Wick calculus using the technique of integration within an ordered product of operators

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We show that the connection between Wick ordered polynomials and Hermite polynomials derived by Wurm and Berg by an inductive method can be directly and concisely obtained using the technique of integration within an ordered product of operators (IWOP).
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In a recent paper Wurm and Berg[1] introduced some ideas and results regarding the normal ordering of operators and functions which are useful in quantum field theory. They established an interesting connection between normal ordering, Wick transformations,[2] and the Hermite polynomials. The latter are denoted by $\mathrm{He}_n(x)$, with $\mathrm{He}_n(x) \equiv 2^{-n/2}H_n(x/\sqrt{2})$ and

$$H_n(x) = \sum_{m=0} \frac{(-1)^m n!}{m! 2^m (n-2m)!} x^{n-2m}. \tag{1}$$

For the (dimensionless) position operator $\hat{q}=a+a^\dagger$, where a and $a^\dagger$ are the boson annihilation and creation operators, respectively, with $[a,a^\dagger]=1$, they noticed that

$$\hat{q}^2 = :\hat{q}^2: + 1, \tag{2a}$$

$$\hat{q}^3 = :\hat{q}^3: + 3:\hat{q}:, \tag{2b}$$

$$\hat{q}^4 = :\hat{q}^4: + 6:\hat{q}^2: + 3, \tag{2c}$$

where : : stands for normal ordering (Wick ordering). The latter means that when all creation operators in a function $F(a^\dagger,a)$ are arranged to be on the left-hand side of all the annihilation operators according to $[a,a^\dagger]=1$, then $F(a^\dagger,a)$ has been converted to normal ordering and is denoted as $F(a^\dagger,a) = :G(a^\dagger,a):$. By recursively replacing normal-ordered terms on the right by expressions on the left that are not normal-ordered (for example,$:\hat{q}^2:$ can be replaced by $\hat{q}^2 - 1$), they inverted Eq. (2c) and obtained

$$:\hat{q}^2: = \hat{q}^2 - 1 = \mathrm{He}_2(\hat{q}), \tag{3a}$$

$$:\hat{q}^3: = \hat{q}^3 - 3\hat{q} = \mathrm{He}_3(\hat{q}), \tag{3b}$$

$$:\hat{q}^4: = \hat{q}^4 - 6\hat{q}^2 + 3 = \mathrm{He}_4(\hat{q}). \tag{3c}$$

They then used an inductive method to obtain

$$:\hat{q}^n: = \mathrm{He}_n(\hat{q}), \tag{4}$$

where $\mathrm{He}_n(\hat{q}) \equiv 2^{-n/2}H_n(\hat{q}/\sqrt{2})$.

In this comment we present a direct and concise method—integration within an ordered product of operators (IWOP)—to derive Eq. (4). The main points of the IWOP technique include the following:[3–5]

(1) The order of the Bose operators a and $a^\dagger$ within a normally ordered product can be permuted. That is, even though $[a,a^\dagger]=1$, we have $:aa^\dagger: = :a^\dagger a: = a^\dagger a$, because $:aa^\dagger:$ is a normal ordering operator, so $:aa^\dagger: = a^\dagger a$; however, $a^\dagger a = :a^\dagger a:$, so $:a^\dagger a: = :aa^\dagger:$.
(2) c numbers can be taken out of the symbol::, which means that $:rG(a^\dagger,a): = r:G(a^\dagger,a):$, where r is a number or a function of numbers.
(3) A normally ordered product can be integrated with respect to a c-number *function* provided that the integration is convergent.

For example, if we use the coordinate eigenket of $\hat{Q}$, $\hat{Q}|q\rangle = q|q\rangle$, $|q\rangle$ in Fock space is expressed as

$$|q\rangle = \pi^{-(1/4)} \exp\left[-\frac{q^2}{2} + \sqrt{2}qa^\dagger - \frac{a^{\dagger 2}}{2}\right]|0\rangle. \tag{5}$$

We use the vacuum projection operator's normal ordering form $|0\rangle\langle 0| = :e^{-a^\dagger a}:$ and recast the completeness relation of the coordinate representation as a Gaussian normally ordered form,

$$\int_{-\infty}^{\infty} dq|q\rangle\langle q| = \int_{-\infty}^{\infty} \frac{dq}{\sqrt{\pi}} e^{-(q^2/2)+\sqrt{2}qa^\dagger-(a^{\dagger 2}/2)}|0\rangle \times\langle 0|e^{-(q^2/2)+\sqrt{2}qa-(a^2/2)} \tag{6a}$$

$$= \int_{-\infty}^{\infty} \frac{dq}{\sqrt{\pi}} e^{-(q^2/2)+\sqrt{2}qa^\dagger-(a^{\dagger 2}/2)}: \times e^{-a^\dagger a}:e^{-(q^2/2)+\sqrt{2}qa-(a^2/2)} \tag{6b}$$

$$=\int_{-\infty}^{\infty}\frac{dq}{\sqrt{\pi}}:e^{-q^2+\sqrt{2}q(a^\dagger+a)-[(a^{\dagger 2}+a^2)/2]-a^\dagger a}: \quad (6c)$$

$$=\int_{-\infty}^{\infty}\frac{dq}{\sqrt{\pi}}:e^{-(q-\hat{Q})^2}:=1, \quad (6d)$$

where

$$\hat{Q}=\frac{a^\dagger+a}{\sqrt{2}}=\frac{\hat{q}}{\sqrt{2}}. \quad (7)$$

If we use the relation[6]

$$\int_{-\infty}^{\infty}e^{-(x-y)^2}H_n(x)dx=\sqrt{\pi}(2y)^n, \quad (8)$$

and the IWOP method,

$$H_n(\hat{Q})=\int_{-\infty}^{\infty}dq|q\rangle\langle q|H_n(q) \quad (9a)$$

$$=\int_{-\infty}^{\infty}\frac{dq}{\sqrt{\pi}}:e^{-(q-\hat{Q})^2}H_n(q):=2^n:\hat{Q}^n:=\sqrt{2^n}:\hat{q}^n:, \quad (9b)$$

or

$$:\hat{q}^n:=2^{-n/2}H_n(\hat{Q})=2^{-n/2}H_n\left(\frac{\hat{q}}{\sqrt{2}}\right)=He_n(\hat{q}), \quad (10)$$

which is Eq. (4). We see that this method is analytic and exact.

We can use this method to derive other useful relations. For example, by using Eq. (5) and the IWOP technique, we obtain

$$e^{\lambda\hat{Q}^2}=\frac{1}{\sqrt{1-\lambda}}:\exp\left[\frac{\lambda\hat{Q}^2}{1-\lambda}\right]:. \quad (11)$$

If we operate $e^{\lambda\hat{Q}^2}$ on the vacuum state, we have the squeezed state (its definition can be found in Ref. 6),

$$e^{\lambda\hat{Q}^2}|0\rangle=\frac{1}{\sqrt{1-\lambda}}\exp\left[\frac{\lambda a^{\dagger 2}}{2(1-\lambda)}\right]|0\rangle. \quad (12)$$

As another example, the normally ordered expansion of $\hat{Q}^m$ is

$$\hat{Q}^m=\int_{-\infty}^{\infty}\frac{dq}{\sqrt{\pi}}q^m:e^{-(q-\hat{Q})^2}: \quad (13a)$$

$$=\frac{1}{\sqrt{\pi}}\sum_{l=0}^{[m/2]}\binom{m}{2l}:\left(\frac{a+a^\dagger}{\sqrt{2}}\right)^{m-2l}:\Gamma\left(\ell+\frac{1}{2}\right), \quad (13b)$$

where $\Gamma(\ell+1/2)$ is the gamma function and is equal to $\sqrt{\pi}2^{-\ell}(2l-1)!!$. We have used the relation[7]

$$\int_{-\infty}^{\infty}\frac{dq}{\sqrt{\pi}}q^m e^{-\sigma(q-\lambda)^2}=\frac{1}{\sqrt{\sigma^{m+1}}}\sum_{k=0}^{[m/2]}\frac{m!}{2^{2k}k!(m-2k)!}\times(\sigma^{1/2}\lambda)^{m-2k}, \quad (\mathrm{Re}\,\sigma>0). \quad (14)$$

We generalize Eq. (9b) to the two-mode case and have

$$H_n\left(\frac{\hat{Q}_1+\hat{Q}_2}{\sqrt{2}}\right)=2^{n/2}:(\hat{Q}_1+\hat{Q}_2)^n:. \quad (15)$$

If we use the completeness relation of the two-mode coordinate eigenstate,

$$\iint_{-\infty}^{\infty}dq_1dq_2|q_1,q_2\rangle\langle q_1,q_2|=1, \quad \hat{Q}_i|q_i\rangle=q_i|q_i\rangle, \quad i=1,2, \quad (16)$$

and Eqs. (5)–(8), we obtain

$$H_n\left(\frac{\hat{Q}_1+\hat{Q}_2}{\sqrt{2}}\right)=\frac{1}{\pi}\int_{-\infty}^{\infty}dq_1\int_{-\infty}^{\infty}dq_2H_n\left(\frac{q_1+q_2}{\sqrt{2}}\right):e^{-(q_1-\hat{Q}_1)^2-(q_2-\hat{Q}_2)^2}: \quad (17a)$$

$$=\frac{1}{\pi}\int_{-\infty}^{\infty}dy_1H_n(y_1)\int_{-\infty}^{\infty}dy_2:e^{-(y_1-\hat{Q}_1+\hat{Q}_2/\sqrt{2})^2-(y_2-\hat{Q}_1-\hat{Q}_2/\sqrt{2})^2}: \quad (17b)$$

$$=2^{n/2}:(\hat{Q}_1+\hat{Q}_2)^n:. \quad (17c)$$

From Eqs. (6)–(17) we can see that if we use the IWOP technique, we can integrate over Dirac's ket-bra integration-form operators, which then helps to derive many operators' normal ordering form.

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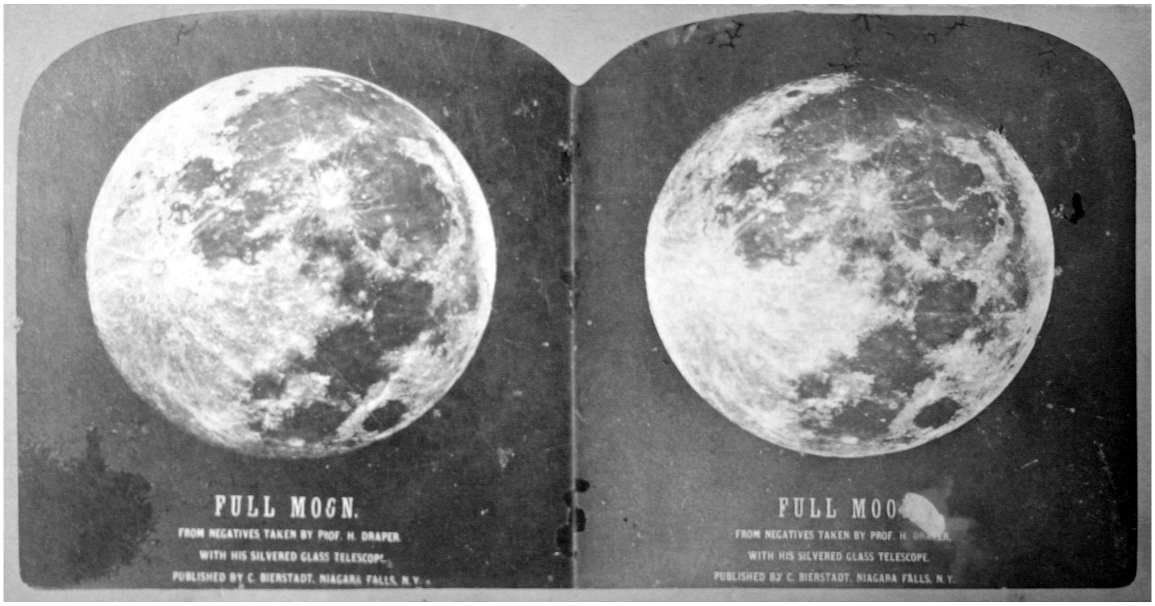

Stereoscopic Picture of the Moon. The stereoscopic effect depends on the fact that the two eyes see a scene from slightly different viewpoints. However, to make a stereoscopic picture of the moon, the two pictures must be taken with two lenses about 100,000 kilometers apart. Charles Wheatstone (1802–1875), who wrote the first paper on the stereoscopic effect in 1838, suggested that the librations of the Moon be used to obtain the necessary baseline. The principal libration, or apparent rocking back and forth of the moon, results from the fact that the moon travels in an elliptical orbit around the earth while spinning at a constant rate about its axis. Thus, pictures taken at different times give the stereoscopic effect. This stereo picture of the moon is labeled "FULL MOON. From negatives taken by Prof. H. Draper with his silvered glass telescope." Draper was the Professor of Chemistry at New York University. Ref: Thomas B. Greenslade, Jr., "The First Stereoscopic Pictures of the Moon," Am. J. Phys. **40**, 536–540 (1972) (Photograph and Notes by Thomas B. Greenslade, Jr., Kenyon College)