Electrons in Flatland

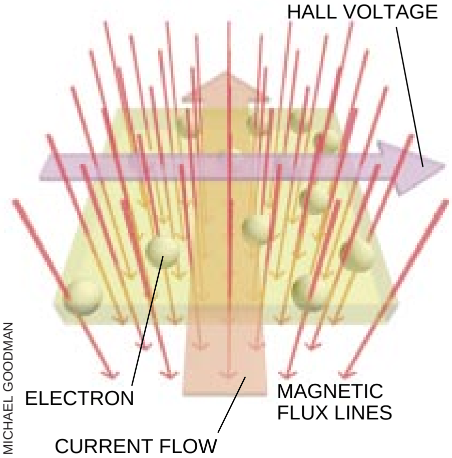
Trapped in a two-dimensional plane, electrons can exhibit the quantum Hall effect, a startling phenomenon now thought to be intimately connected to superconductivity

by Steven Kivelson, Dung-Hai Lee and Shou-Cheng Zhang

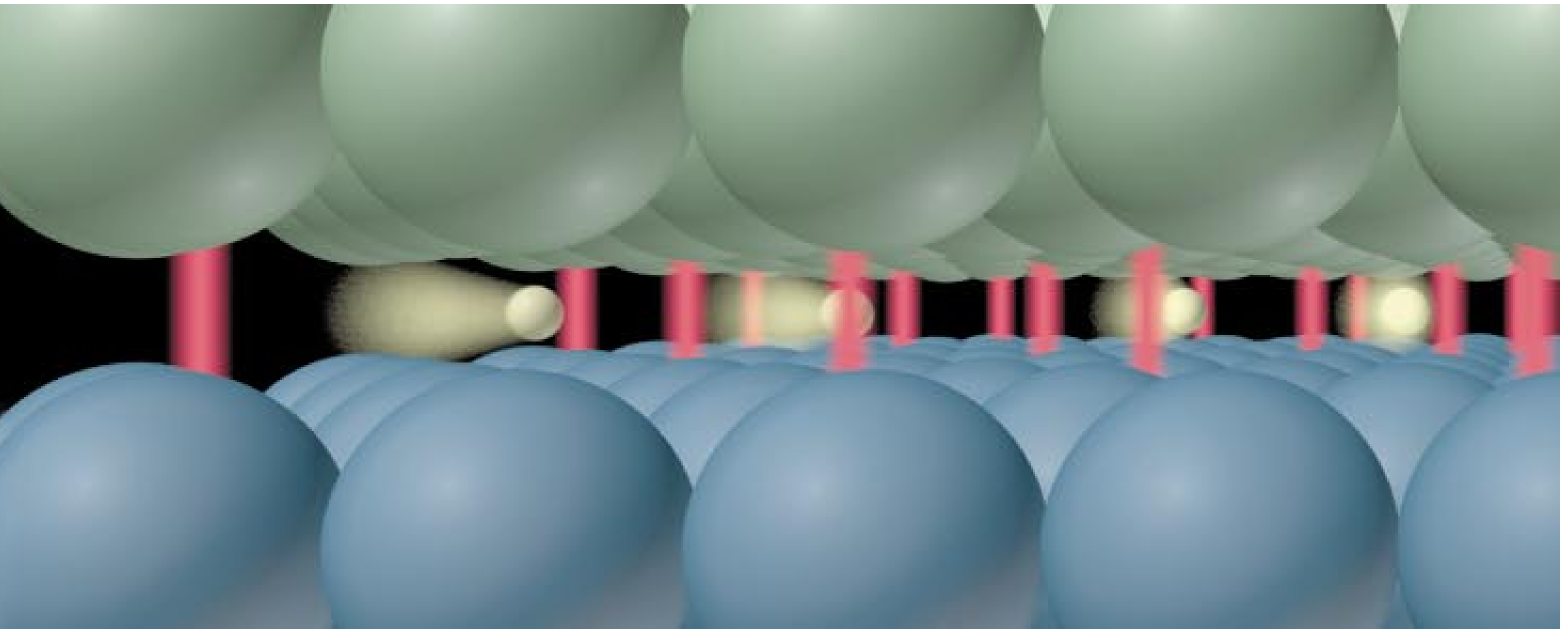
Since the time of the ancient Greeks, a central goal of all scientific disciplines has been to find a minimal set of basic principles that underlie diverse natural phenomena. This reductionist philosophy has succeeded well in some areas, such as high-energy physics—the study of the fundamental particles of force and matter. Here theorists have grouped all particles into a few families and expressed the basic laws of physics in terms of the interactions among them.

The situation is quite different in condensed-matter physics, which is the study of solids and liquids. Research in this century into the behavior of electrons in solids has uncovered various states of matter in which electrons organize themselves in myriad remarkable ways. For example, solids are typically either insulators (they strongly resist the flow of electric current) or metals (they conduct current well but still have a small amount of resistance). Yet under some circumstances, certain solids can enter a superconducting state, in which electric current flows without any resistance at all. The theoretical characterizations of these different states have been as diverse as the states themselves.

That may soon change. Researchers have found a deep connection between superconductivity and another intensely studied subject in condensed-matter physics: the quantum Hall effect. This phenomenon occurs when electrons are subjected to three specific conditions at once: they are trapped at the interface between two semiconductor crystals, so that they can move only in a two-dimensional "flatland"; they are cooled to temperatures near absolute zero; and they are subjected to a high magnetic



QUANTUM HALL EFFECT takes place in the plane between two semiconductors cooled to near absolute zero (*below*); the atoms on the surface of the semiconductors are depicted as green and blue spheres. When a magnetic field (*red lines*) is applied, electrons in a current (*yellow, traveling into page*) are redistributed so that more electrons are on one side (the right) than the other. This redistribution of electrical charge produces a measurable voltage (the Hall voltage) and conductance perpendicular to the current flow (*top view at left*). The quantum Hall effect refers to stepwise increases in conductance as the magnetic field rises.



field. The magnetic field causes the electrons to drift sideways to the direction of the current flow. As a result, a sideways voltage, or force that pushes electrons, develops. If the magnetic field increases, this voltage also increases, but not linearly; rather it increases in a precise, stepwise fashion. This phenomenon is the quantum Hall effect and is considered the signature of a new, distinct phase of matter.

When the quantum Hall effect was discovered in 1980, physicists recognized that the properties of electrons in this extraordinary state differ fundamentally from those in all other known states of matter. The latest explorations in this field, however, have uncovered a striking relation between the quantum Hall effect and the more familiar phenomenon of superconductivity. Studies of this link have even led to predictions of other new phases of matter, which experiments have recently confirmed.

Although the quantum Hall effect may have no immediate practical value, its study has fostered the development of new concepts and theoretical tools. These tools are likely to have broad implications for physics, in much the same way that the theory of superconductivity has helped advance elementary particle physics and that the study of phase transitions has greatly enhanced the understanding of the early universe.

Explorations of the quantum Hall effect also afford a glimpse into the astonishing ways in which the subatomic world operates; these studies are thereby spurring theorists to formulate a more complete view of the physical universe. Moreover, the principles involved may well prove important in future generations of semiconductor microelectronic devices. As these devices are made ever smaller, they will eventually reach dimensions at which quantum mechanics and the interactions among electrons will become crucial to design considerations.

Discovery of the Quantum Hall Effect

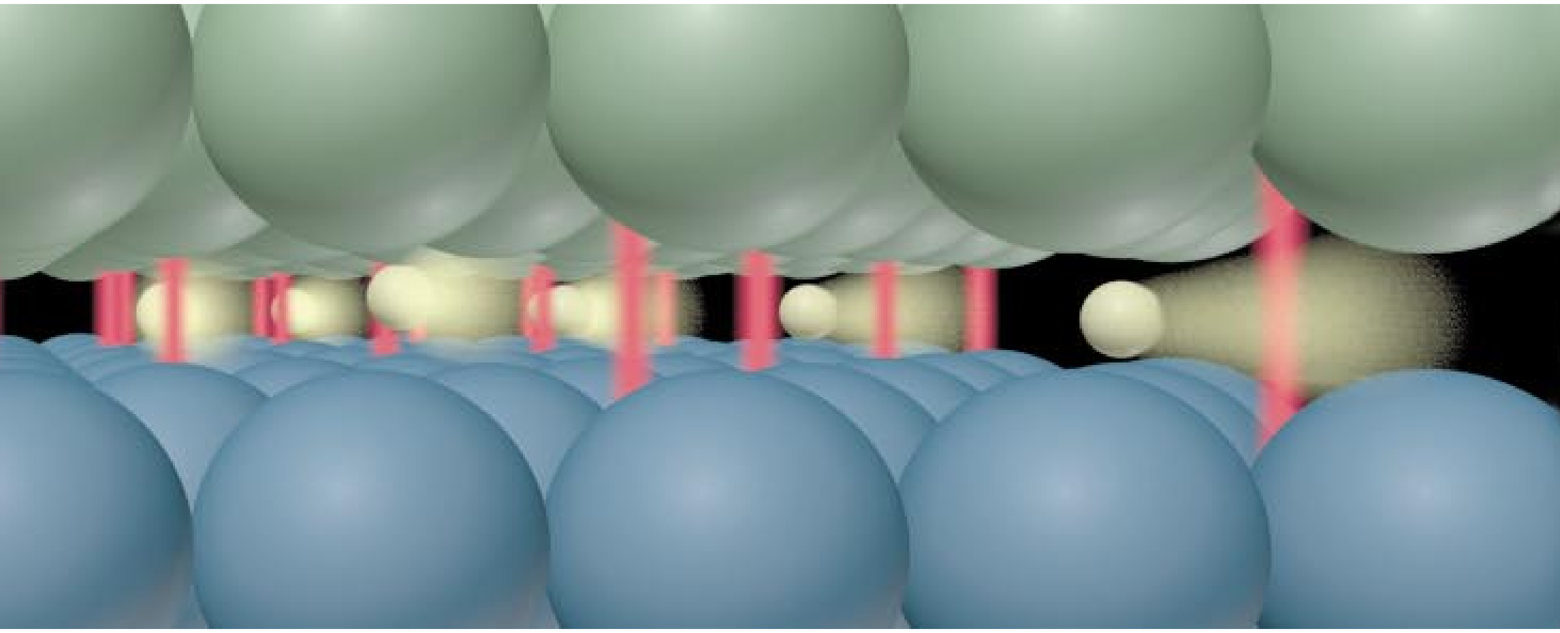
The quantum Hall effect turns out to be an unusual manifestation of a well-known, more general phenomenon in electrical conduction that was discovered by the American physicist Edwin H. Hall in the 19th century. When a voltage is applied between the ends of a wire, a current begins to flow. If the wire is then subjected to a magnetic field, the flowing electrons experience a sideways force. This force redistributes the electrons nonuniformly—more electrons end up on the right side of the wire and fewer on the left.

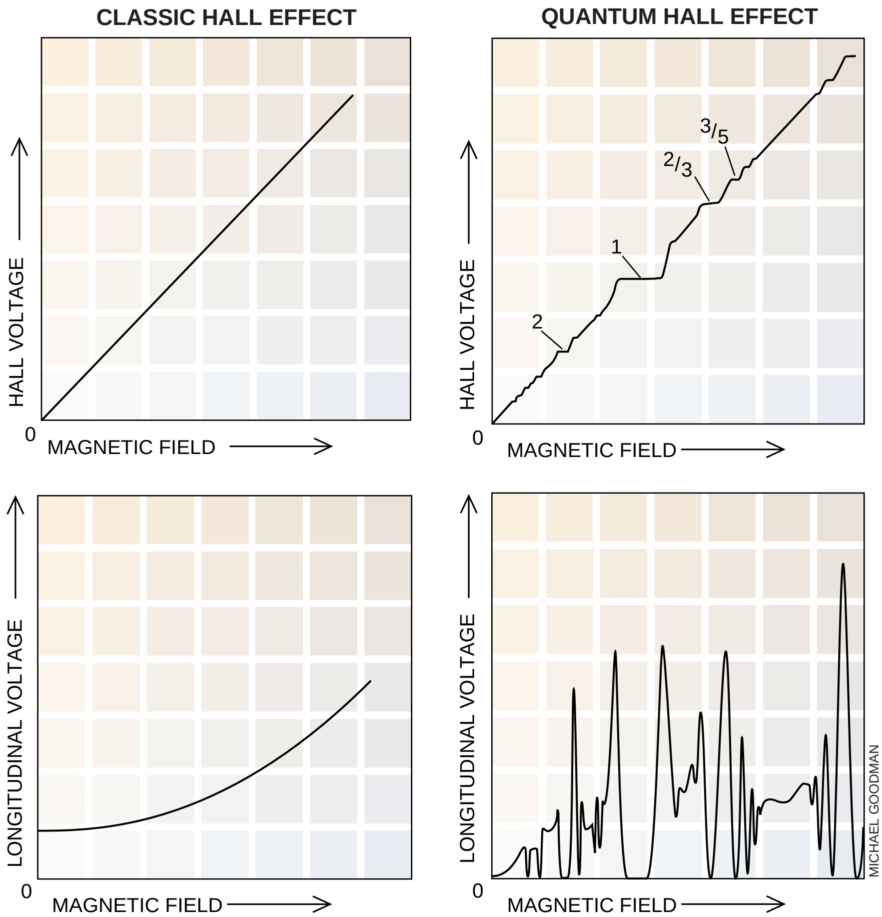
The nonuniform distribution in turn produces an electrical voltage perpendicular to the current direction. To detect this sideways voltage (the Hall voltage), one simply attaches the probes of a standard measuring device, such as a voltmeter, to the sides of the wire. For a fixed magnitude of current, the Hall voltage rises smoothly with the strength of the magnetic field. This phenomenon is now commonly referred to as the classic Hall effect.

In 1980 Klaus von Klitzing, then at the High Magnetic Field Laboratory of the Max Planck Institute in Grenoble, Michael Pepper of the University of Cambridge and Gerhardt Dordda of Siemens Research Laboratory in Munich discovered that under special circumstances, the Hall effect does not obey the usual rules. Thanks to advances in semiconductors, it is possible to trap a collection of electrons between two crystalline semiconductors such that the electrons can move only in a single plane. When the investigators chilled these trapped electrons to within a degree or two Celsius of absolute zero, they found that the Hall voltage did not rise smoothly as the strength of the magnetic field increased.

Instead the Hall voltage rose in steps, with values that did not vary at all over a small range of magnetic-field strengths [*see illustration on next page*]. In addition, the longitudinal voltage—that is, the voltage necessary to maintain the flow of current—nearly vanished when these plateaus in the Hall voltage were reached. In other words, the electrons in flatland became "perfectly conducting." (They are not technically superconducting; superconducting electrons can also expel a magnetic field. These perfectly conducting electrons do not.)

Perhaps more astonishing is that at each plateau, a quantity called the Hall conductance was found to have a special value. The Hall conductance is the ratio between the amount of longitudinal current and the value of the Hall voltage. Von Klitzing and his colleagues concluded that at each plateau the Hall conductance equaled an integer multiple of the quantum of conductance, a unit equal to 1/25,812.8 inverse ohms (conductance is the inverse of resistance). The quantum of conductance is represented as e^2/h. (The e stands for





SIGNATURES of the quantum Hall effect are seen when certain measurements are compared with those relating to the classic (nonquantized) Hall effect. In the classic Hall effect the voltage in the sideways direction (the Hall voltage) varies smoothly with the magnetic field; in the quantum version the voltage plateaus at conductances equal to certain integer and fractional multiples of a fundamental constant (only a few multiples are noted). Parallel to the current, the longitudinal voltage varies smoothly with the magnetic field in the classic effect; in the quantum version the voltage disappears when the Hall voltage plateaus.

the charge of an electron; *h* is Planck's constant, which relates the frequency of a light ray to the smallest amount of energy it can carry.) For his discovery of this "integral quantum Hall effect," von Klitzing won the 1985 Nobel Prize in Physics.

In 1982 Daniel C. Tsui, now at Princeton University, Horst L. Störmer of AT&T Bell Laboratories and Arthur C. Gossard, now at the University of California at Santa Barbara, encountered another unexpected property of the quantum Hall effect. They discovered that the Hall voltage plateaued more often than was originally thought. It leveled off at specific fractional values, such as $^1/_3$, $^2/_5$ and $^3/_7$, of the quantum of conductance. The name of this phenomenon, not surprisingly, is the fractional quantum Hall effect.

So far no experiment has revealed any deviation at all between the measured Hall conductances and the quantized values. They are the same to within at least one part in 10 million (seven decimal places); indirect evidence suggests they are the same to within at least one part in 100 billion. Because of this accuracy, the National Institute of Standards and Technology has adopted the quantum Hall effect as the standard to calibrate resistance-measuring devices.

Magic Filling Factors

Why does the Hall conductance adopt these "magic" values? Investigators spent years trying to solve this puzzle. The answer, as will be seen, lies with the strength of the magnetic field impinging on each electron.

To understand the solution, one needs to know three things about how physicists describe magnetic fields. First, quantum mechanics represents the amount of magnetic field acting on a sample in terms of a unit called the magnetic flux quantum. One way to picture a flux quantum is to imagine it as an arrow. To measure the strength of the magnetic field, one simply counts the number of flux quanta—arrows—poking through a sample in a given area.

Second, an important quantity related to the magnetic-field strength is called the filling factor. This is the number of electrons in a sample divided by the number of magnetic flux quanta penetrating the sample. When the filling factor is one, there is one flux quantum per electron; when the filling factor is $^1/_3$, there are three flux quanta for each electron.

Third, there is a correlation between the quantized values of the Hall conductance and the corresponding filling factors (known as magic filling factors). When the filling factor is 1, the Hall conductance is found to be $1e^2/h$; when the filling factor is $^1/_3$, the Hall conductance is $^1/_3\ e^2/h$, and so on.

Robert B. Laughlin, now at Stanford University, first explained the plateaus in the Hall conductance using separate, idealized mathematical models for the integer and the fractional filling factors. His groundbreaking explanations (and those of others) relied on wave functions: mathematical functions that describe everything there is to know about the state of quantum particles.

Successful as Laughlin's approach was, it left several questions unanswered. It relied on certain simplifications that are difficult to apply to real-world materials, which contain many imperfections. Wave functions are abstract, so Laughlin's explanations were rather tough to picture. His approach did not indicate whether any relation exists among the quantum Hall effect and other kinds of electronic activity in solids. Finally, considering the similarity between the integer and fractional quantum Hall effects, it would seem they should be treated on an equal footing and not separately.

By exploiting a precise mathematical analogy between the quantum Hall effect and superconductivity, we have developed a new way to comprehend the quantum Hall effect. Besides unifying two seemingly disparate phenomena, the analogy enables physicists to apply knowledge of superconductivity to the quantum Hall effect. This approach complements Laughlin's, and it embodies many of the insights gained from his work. But it is distinctly separate, focusing on the real, macroscopic observables of the physical system rather than on the hard-to-visualize, microscopic properties of an ideal system.

The first steps in this new direction

occurred in 1987. Steven M. Girvin and Allan H. MacDonald, now both at Indiana University, recognized that the wave functions used to explain the quantum Hall effect could be viewed as representing the superconducting state of a new, imaginary type of particle, called a composite boson. A similar observation was made somewhat later by Nicholas Read, now at Yale University.

Bosons and Fermions

Bosons are one of two families into which physicists group all particles on the basis of their "statistics," or group behavior. The wave function describing a collection of bosons remains the same when two of the particles exchange places. The other family of particles is the fermions; their wave function changes sign (from positive to negative, or vice versa) when two particles are swapped.

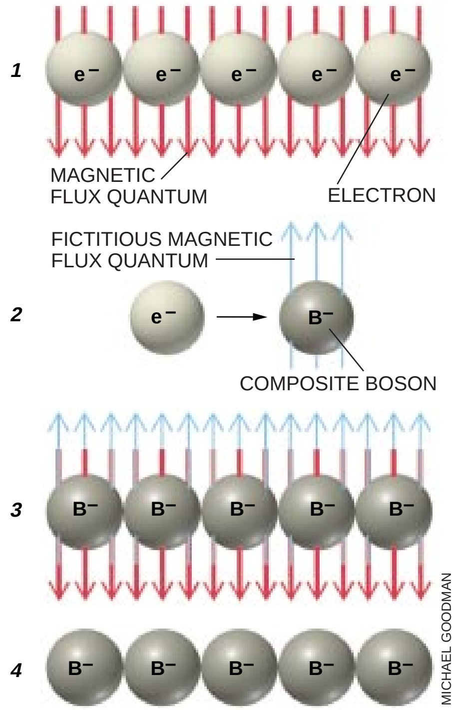
Electrons, protons and neutrons are all fermions. An atom, which contains all three, can also be treated as a single (composite) particle. Whether it is a boson or fermion depends on the net number of its constituents: if this number is odd, the atom is a fermion; if even, the atom is a boson. The isotope called helium 4, for example, contains two electrons, two protons and two neutrons, making it a boson. In contrast, the isotope helium 3 has two electrons, two protons and only one neutron; therefore, it is a fermion.

Bosons and fermions differ in many ways. Most relevant to this article are the rules governing the occupation of quantum-mechanical states. Fermions obey the Pauli exclusion principle, which forbids two fermions from occupying the same state—essentially, they cannot be in the same place at the same time. This rule does not apply to bosons; many bosons can exist in exactly the same state.

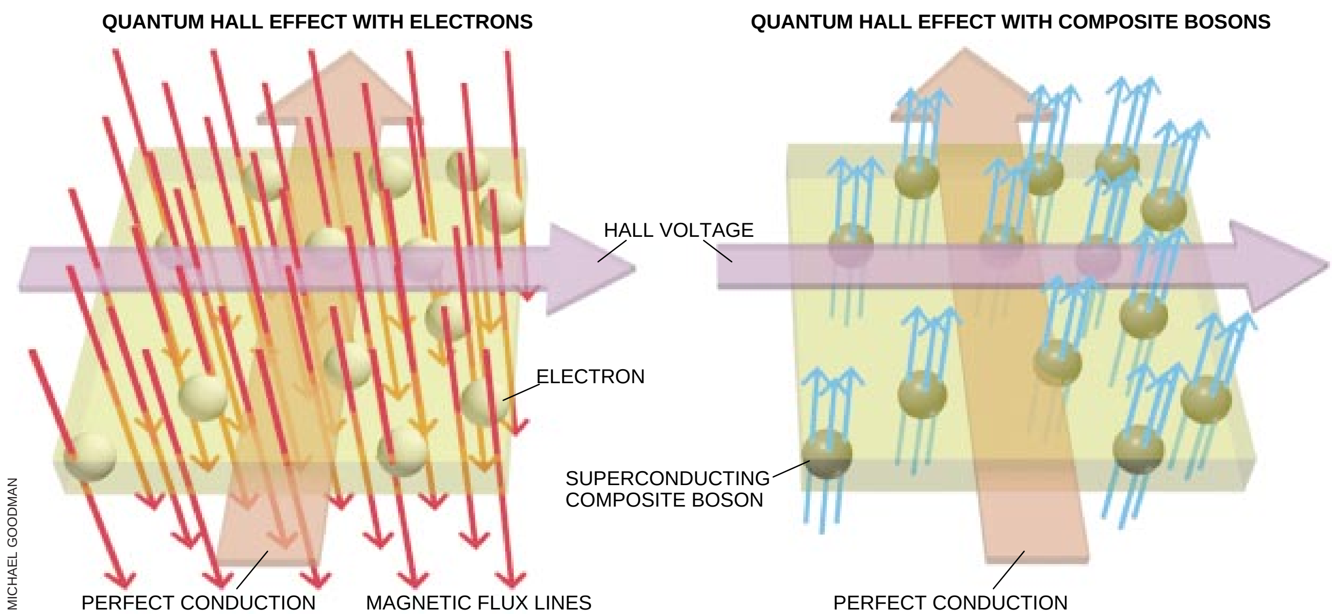
These two fundamentally different properties of fermions and bosons account for many observations in physics. A good example is the dramatic difference between a superconductor and an ordinary metal. Electrical conduction in ordinary metals can be readily understood in terms of the properties of fermions (specifically, electrons); in contrast, superconductivity is a property of bosons.

How can this be, considering that the electric current carriers in all solids are electrons, which are fermions? The answer is that in the superconducting phase, the electrons bypass the rules for fermions by pairing up. Each pair then acts as a boson; all these pairs can condense into the same quantum state to produce superconductivity. In the ordinary metallic state, however, electrons retain their single, fermion identities. As fermions, they exist in different states, as demanded by the Pauli exclusion principle, and fail to superconduct.

The theory that uses composite bosons to explain the quantum Hall effect



COMPOSITE BOSONS can represent the electrons in the quantum Hall effect. For example, at what is called filling factor $1/3$, there are three flux quanta (a measure of magnetic-field intensity) for every electron (*1*). The authors envisioned this condition with composite bosons, or charged particles having three (fictitious) magnetic flux quanta (*2*). Orienting the fictitious flux against the real magnetic flux (*3*) eliminates the magnetic field each boson "sees" (*4*), easing the modeling of the quantum Hall effect.



EXPLANATION OF QUANTUM HALL EFFECT with electrons at filling factor $1/3$ (*left*) follows once the composite bosons cancel out the external magnetic field (*right, where the canceled field has been left out for clarity*). Cold, charged bosons in the absence of a magnetic field become superconducting, accounting for the perfect conduction in the longitudinal direction. The Hall voltage develops because of induction: the moving fictitious magnetic flux produces a sideways voltage.

Comparing Superconductivity with the Quantum Hall Effect

SUPERCONDUCTIVITY	QUANTUM HALL EFFECT
Paired electrons (Cooper pairs) are the basic carriers of charge	Composite bosons are the basic carriers of charge
Perfect conduction	Perfect conduction in longitudinal direction
Persistence in a magnetic field and in materials with imperfections	Quantized Hall plateaus, whose values persist over a small range of magnetic-field strengths
Electrons exclude a weak magnetic field	Electrons resist change in density at a fixed magnetic field, a property called incompressibility
Flux quantization (a property of superconducting rings, which must enclose an integral number of magnetic flux quanta)	Fractional charge (the quantization of electrical charge in units that are fractions of the electron's charge)

was introduced in 1989 by two of us (Zhang and Kivelson), along with T. Hans Hansson, now at the University of Stockholm. Loosely summarizing, we proposed that electrons moving in two dimensions in a strong magnetic field are mathematically equivalent to a collection of composite bosons in a much weaker magnetic field. Under particular circumstances—namely, when the electron filling factor reaches a magic value (specifically, 1, $^1/_3$ or $^1/_5$)—the magnetic field that composite bosons experience is actually zero. In that case, we argued, composite bosons would, under a broad range of circumstances, become superconducting. We then showed that when composite bosons become superconducting, they give rise to the quantized Hall conductances.

Along with Matthew P. A. Fisher, now at the University of California at Santa Barbara, one of us (Lee) logically extended this theory to account for all the other, more complicated quantized Hall plateaus, such as $^2/_5$ and $^3/_7$. These works formed the basis of the subsequent investigations that the three of us carried out to study the quantum Hall effect under a variety of conditions.

Electrons as Composite Bosons

The theory of composite bosons is based on a mathematical equivalence between electrons moving in two dimensions and a collection of bosons carrying a bundle of fictitious magnetic flux. It turns out that in order for the composite boson to mimic the electron's Fermi statistics, each boson must carry an odd number of fictitious magnetic flux quanta. (A more rigorous justification for this representation appears in *Scientific American*'s area on America Online.)

Perhaps an example best explains the effects of fictitious magnetic flux. Consider one of the magic filling factors at which a plateau in the Hall voltage appears—say, $^1/_3$. This filling factor means that there are three quanta of real magnetic flux per electron. Let's consider each electron not as a fermion but as a composite boson bound to three quanta of fictitious flux. Now point these three flux quanta in the direction opposite to the external magnetic field. The net flux seen by the bosons is the sum of the real and fictitious fluxes. Because we have pointed the fictitious flux so that it cancels the real flux, the boson sees no net magnetic flux whatsoever. At low temperatures, bosons in the absence of a magnetic field are known to superconduct, so we expect the same to be true of cold composite bosons at filling factor $^1/_3$.

Why should the superconductivity of composite bosons imply perfect conduction in the current direction and a quantized Hall conductance in the perpendicular direction? The first part is easy. Because the composite bosons are superconducting, no voltage is needed to sustain the current flow; one thus finds perfect conduction.

The second part is more subtle. Recall that each flowing composite boson carries an odd number of fictitious magnetic flux quanta. Therefore, if the bosons are flowing, fictitious magnetic flux quanta must flow with them. But moving magnetic fluxes (even fictitious ones) generate an electrical voltage perpendicular to the flow (this property is known as Faraday's law of electromagnetic induction). Furthermore, this sideways voltage is proportional to the total amount of fictitious flux flowing through the sample every second. So for filling factor $^1/_3$, the magnetic flux current is three times the electric current. That in turn accounts for the Hall conductance being equal to $^1/_3$ of the quantum of conductance.

From this viewpoint, the only difference between the various magic filling factors—be they 1, $^1/_3$ or $^1/_5$—lies in the number of fictitious magnetic flux quanta each composite boson carries. In addition, the quantized Hall conductances (such as 1, $^1/_3$, $^2/_5$ and so on, multiplied by e^2/h) depend only on the ratio between charge and flux in the composite boson and not on the details of the material in which they are observed.

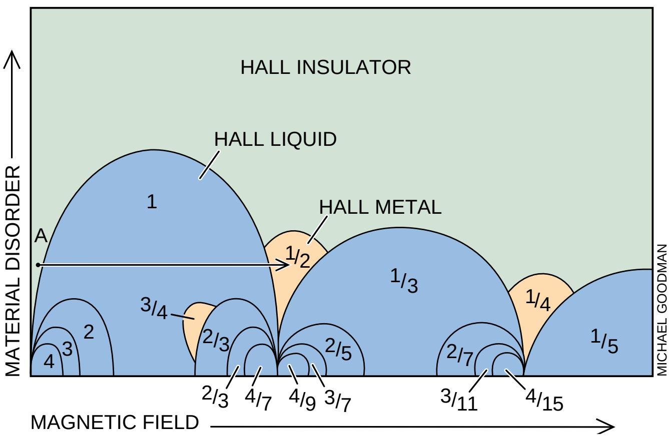
The model using composite bosons also explains why the Hall conductance remains unchanged even when the filling factor deviates slightly from a magic value. Consider the situation in which the electron filling factor is slightly more than $^1/_3$. In that case, the fictitious flux cancels the real flux only partially, and the composite bosons experience a small net magnetic field. But like a real superconductor, the composite boson superconductor can tolerate a small magnetic field. Consequently, the Hall conductance is unchanged within a finite window around filling factor $^1/_3$.

The analogy between superconductivity and the quantum Hall effect goes much further. For instance, the ability of a superconductor to repel a magnetic field translates into the ability of the electrons displaying the quantum Hall effect to resist any change in the overall area they occupy (the Hall effect electrons are said to be "incompressible"). Other, more obscure aspects of superconductivity also have direct analogues in the quantum Hall effect.

A Map for Flatland

Using the composite boson theory, the three of us studied the quantum Hall effect under a broad range of circumstances. The outcome of that study is represented in a so-called phase diagram. Physicists often use a phase diagram to summarize the behavior of a material under various conditions. For example, at different pressures and temperatures, a collection of water molecules can become a liquid, ice or steam. A diagram illustrating these phases can be constructed to indicate the physical state of the water molecules over a range of pressures and temperatures.

Rather than pressure and temperature, the phase diagram for the electrons in flatland uses as parameters the strength of the magnetic field and the degree of imperfection, or disorder, in the semiconductor crystals that trap the electrons. We obtained such a diagram from the known phase diagram of superconductors. Mapping information from the superconducting phase diagram produced a beautifully nested structure [*see illustration at right*].



NEW STATES OF MATTER for electrons in flatland are shown in a phase diagram. At a given magnetic field and level of disorder (*point A*), the electrons act as a "Hall insulator" (*green*), which has both insulating and metallic characteristics. At higher magnetic fields, the electrons turn into a "Hall liquid" (*blue*)—that is, they show the quantum Hall effect—and then become a "Hall metal" (*beige*). The numbers signify the integer and fractional values of the quantized Hall conductance.

The theory of composite bosons in the quantum Hall effect also led to the prediction of an unexpected state, in which the electrons adopt the properties of an insulator and a metal simultaneously. The prediction of such a Hall insulator was confirmed in a recent experiment carried out by Hong-Wen Jiang and Kang-Lung Wang of the University of California at Los Angeles and Scott T. Hannahs of Florida State University. When they increased the degree of imperfection in the semiconductors beyond a certain point, a very large voltage became necessary to sustain the current flow. The need for more voltage increased steadily as the temperature fell toward absolute zero—characteristics of an insulator. In contrast, the Hall voltage remained independent of temperature and increased with the strength of the magnetic field—characteristics of a metal.

Experiments by Jiang, Tsui, Störmer and Loren N. Pfeiffer and Ken W. West of AT&T Bell Laboratories and others have revealed another surprise—this time, near filling factor $^1/_2$. In that case, physicists discovered that electrons acted largely as though they were in an ordinary metal and not in a magnetic field. Among other features, the Hall conductance was not quantized but depended linearly on the magnetic field.

An intriguing explanation for this so-called Hall metal relies on the idea that an electron can be viewed as a composite fermion. A composite fermion resembles a composite boson, except that it carries an even number of fictitious magnetic flux quanta and, as a result, obeys Fermi statistics. Several researchers have put forward such ideas, based in part on a concept first introduced by Jainendra K. Jain of the State University of New York at Stony Brook. These workers include Read, Bertrand I. Halperin of Harvard University, Patrick A. Lee of the Massachusetts Institute of Technology and, independently, Vadim Kalmeyer, formerly at the IBM Almaden Research Center, and one of us (Zhang).

The virtue of both the composite bosons and composite fermions is that seemingly strange behavior of electrons in flatland can be related to the familiar behavior of composite particles. A question often raised is whether these composite particles are real or are, like quarks in high-energy physics, useful constructs that cannot be isolated and studied individually. This debate has sparked considerable research, but definitive results have yet to be garnered.

Sixteen years after its discovery, the quantum Hall effect remains one of the most exciting areas of research in condensed-matter physics. The rich variety of phenomena has provided a testing ground for many theoretical ideas. A global picture has emerged that unifies the understanding of these phenomena and others in condensed-matter systems. Yet despite the progress, critical issues remain unsolved. For example, the Hall insulator and the Hall metal are still far from completely understood. How other properties of electrons, such as their spin, fit into the picture is also not fully known.

In an earlier *Scientific American* article on this subject, in 1986, Halperin remarked that "the true importance of the quantized Hall effect does not lie in any… applications, but rather in the new insight physicists have gained into the peculiar properties of systems of electrons in strong magnetic fields and into the hidden regularities implied by the mathematical laws of quantum mechanics. Nature may well hold in store other surprising states of matter that none of us yet imagine." Ten years later physicists have found some of those states, and we look forward to discovering more.

The Authors

STEVEN KIVELSON, DUNG-HAI LEE and SHOU-CHENG ZHANG collaborated in helping to draw the connection between superconductivity and the quantum Hall effect. Kivelson, who received his doctorate from Harvard University, held positions at several institutions before taking on his current professorship of physics at the University of California, Los Angeles. Lee earned his Ph.D. from the Massachusetts Institute of Technology and worked at the IBM Thomas J. Watson Research Center before joining the faculty of the University of California, Berkeley. Zhang, currently an associate professor at Stanford University, received his Ph.D. from the State University of New York at Stony Brook and previously held positions at the University of California, Santa Barbara, and at the IBM Almaden Research Center.

Further Reading

QUANTIZED HALL EFFECT. Bertrand I. Halperin in *Scientific American,* Vol. 254, No. 4, pages 52–60; April 1986.

ANYONS. Frank Wilczek in *Scientific American,* Vol. 264, No. 5, pages 58–65; May 1991.

THE CHERN-SIMONS-LANDAU-GINZBURG THEORY OF THE FRACTIONAL QUANTUM HALL EFFECT. Shou-Cheng Zhang in *International Journal of Modern Physics B,* Vol. 6, No. 1, pages 25–58; January 1992.

GLOBAL PHASE DIAGRAM IN THE QUANTUM HALL EFFECT. S. Kivelson, D.-H. Lee and S.-C. Zhang in *Physical Review B,* Vol. 46, No. 4, pages 2223–2238; July 15, 1992.