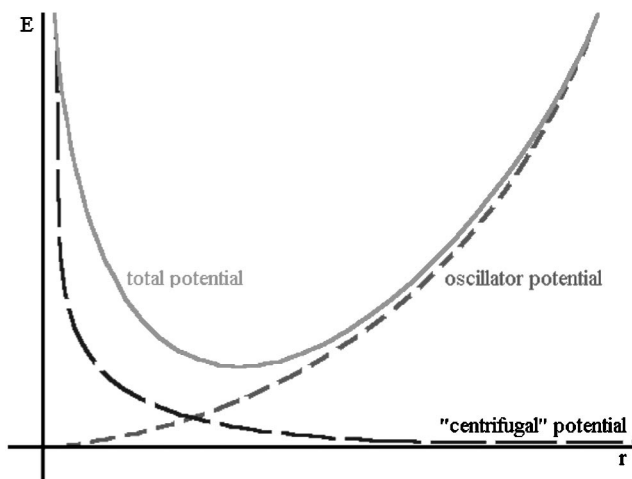


Fig. 2. The 3D harmonic oscillator—the oscillator potential is the dotted line, the “centrifugal” potential is the dashed line, and the total potential is the solid line.

where  $\gamma = +1$  for  $so(3)$  and  $\gamma = -1$  for  $so(2,1)$ . Using these equations enables us to translate much of our knowledge of  $so(3)$  into  $so(2,1)$ . For example, the raising and lowering operators are  $T_{\pm} = T_1 \pm iT_2$ , and produce

$$[T_+, T_-] = 2\gamma\hbar T_3, \quad (29)$$

$$[T_3, T_{\pm}] = \pm\hbar T_{\pm}, \quad (30)$$

$$\begin{aligned} T^2 &= \gamma(T_1^2 + T_2^2) + T_3^2 = \gamma T_+ T_- + T_3^2 - \hbar T_3 \\ &= \gamma T_- T_+ + T_3^2 + \hbar T_3, \end{aligned} \quad (31)$$

$$[T^2, T_{\kappa}] = 0, \quad \kappa = 1, 2, 3. \quad (32)$$

Simultaneous eigenkets of  $T^2$  and  $T_3$  exist and obey

$$T^2|Qq\rangle = Q|Qq\rangle, \quad (33)$$

$$T_3|Qq\rangle = q|Qq\rangle, \quad (34)$$

$$T_3 T_{\pm}|Qq\rangle = (q \pm \hbar) T_{\pm}|Qq\rangle. \quad (35)$$

Thus the  $T_{\pm}$  operators perform ladder operations on the eigenvectors of  $T_3$ , and therefore the eigenvalue–eigenvector spectrum of  $T_3$  is obtained. So, to find the constraints on the eigenvalues, we consider

$$\begin{aligned} \langle Qq|(T^2 - T_3^2)|Qq\rangle &= Q - q^2, \\ &= \frac{\gamma}{2} \langle Qq|(T_+ T_- + T_- T_+)|Qq\rangle. \end{aligned} \quad (36)$$

By rewriting Eq. (36) using

$$T_+|Qq\rangle = |\chi\rangle, \quad (37)$$

$$T_-|Qq\rangle = |\psi\rangle, \quad (38)$$

we obtain<sup>4</sup>

$$\frac{1}{2} \langle Qq|T(T_+ T_- + T_- T_+)|Qq\rangle = \langle \chi|\chi\rangle + \langle \psi|\psi\rangle \geq 0, \quad (39)$$

which means, for  $\gamma = +1$ ,

$$Q - q^2 \geq 0, \quad (40)$$

$$q \leq \sqrt{Q}. \quad (41)$$

Equation (41) is the result we expect for  $so(3)$ . The eigenvalues of  $T_3$  ( $L_z$ ) are bounded above and below, creating a range of values for  $q(m_l)$ . (Moreover, from these bounds it follows that  $m_l = l, l-1, \dots, 1, 0, -1, \dots, -l$ .) However, for  $\gamma = -1$ ,

$$Q - q^2 \leq 0, \quad (42)$$

$$q \geq \sqrt{Q}. \quad (43)$$

Either  $q$  has a lower bound or an upper bound, but not both. [Because of the infinite nature of the eigenvalues,  $so(2,1)$  is called a noncompact algebra.] We choose for  $q$  to have a lower bound; the motivation for this choice will become evident later. We define the lowest eigenstate as

$$T_-|Qq_0\rangle = 0, \quad (44)$$

and find

$$\begin{aligned} T^2|Qq_0\rangle &= (-T_+ T_- + T_3^2 - \hbar T_3)|Qq_0\rangle \\ &= (q_0^2 - q_0\hbar)|Qq_0\rangle = q_0(q_0 - \hbar)|Qq_0\rangle. \end{aligned} \quad (45)$$

Just as the irreducible representations<sup>5</sup> of  $so(3)$  are labeled by  $l$  and the eigenvalues of  $L^2$  are  $l(l+1)$ , so the irreducible representations of  $so(2,1)$  are labeled by  $q_0$ , and the eigenvalues of  $T^2$  are  $q_0(q_0 - \hbar)$ . The only difference between

the irreps of the two groups is that the irreducible representations of  $so(3)$  consist of a finite number of states, whereas the irreducible representations of  $so(2,1)$  consist of an infinite number of states.

To illuminate the nature of  $q_0$ , consider

$$\begin{aligned} T^2 &= -T_1^2 - T_2^2 + T_3^2 = (T_3 - T_1)(T_3 + T_1) - [T_3, T_1] - T_2^2 \\ &= V_1(V_3 + \tau V_1^{-1}) - i\hbar V_2 - V_2^2. \end{aligned} \quad (46)$$

Then, using

$$V_2^2 = \frac{1}{a^2} \left[ r^2 p^2 - ia\hbar r p - \left( \frac{a-1}{2} \hbar \right)^2 \right] \quad (47)$$

[from Eqs. (8) and (13)], and Eqs. (12), (13), and (14), we can simplify Eq. (46) to

$$T^2 = \tau + \frac{\hbar^2}{4a^2} (1 - a^2). \quad (48)$$

From

$$\langle Qq_0 | T^2 | Qq_0 \rangle = q_0(q_0 - \hbar), \quad (49)$$

we obtain

$$q_0^2 - q_0 \hbar - \left[ \tau + \frac{\hbar^2}{4a^2} (1 - a^2) \right] = 0, \quad (50)$$

which implies that

$$q_0 = \frac{\hbar}{2} \left( 1 \pm \sqrt{\frac{4\tau}{\hbar^2} + \frac{1}{a^2}} \right). \quad (51)$$

#### IV. WAVE FUNCTIONS

To find the ground state wave function, consider

$$(T_- - T_3) | Qq_0 \rangle = -q_0 | Qq_0 \rangle. \quad (52)$$

Using  $T_- = T_1 - iT_2$  and Eqs. (20), (21), (22), we obtain

$$(V_1 + iV_2 - q_0) | Qq_0 \rangle = 0, \quad (53)$$

which simplifies to

$$\left[ r^a + \frac{irp}{a} + \frac{\hbar}{2} \frac{a-1}{a} - q_0 \right] | Qq_0 \rangle = 0. \quad (54)$$

If we express Eq. (54) in the position representation [ $r \rightarrow \alpha^{-1}r, p \rightarrow -i\hbar \alpha d/dr$ ,  $|Qq\rangle \rightarrow \Psi_0(r)$ ], we obtain the differential equation

$$r \frac{d\psi_0(r)}{dr} + \left[ \frac{a}{\hbar} \left( \frac{r}{\alpha} \right)^a + \frac{a-1}{2} - \frac{aq_0}{\hbar} \right] \psi_0(r) = 0, \quad (55)$$

where  $\alpha$  permits a scaling of the position coordinate. It directly follows that Eq. (55) is separable, and thus  $\Psi_0(r)$  can be written as

$$\psi_0(r) = Ar^C e^{-(1/\hbar)(r/\alpha)^a}, \quad (56)$$

where  $C = aq_0 \hbar^{-1} - 1/2(a-1)$ . The substitution of  $q_0$  from Eq. (51) gives

$$C = \frac{1}{2} \left[ 1 \pm \sqrt{\frac{4a^2\tau}{\hbar^2} + 1} \right]. \quad (57)$$

The excited state wave functions can be obtained using

$$[T_+ - (T_3 - q_n)] \psi_n(r) = \kappa_{n+1} \psi_{n+1}(r), \quad (58)$$

where  $\kappa_n$  is a normalization constant and  $q_n = q_0 + n_r \hbar$ . Through a process similar to that leading from Eqs. (52) to (55), we write Eq. (58) as

$$\begin{aligned} r \frac{d\psi_n(r)}{dr} + \left[ -\frac{a}{\hbar} \left( \frac{r}{\alpha} \right)^a + \frac{a-1}{2} + \frac{aq_n}{\hbar} \right] \psi_n(r) \\ = \kappa_{n+1} \psi_{n+1}(r). \end{aligned} \quad (59)$$

The actual calculation of the excited state wave functions (and of their normalization constants) is left as an exercise for the reader. Note that we have proceeded thus far by treating Eq. (2) as a one-dimensional Hamiltonian. However, because radial integrals have an extra factor of  $r^2$  compared with one-dimensional integrals, the three-dimensional wave functions are related to the one-dimensional wave functions we have found, viz.,

$$\psi_n(r) \equiv r \phi_n(r), \quad (60)$$

where  $\phi_n(r)$  is the radial portion of the wave function that solves the original three-dimensional problem, and  $\Psi(r)$  is the wave function that solves the simplified one-dimensional radial problem stated in Eq. (2).

#### V. THE LIMITS OF $so(2,1)$ AS APPLIED TO CENTRAL FORCE PROBLEMS

All of the problems under consideration are based upon the connection between the  $so(2,1)$  operator  $T_3$  and the Hamiltonian operator, as follows:

$$(T_3 - q_n) = \alpha r^\beta (H - E). \quad (61)$$

If we expand each side of Eq. (61) and use Eqs. (20), (21), and (22), we obtain

$$\begin{aligned} \frac{1}{2} \left( \frac{1}{a^2} r^{2-a} p^2 + \frac{\tau}{r^a} + r^a \right) - q_n \\ = \alpha r^\beta \left[ \frac{p^2}{2m} + \frac{l(l+1)\hbar^2}{2mr^2} + V(r) - E \right]. \end{aligned} \quad (62)$$

Immediately the terms in  $p^2$  can be equated, and the result is  $\alpha = ma^{-2}$  and  $\beta = 2 - a$ . Therefore, Eq. (62) reduces to

$$\frac{1}{2} \left( \tau - \frac{l(l+1)\hbar^2}{a^2} \right) + \frac{r^{2a}}{2} - r^a q_n - \frac{m}{a^2} r^2 V(r) + \frac{m}{a^2} r^2 E = 0. \quad (63)$$

We can make a power series expansion of  $r^2 V(r)$  in  $r$ , but the only terms with nonzero coefficients will be terms of the same powers of  $r$  that we see in Eq. (63), except for an  $r^2$  term (we do not wish to build the energy  $E$  into our potential).

Thus, by inspection we write

$$r^2 V(r) = A + Br^{2a} + Dr^a. \quad (64)$$

If we substitute Eq. (64) into Eq. (63), we find

$$\begin{aligned} \frac{1}{2} \left( \tau - \frac{l(l+1)\hbar^2}{a^2} - \frac{m}{a^2} A \right) + \left( \frac{1}{2} - \frac{m}{a^2} B \right) r^{2a} \\ - \left( q_0 + \frac{m}{a^2} D \right) r^a + \frac{m}{a^2} r^2 E = 0. \end{aligned} \quad (65)$$

It is impossible for the above equality to hold for all values of  $a$  without  $E$  being identically zero. However, for certain choices of  $a$ , one of the other terms will cancel the  $r^2 E$  term.

The first term, having no  $r$  dependence at all, cannot provide this cancellation. This leads to two possible cases:  $a = 1$  and  $a = 2$ .

By rearranging Eq. (64), viz.,

$$V(r) = \frac{A}{r^2} + Br^{2a-2} + Dr^{a-2}, \quad (66)$$

we see that  $A = 0$  and  $a = 1$  gives the Coulomb potential and  $A = 0$  and  $a = 2$  gives the harmonic oscillator potential. Note that the *only* other possible nonrelativistic central force problems solvable with the  $so(2,1)$  algebra are the modified Coulomb potential (where  $A \neq 0$ ), and the Davidson<sup>6</sup> potential [the three-dimensional (3-D) harmonic oscillator with  $A \neq 0$ ].

## VI. THE HYDROGEN ATOM

The energy eigenvalue equation for the hydrogen atom is

$$\left[ \frac{p^2}{2\mu} - \frac{e^2}{4\pi\epsilon_0 r} + \frac{l(l+1)\hbar^2}{2\mu r^2} - E \right] |Elm_l\rangle = 0. \quad (67)$$

If we multiply through by  $\mu\alpha r$ , we obtain

$$\left[ \frac{\alpha r p^2}{2} - \frac{\mu\alpha e^2}{4\pi\epsilon_0} + \frac{\alpha l(l+1)\hbar^2}{2r} - \mu\alpha r E \right] |Elm_l\rangle = 0. \quad (68)$$

The substitution  $R$  for  $\alpha^{-1}r$  and  $P$  for  $\alpha p$  in Eq. (68) produces

$$\left[ \frac{RP^2}{2} - \frac{\mu e^2 \alpha}{4\pi\epsilon_0} + \frac{l(l+1)\hbar^2}{2R} - \mu\alpha^2 R E \right] |Elm_l\rangle = 0. \quad (69)$$

By using Eqs. (12), (13), and (14), we rewrite Eq. (69) as

$$\frac{1}{2} \left[ V_3 - \frac{\mu e^2 \alpha}{2\pi\epsilon_0} + l(l+1)\hbar^2 V_1^{-1} - 2\mu\alpha^2 V_1 E \right] |Elm_l\rangle = 0, \quad (70)$$

where  $a = 1$  so that the powers of  $R$  match. If we examine Eq. (70) closely, we see that three terms match the definition of  $T_3$  in Eq. (22). In fact, if

$$\tau = l(l+1)\hbar^2, \quad (71)$$

$$2\mu\alpha^2 E = -1, \quad (72)$$

$$q = \frac{\mu e^2 \alpha}{4\pi\epsilon_0}, \quad (73)$$

then Eq. (70) reduces to

$$[T_3 - q] |elm_l\rangle = 0. \quad (74)$$

Note that the values of  $q$  are the eigenvalues of  $T_3$  and, as we saw in Sec. III, should be indexed by and increase from the eigenvalue  $q_0$ . From Eq. (51) we deduce that

$$q_0 = \frac{\hbar}{2} (1 \pm (2l+1)). \quad (75)$$

Because we want  $q_0$  to be positive, we take the positive sign, and therefore obtain

$$q_0 = \hbar(l+1), \quad (76)$$

$$T^2 = l(l+1)\hbar^2. \quad (77)$$

Thus, if we label increments of  $q$  by  $n_r \hbar$ , we find

$$q = q_0 + n_r \hbar = (l+1)\hbar + n_r \hbar = \frac{\mu e^2 \alpha}{4\pi\epsilon_0}, \quad (78)$$

or

$$\alpha = \frac{4\pi\epsilon_0 \hbar}{\mu e^2} [(l+1) + n_r]. \quad (79)$$

Finally,

$$E = -\frac{\mu e^4}{32\pi^2 \epsilon_0^2 \hbar^2} \frac{1}{(l+1+n_r)^2}. \quad (80)$$

With the substitution  $n = l+1 + n_r$ , we get the familiar energy levels of the hydrogen atom,

$$E = -\frac{Ry}{n^2}. \quad (81)$$

Note that here the irreducible representations of  $so(2,1)$  consist of energy eigenstates of a particular angular momentum, and therefore each irreducible representation is labeled by a value of angular momentum. The raising and lowering operators,  $T_{\pm}$ , change the energy eigenstate within a particular irreducible representation, but do not move between irreducible representations. In other words, we can use  $T_{\pm}$  to change  $n$ , but it does not change  $l$ . Another interesting point is that the Hamiltonian does not commute with all of the elements of  $so(2,1)$ ; unlike  $so(3)$ ,  $so(2,1)$  is not a symmetry group of the hydrogen atom. Instead, because the Hamiltonian is simply related to one of the generators,  $so(2,1)$  is called a dynamical symmetry group of the hydrogen atom.

We immediately write the wave functions using Eq. (56), viz.,

$$\psi_0(r) = Ar^C e^{-r/\alpha\hbar}. \quad (82)$$

If we substitute for  $\alpha$  from Eq. (79), we obtain

$$\psi_0(r) = Ar^C e^{-\mu e^2 r / 4\pi\epsilon_0 \hbar^2 n}, \quad (83)$$

which simplifies to

$$\psi_0(r) = Ar^C e^{-r/na_0}, \quad (84)$$

where  $a_0$  is the Bohr radius. From Eqs. (57) and (71) we calculate

$$C = \frac{1}{2} [1 \pm \sqrt{4l(l+1)+1}], \quad (85)$$

and taking the positive value, as we did before, we obtain

$$C = l+1. \quad (86)$$

Finally, recalling Eq. (60), we obtain the ground state wave functions for each irreducible representation,

$$\phi_0(r) = Ar^l e^{-r/na_0}. \quad (87)$$

The energy spectrum is depicted in Fig. 3, along with the action of  $T_+$  and  $T_-$ .

## VII. THE THREE-DIMENSIONAL ISOTROPIC HARMONIC OSCILLATOR

The 3-D isotropic harmonic oscillator can be solved in much the same way as the hydrogen atom. First, the energy eigenvalue equation is

$$\left[ \frac{p^2}{2m} + \frac{1}{2} m \omega^2 r^2 + \frac{l(l+1)\hbar^2}{2mr^2} - E \right] |Elm_l\rangle = 0. \quad (88)$$

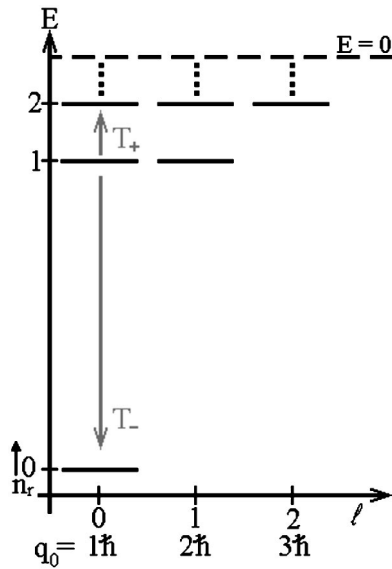


Fig. 3. The hydrogen atom energy spectrum—the raising and lowering operators move between states in an irrep labelled by 1.

Then, if we multiply through by  $\beta^2/4$  and substitute  $R = \beta^{-1}r$  and  $P = \beta p$ ,

$$\left[ \frac{P^2}{8m} + \frac{1}{8}m\omega^2\beta^4R^2 + \frac{l(l+1)\hbar^2}{8mR^2} - \frac{\beta^2E}{4} \right] |Elm_l\rangle = 0, \quad (89)$$

$$\left[ \frac{1}{2} \left( \frac{P^2}{4m} + \frac{1}{4}m\omega^2\beta^4R^2 + \frac{l(l+1)\hbar^2}{4mR^2} \right) - \frac{\beta^2E}{4} \right] |Elm_l\rangle = 0. \quad (90)$$

We can again use Eqs. (12), (13), and (14), and the condition that  $a=2$  to obtain

$$\left[ \frac{1}{2m} \left( V_3 + \frac{1}{4}m^2\omega^2\beta^4V_1 + \frac{l(l+1)\hbar^2}{4V_1} \right) - \frac{\beta^2E}{4} \right] |Elm_l\rangle = 0. \quad (91)$$

Then if

$$\frac{1}{4}m^2\omega^2\beta^4 = 1, \quad (92)$$

$$\tau = \frac{l(l+1)\hbar^2}{4}, \quad (93)$$

we find that the eigenvalue equation simplifies to

$$\left[ T_3 - \frac{\beta^2mE}{4} \right] |Elm_l\rangle = 0. \quad (94)$$

The values for the energy can be obtained from the eigenvalues of  $T_3$ , once we have determined the values for  $q_0$ ,

$$\begin{aligned} q_0 &= \frac{1}{2} \left( \hbar \pm \sqrt{4\tau + \frac{\hbar^2}{a^2}} \right) = \frac{1}{2} \left( \hbar \pm \hbar \sqrt{l(l+1) + \frac{1}{4}} \right) \\ &= \frac{1}{2} (\hbar \pm \hbar(l + \frac{1}{2})). \end{aligned} \quad (95)$$

In anticipation of positive energy values, we use the positive sign, whence

$$q_0 = \frac{\hbar}{2} \left( l + \frac{3}{2} \right). \quad (96)$$

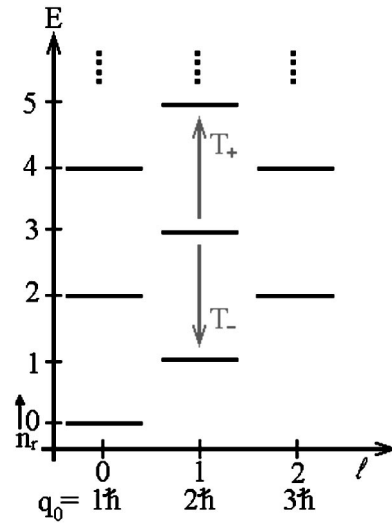


Fig. 4. The 3D harmonic oscillator energy spectrum—the raising and lowering operators move between states in an irrep labelled by 1.

Therefore,

$$E = (2n_r + l + \frac{3}{2})\hbar\omega, \quad (97)$$

or, for  $N \equiv 2n_r + l$ ,

$$E = (N + \frac{3}{2})\hbar\omega. \quad (98)$$

As before, we can immediately write the wave functions using Eq. (56), viz.,

$$\psi_0(r) = Ar^C e^{-r^2/2b_0^2}, \quad (99)$$

and from Eq. (92) we obtain

$$\psi_0(r) = Ar^C e^{-m\omega r^2/2\hbar}, \quad (100)$$

or

$$\psi_0(r) = Ar^C e^{-r^2/2b_0^2}, \quad (101)$$

where  $b_0 = (\hbar/m\omega)^{1/2}$  is a characteristic length of the oscillator. From Eqs. (57) and (93) we calculate

$$C = \frac{1}{2} [1 \pm \sqrt{4l(l+1) + 1}], \quad (102)$$

and, taking the positive value, we obtain

$$C = l + 1. \quad (103)$$

Finally, considering Eq. (60), we obtain the ground state wave functions for each irreducible representation,

$$\phi_0(r) = Ar^l e^{-r^2/2b_0^2}. \quad (104)$$

The energy spectrum is depicted in Fig. 4, along with the action of  $T_+$  and  $T_-$ .

## VIII. CLOSING REMARKS

We find the foregoing both rewarding and limiting. Limiting because the exactly solvable cases are few<sup>7</sup> (but not trivial). Rewarding because it introduces a range of new algebraic concepts in a way that is accessible to students who have mastered the angular momentum algebra. Indeed, these algebraic concepts reveal a simple underlying unity to the exactly solvable cases, which is not evident when using other methods.

For the adventurous student who wishes to make a more in-depth study of algebraic methods as applied to familiar



quantum mechanical problems, we note (with no attempt at completeness) the following selections: Adams,<sup>8</sup> de Lange and Raab,<sup>9</sup> and Frank and van Isacker.<sup>10</sup>

We are also aware of two introductory texts (Harris and Loeb<sup>11</sup> and Ohanian<sup>12</sup>) that introduce (other) algebraic techniques for solving central force problems. Although we encourage the student to look at these texts, we point out that although the techniques are simply defined (they involve factorization of the radial Schrödinger equation), they are not familiar structures. Again, for the adventurous student, we note that these structures can be classified as supersymmetric or as isospectral, details of which are developed, for example, in de Lange and Raab's book<sup>9</sup> and in an introductory text by Schwabl.<sup>13</sup> Suggested problems for students are given in the Appendix.

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## APPENDIX: SUGGESTED PROBLEMS FOR STUDENTS

(1) Show for the hydrogen atom that

(a)

$$\begin{aligned}\psi_0(r) &= \left[ \frac{2^k}{(l+1)^k a_0^k (k-1)!} \right]^{1/2} r^{l+1} e^{-r/a_0(l+1)} \\ &= r \phi_0(r) \equiv r R_{n,l=n-1}(r);\end{aligned}$$

(b)

$$\begin{aligned}\psi_1(r) &= \left[ \frac{2^k(l+1)}{(l+2)^k a_0^k (k-2)!} \right]^{1/2} \\ &\quad \times \left( 1 - \frac{r}{(l+1)(l+2)a_0} \right) r^{l+1} e^{-r/a_0(l+1)} \\ &= r \phi_1(r) \equiv r R_{n,l=n-2}(r).\end{aligned}$$

[Hint: Equations (59), (79), (84), and (86) can be used with the precaution that the value of  $n_r$  should be for the final (raised) state.]

(2) Show for the three-dimensional isotropic harmonic oscillator that

(a)

$$\phi_0(r) = b_0^{-3/2} \left[ \frac{2^{l+2}}{\pi^{1/2}(2l+1)!!} \right]^{1/2} \left( \frac{r}{b_0} \right)^l e^{-r^2/2b_0^2};$$

(b)

$$\begin{aligned}\phi_1(r) &= b_0^{-3/2} \left[ \frac{2^{l+2}}{2\pi^{1/2}(2l+3)!!} \right]^{1/2} \left[ 2l+3 - \frac{2r^2}{b_0^2} \right] \\ &\quad \times \left( \frac{r}{b_0} \right)^l e^{-r^2/2b_0^2};\end{aligned}$$

(c)

$$\begin{aligned}\phi_n(r) &= b_0^{-3/2} \left[ \frac{2^{l+2}}{2^{n-1}\pi^{1/2}(n-1)!(2l+2n-1)!!} \right]^{1/2} \\ &\quad \times G_{nl} \left( \frac{r}{b_0} \right) \left( \frac{r}{b_0} \right)^l e^{-r^2/2b_0^2},\end{aligned}$$

where

$$G_{nl}(x) = \sum_{k=0}^{n-1} \frac{(-1)^k 2^k (n-1)!(2l+2n-1)!!}{(n-k-1)!k!(2l+2k+1)!!} x^{2k}.$$

Note:  $(2l+1)!! \equiv (2l+1)(2l-1)(2l-3)\cdots(5)(3)(1)$ .

(3) Calculate the energy eigenvalue spectrum and the normalized ground state wave functions of the Davidson potential, as given by

$$V_{\text{Davidson}} = \frac{A}{r} + \frac{B}{r^2}.$$

(Hint: Proceed in a manner similar to the development of the hydrogen atom given in the text.)

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<sup>1</sup>We are entirely indebted to a paper by J. Cizek and J. Paldus, "An algebraic approach to bound states of simple one-electron systems," *Int. J. Quantum Chem.* **12**, 875–896 (1977), for this idea.

<sup>2</sup>Although we name the algebraic features that arise in the present work by their technical names, this is done to acquaint the reader with terminology and does not require previous exposure to these terms. For an introduction to Lie algebras that would be suitable for the application to physics problems, we suggest H. J. Lipkin, *Lie Groups for Pedestrians* (North Holland, Amsterdam, 1965).

<sup>3</sup>This realization was made by Cizek and Paldus, Ref. 1.

<sup>4</sup>A subtle issue arises here in the definition of a Hermitian inner product, for example, the formation of  $\langle \chi | \chi \rangle$  from  $|\chi\rangle = T_+ |Qq\rangle$  and  $\langle \chi | = \langle Qq | T_-$ . The Hermitian inner product for any operator  $\mathbf{O}$  that can be expressed in terms of  $V_1$ ,  $V_2$ , and  $V_3$  [Eqs. (12)–(14)], is given by  $\langle \psi | \mathbf{O} | \chi \rangle = \int \psi^*(r^{a-2}\mathbf{O})\chi d\Omega$ , where the integration extends over the whole three-dimensional (physical) space. Although the point is one of considerable sophistication, in practical terms it means that  $r^{a-2}\mathbf{O}$ , where  $\mathbf{O}$  is any function of  $V_1$ ,  $V_2$ , and  $V_3$ , is self-adjoint. The point is unfamiliar in much of standard quantum mechanics, where the operators describing the system are Hermitian.

<sup>5</sup>An irrep (or irreducible representation) is, in the context of this present discussion, a set of eigenvectors that are related to each other by the action of the ladder operators  $T_{\pm}$ . For angular momentum, they are the set of states with a common value of  $l$  and different values of  $m_l$ ; physically, this is the set of angular momentum states that differ only in their directional components. The length of the angular momentum vector is unchanging, or irreducible, for the set.

<sup>6</sup>P. M. Davidson, "Eigenfunctions for calculating electronic vibrational intensities," *Proc. R. Soc. London* **130**, 459–472 (1932); See also D. J. Rowe and C. Bahri, "Rotational–vibrational spectra of diatomic molecules and nuclei with Davidson interactions," *J. Phys. A* **31**, 4947–4961 (1998).

<sup>7</sup>The Dirac–Coulomb problem is also exactly solvable by this method.

<sup>8</sup>B. G. Adams, *Algebraic Approach to Simple Quantum Systems* (Springer-Verlag, Berlin, 1994). This monograph elaborates considerably on the present method.

<sup>9</sup>O. L. de Lange and R. E. Raab, *Operator Methods in Quantum Mechanics* (Clarendon, Oxford, 1991).

<sup>10</sup>A. Frank and P. Van Isacker, *Algebraic Methods in Molecular and Nuclear Structure Physics* (Wiley, New York, 1994).

<sup>11</sup>L. Harris and A. L. Loeb, *Introduction to Wave Mechanics* (McGraw-Hill, New York, 1963).

<sup>12</sup>H. C. Ohanian, *Principles of Quantum Mechanics* (Prentice–Hall, Englewood Cliffs, NJ, 1990).

<sup>13</sup>F. Schwabl, *Quantum Mechanics* (Springer-Verlag, Berlin, 1992).