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A expressão $\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}')}{\eta} dv'$ podem ser reescrita p/ diferentes casos como.

$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{I}}{\eta} d\vec{l}' = \frac{\mu_0 I}{4\pi}\int \frac{1}{\eta} d\vec{l}' \text{ ; ou} \quad \left(\underline{\text{caso de fios}}\right)$$

$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{K}}{\eta} da' \quad \left(\text{caso p/ superfícies}\right)$$

<u>Problema 5.27</u>: Podemos verificar a consistência das

a) Equações acima com $\nabla \cdot B = 0$;

$$\vec{\nabla}\cdot\vec{A} = \frac{\mu_0}{4\pi}\int \nabla\cdot\left(\frac{\vec{J}(\vec{r}')}{\eta}\right) dv'$$

$$\nabla\cdot\left(\frac{\vec{J}}{\eta}\right) = \frac{1}{\eta}\underbrace{\left(\nabla\cdot\vec{J}(\vec{r}')\right)}_{0} + \vec{J}\cdot\nabla\left(\frac{1}{\eta}\right)$$

Pois $\vec{J}(\vec{r}')$ é função apenas de $\vec{r}'$

$$\vec{\eta} = \vec{r} - \vec{r}' \Rightarrow \nabla\left(\frac{1}{\eta}\right) = -\nabla'\left(\frac{1}{\eta}\right) \therefore$$

$$\vec{\nabla}\cdot\left(\frac{\vec{J}(\vec{r}')}{\eta}\right) = -\vec{J}\cdot\nabla'\left(\frac{1}{\eta}\right)$$

Podemos olhar agora p/ quanto vale $\vec{\nabla}'\cdot\left(\frac{\vec{J}(\vec{r}')}{\eta}\right)$

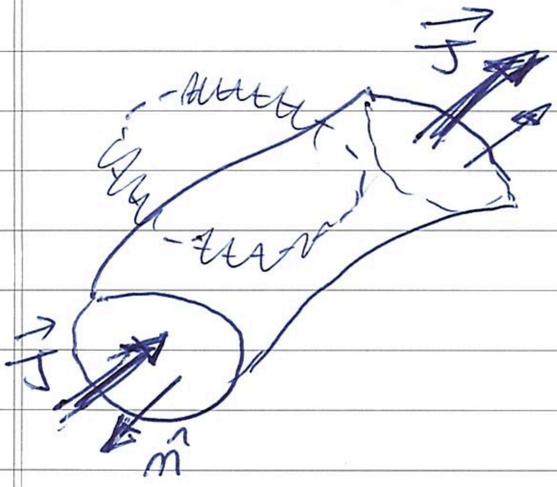
$$\vec{\nabla}'\cdot\left(\frac{\vec{J}(\vec{r}')}{\mathscr{r}}\right) = \frac{1}{\mathscr{r}}\left(\underbrace{\vec{\nabla}'\cdot\vec{J}}_{0}\right) + \vec{J}\cdot\nabla'\left(\frac{1}{\mathscr{r}}\right)$$

Mas $\vec{\nabla}'\cdot\vec{J} = 0$ em Magnetostática por $\frac{\partial \rho}{\partial t} = 0$ (corrente constante!). Veja Eq 5.31 Griffiths

$$\therefore \boxed{\vec{\nabla}\cdot\left(\frac{\vec{J}}{\mathscr{r}}\right) = -\vec{\nabla}'\cdot\left(\frac{\vec{J}}{\mathscr{r}}\right)}$$

usando o Teo do Divergência

$$\nabla\cdot\vec{A} = -\frac{\mu_0}{4\pi}\int \vec{\nabla}'\cdot\left(\frac{\vec{J}}{\mathscr{r}}\right) dv' = \frac{-\mu_0}{4\pi}\oint \frac{\vec{J}}{\mathscr{r}}\cdot d\vec{a}' = 0$$



Mas note que a área é aquela limite que envolve o volume onde existe corrente. Neste área (fechada) não tem corrente $\vec{J}$ nenhuma!

$$\oint \frac{\vec{J}}{\mathscr{r}}\cdot da' = 0$$

Desta forma $\boxed{\nabla\cdot\vec{A} = 0}$

b) Podemos verificar que $\nabla\times\vec{A} = \vec{B}$ diretamente!

$$\vec{\nabla}\times\vec{A} = \frac{\mu_0}{4\pi}\int \nabla\times\left(\frac{\vec{J}}{\mathscr{r}}\right) dv' = \frac{\mu_0}{4\pi}\int\left[\frac{1}{\mathscr{r}}\nabla\times\vec{J} - \vec{J}\times\nabla\left(\frac{1}{\mathscr{r}}\right)\right.$$

Mas o primeiro termo $\nabla \times J = 0$ pois $J(\vec{r}\,')$ é ~~função~~

$\vec{r}\,'$ e não de $\vec{r}$. Já é $\vec{\nabla}\left(\frac{1}{r_\varphi}\right) = -\frac{\hat{r}_\varphi}{r_\varphi^2}$ ∴

$$\boxed{\nabla \times \vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{J} \times \hat{r}}{r_\varphi^2}\, dv' = \vec{B}}$$ (Definição p/ Biot-Savart)

Finalmente c) $\nabla^2 \vec{A} = \frac{\mu_0}{4\pi}\int \nabla^2\left(\frac{\vec{J}}{r_\varphi}\right) dv'$

$\nabla^2\left(\frac{\vec{J}}{r_\varphi}\right) = \vec{J}\,\nabla^2\left(\frac{1}{r_\varphi}\right)$ Novamente porque $\vec{J}$ é constante; Além do fato que o diferencial seria com respeito à $\vec{r}$ e não ~~$\vec{r}$~~ $\vec{r}\,'$!

Mas $\nabla^2\left(\frac{1}{r_\varphi}\right) = -4\pi\, \delta^3(\vec{r_\varphi}) = \vec{\nabla}\cdot\nabla\left(\frac{1}{r_\varphi}\right) = \vec{\nabla}\cdot\left(\frac{-\hat{r}_\varphi}{r_\varphi^2}\right)$

∴ $\nabla^2 \vec{A} = \frac{\mu_0}{4\pi}\int \vec{J}(\vec{r}\,')\left[-4\pi\, \delta^3(\vec{r} - \vec{r}\,')\right] dv'$

$$\boxed{\nabla^2 \vec{A} = -\mu_0 \vec{J}(\vec{r})} \longrightarrow \text{densidade de corrente!}$$

Veja como o nome Vetor potencial magnético é apropriado!

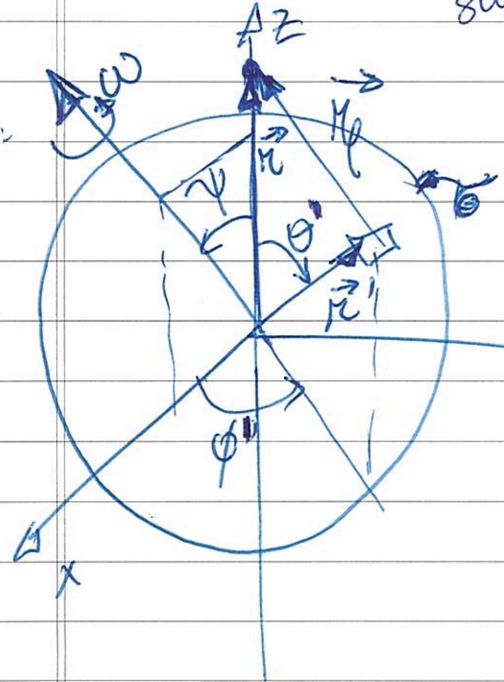
No caso da eletrostática $\nabla^2 V = -\frac{\rho}{\varepsilon_0}$ ← densidade de carga

Finalmente assumindo que $\vec{J} \to 0$ p/ $\vec{r} \to \infty$

$$\boxed{\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}')}{\mathscr{r}}\, dv'}$$

Veja Primeiro pg 205.a

Exemplo 5.11 Carga na sup. girando $\to \vec{\omega}$ no plano xz



$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{K}(\vec{r}')}{\mathscr{r}}\, da'$$

$$\vec{K} = \sigma \vec{v} \quad e \quad \vec{v} = \vec{\omega} \times \vec{r}'$$

$$\mathscr{r} = \sqrt{R^2 + r^2 - 2Rr\cos\theta'}$$

Usando $\vec{\mathscr{r}} = \vec{r} - \vec{r}'$ e

$$\mathscr{r} = \sqrt{\vec{\mathscr{r}} \cdot \vec{\mathscr{r}}} \quad (\text{Note } |\vec{r}'| = R)$$

$$da' = R^2 \operatorname{sen}\theta'\, d\theta'\, d\phi'$$

$$\vec{v} = \vec{\omega} \times \vec{r}' = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \omega \sin\psi & 0 & \omega\cos\psi \\ R\sin\theta'\cos\phi' & R\sin\theta'\sin\phi' & R\cos\theta' \end{vmatrix}$$

!!! $\boxed{\text{Note: é } \vec{r}' \text{ e não } \vec{r} \text{ ou } \vec{\mathscr{r}}}$

$$\vec{v} = R\omega\Big[-(\cos\psi\sin\theta'\sin\phi')\hat{x} + (\cos\psi\sin\theta'\cos\phi' - \underline{\sin\psi\cos\theta'})\hat{y} + (\sin\psi\sin\theta'\sin\phi')\hat{z}\Big]$$

único que sobrará!

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Todos os termos menos um envolve $\sin\phi'$ ou $\cos\phi'$

$$\therefore \int_0^{2\pi} \sin\phi' d\phi' = \int_0^{2\pi} \cos\phi' d\phi' = 0$$

$\therefore$ Sobra apenas.

$$\vec{A} = \frac{\mu_0}{4\pi} \int \frac{\sigma(R\omega)(-\sin\psi \cos\theta')\hat{y} R^2 \sin\theta' d\theta' d\phi'}{\sqrt{R^2 + r^2 - 2Rr\cos\theta'}}$$

$$\vec{A}(\vec{r}) = -\frac{\mu_0 R^3 \sigma\omega \sin\psi}{2} \left(\int_0^{\pi} \frac{\cos\theta' \sin\theta' \, d\theta'}{\sqrt{R^2 + r^2 - 2Rr\cos\theta'}} \right) \hat{y}$$

Fazendo $u = \cos\theta'$ $\quad du = -\sin\theta' d\theta'$

$$\therefore \int_{-1}^{+1} \frac{u \, du}{\sqrt{R^2 + r^2 - 2Rr u}} = \int_{-1}^{+1} \frac{u \, du}{\sqrt{\underbrace{a - \gamma u}_{x}}}$$

Onde $a = R^2 + r^2$ e $\gamma = 2Rr$

$x = a - \gamma u \ \therefore \ dx = -\gamma du$ e

e $u = \frac{a - x}{\gamma}$

$$\int_{?}^{?} \frac{(a-x)}{\gamma} \frac{1}{x^{1/2}} \left(\frac{-dx}{\gamma} \right) = \int_{?}^{?} \frac{-a \, dx}{\gamma^2 x^{1/2}} + \int_{?}^{?} \frac{x^{1/2} dx}{\gamma^2}$$

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$$= -\frac{2a}{r^2} x^{1/2} + \frac{2}{3}\frac{x^{3/2}}{r^2}$$

$$= -\frac{2a}{r^2}(a - ru)^{1/2} + \frac{2}{3r^2}(a - ru)^{3/2}$$

$$= \left[-\frac{2a}{r^2} + \frac{2}{3r^2}(a - ru)\right](a - ru)^{1/2}$$

$$= \left[\frac{-6a + 2a - 2ru}{3r^2}\right](a - ru)^{1/2}$$

$$-\frac{2}{3}\frac{(2a + ru)}{r^2}(a - ru)^{1/2}$$

$$\therefore \int_{-1}^{+1} \frac{u\,du}{\sqrt{R^2 + r^2 - 2Rru}} = -\frac{(R^2 + r^2 + Rru)}{3R^2r^2}\sqrt{R^2 + r^2 - 2Rru}\,\Big|_{-1}^{+1}$$

$$= -\frac{1}{3R^2r^2}\left[(R^2 + r^2 + Rr)|R - r| - (R^2 + r^2 - Rr)(R + r)\right]$$

se $\vec{r}$ dentro da esfera, $\therefore R > r \therefore$

$$= \boxed{\frac{2r}{3R^2}}$$

b)

$\vec{r}$ fora da esfera $R < r \therefore$

$$= \boxed{\frac{2R}{3r^2}}$$

[signature]

Sabendo que $\vec{\omega} \times \vec{r} = -\omega r \sin\psi \, \hat{y}$

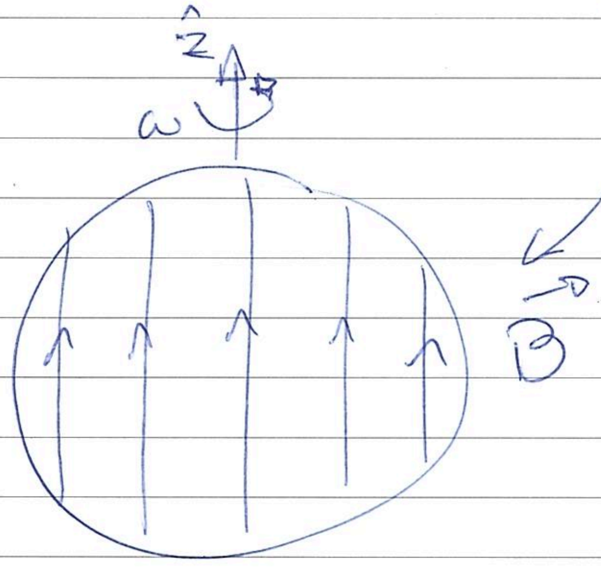
$$\therefore \vec{A}(\vec{r}) = \begin{cases} \dfrac{\mu_0 R \sigma}{3} (\vec{\omega} \times \vec{r}) & \text{p/ pontos dentro da esfera} \\ \dfrac{\mu_0 R^4 \sigma}{3 r^3} (\vec{\omega} \times \vec{r}) & \text{p/ pontos fora da esfera.} \end{cases}$$

$$\vec{A}(r,\theta,\phi) = \begin{cases} \dfrac{\mu_0 R \omega \sigma}{3} r \sin\theta \, \hat{\phi} & (r \leq R) \\ \dfrac{\mu_0 R^4 \omega \sigma}{3} \dfrac{\sin\theta}{r^2} \hat{\phi} & (r \geq R) \end{cases}$$

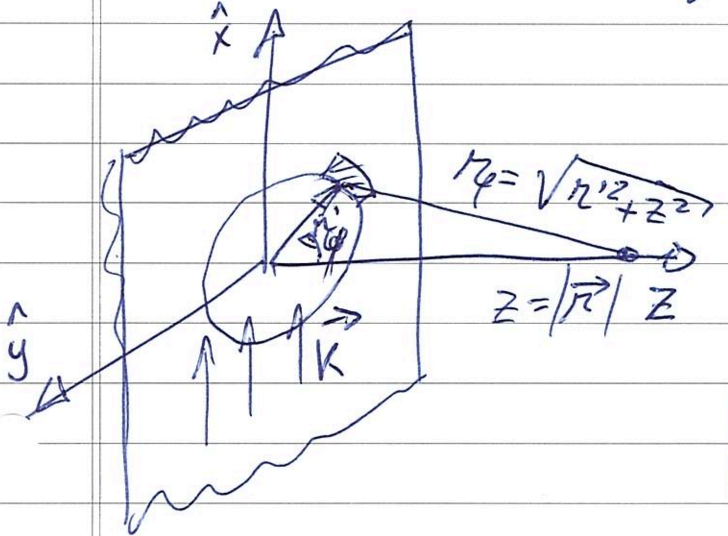
$$\vec{B} = \nabla \times \vec{A} = \frac{2\mu_0 R \omega \sigma}{3} (\cos\theta \, \hat{r} - \sin\theta \, \hat{\theta}) = \frac{2}{3} \mu_0 \sigma R \omega \hat{z}$$

$$= \boxed{\frac{2}{3} \mu_0 \sigma R \vec{\omega}}$$

$\rightarrow$ uniforme dentro da esfera!



Exemplo: Potencial vetor p/ plano infinito com corrente superficial $\vec{K} = K\hat{x}$



$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{K\hat{x}\,\overbrace{r'dr'd\varphi'}^{da'}}{r_\varphi}$$

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} K\hat{x}\, 2\pi \int \frac{r'dr'}{\sqrt{r'^2+z^2}}$$

$$\vec{A}(\vec{r}) = \frac{\mu_0 K\hat{x}}{2} \frac{\cancel{2}}{\cancel{2}}\sqrt{r'^2+z^2}\Big|_{r=0}^{r=R}$$

Note que se fizermos a integral de $R\to\infty$ teremos problemas pois a corrente se estende neste caso até lá. Mas sabemos encontrar $\vec{B}$ p/ o plano infinito. Este é uma função que aponta na direção $\hat{y}$.

$$\therefore \vec{A}(\vec{r}) = \frac{\mu_0 K}{2}\left[\sqrt{R^2+z^2} - z\right]\hat{x} = f(z)\hat{x}$$

$$\vec{B}(z) = \vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f(z) & 0 & 0 \end{vmatrix} = \frac{\partial f(z)}{\partial z}\hat{y}$$

$$\vec{B}(z) = \frac{\mu_0 K}{2}\left[\frac{1}{2}\frac{2z}{\sqrt{R^2+z^2}} - 1\right]\hat{y}$$

p/ $\mathcal{R} \to \infty$ $\quad \boxed{\vec{B}(z) = -\dfrac{\mu_0 K}{2}\,\hat{y}}$ todo os pontos $z > 0$

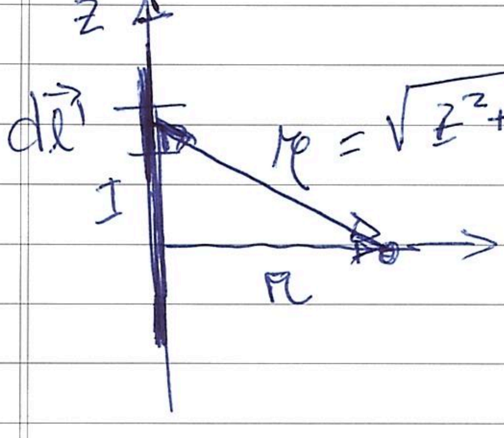
E vemos que é exatamente o valor que havíamos obtido no Exemplo 5.8 com a Lei de Ampère (pg 196).

Escolhendo $\vec{A} = \underbrace{-\dfrac{\mu_0 K}{2} z}_{f(z)}\,\hat{x}$ resolvemos p/ $z > 0$

p/ todo Espaço $\quad \boxed{\vec{A} = -\dfrac{\mu_0 K}{2}\,|z|\,\hat{x}}$

Note que nos dois exemplos que fizemos, e de forma geral, $\vec{A}$ estará na direção da corrente!

Problema 5.22 || $\vec{B}$ p/ um segmento de fio reto com corrente I. Verificar ao final $\vec{B}$



$$r_\varphi = \sqrt{z^2 + r^2}$$

$$\vec{A} = \frac{\mu_0 I}{4\pi}\int \frac{d\vec{l}'}{r_\varphi} = \frac{\mu_0 I\,\hat{z}}{4\pi}\int \frac{dl'}{r_\varphi}$$

$$\vec{A} = \frac{\mu_0 I\,\hat{z}}{4\pi}\int_{z_1}^{z_2} \frac{dz}{\sqrt{z^2 + r^2}}$$

fazendo $\frac{z}{r} \equiv tg\theta \therefore dz = r\sec^2\theta\, d\theta$; $1 + tg^2\theta = \sec^2\theta$

$$\vec{A} = \frac{\mu_0 I}{4\pi}\hat{z}\int_{\theta_1}^{\theta_2} \frac{\cancel{r}\sec^{\cancel{2}}\theta}{\cancel{r}\cancel{\sec\theta}}\, d\theta \quad \therefore \quad \vec{A} = \frac{\mu_0 I}{4\pi}\hat{z}\int_{\theta_1}^{\theta_2} \sec\theta\, d\theta$$

Não trivial !!

$$\int \sec x\, dx = \int \sec x \left(\frac{\sec x + tg x}{\sec x + tg x}\right) dx$$

$$\xi = \sec x + tg x \quad \therefore \quad d\xi = \sec x\, tg x + \sec^2 x)\, dx$$

$$\int \frac{(\sec x^2 + \sec x\, tg x)\, dx}{(\sec x + tg x)} = \int \frac{d\xi}{\xi}$$

$$\boxed{= \ln|\xi| + C}$$

$$\therefore \int \sec x\, dx = \ln|\sec x + tg x| + C$$

$$\therefore \vec{A} = \frac{\mu_0 I}{4\pi}\hat{z}\ln|\sec\theta + tg\theta|\Big|_{\theta_1}^{\theta_2}$$

Mas $\sec\theta = \frac{\sqrt{r^2 + z^2}}{r}$ e $tg\theta = \frac{z}{r}$

$$\therefore |\sec\theta + tg\theta| = \left|\frac{\sqrt{r^2+z^2} + z}{r}\right|$$

$$\therefore \boxed{\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi}\hat{z}\ln\left[\frac{\sqrt{r^2+z_2^2} + z_2}{\sqrt{r^2+z_1^2} + z_1}\right]}$$

$\vec{B} = \nabla \times \vec{A}$ (Em coordenadas cilíndricas).

$$\nabla \times \vec{A} = \left(\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right)\hat{r} + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\hat{\phi}$$
$$+ \frac{1}{r}\left(\frac{\partial}{\partial r}(r A_\phi) - \frac{\partial}{\partial \phi}A_r\right)\hat{z}$$

$$\vec{A} = (0, 0, A_z) \therefore \boxed{\vec{B} = -\frac{\partial}{\partial r}A_z\hat{\phi}}$$

$$B_\phi \frac{4\pi}{\mu_0 I} = -\frac{\partial A_z}{\partial r} = -\frac{d}{dr}\left[\ln\left(z_2 + \sqrt{z_2^2 + r^2}\right) - \ln\left(z_1 + \sqrt{z_1^2 + r^2}\right)\right]$$

$$= \left[\frac{1}{\left(z_2 + \sqrt{z_2^2 + r^2}\right)}\frac{1}{\cancel{2}}\frac{\cancel{2}r}{\sqrt{z_2^2 + r^2}} - \frac{1}{z_1 + \sqrt{z_1^2 + r^2}}\frac{1}{\cancel{2}}\frac{\cancel{2}r}{\sqrt{z_1^2 + r^2}}\right]$$

$$B_\phi = -\frac{\mu_0 I r}{4\pi}\left[\frac{z_2 - \sqrt{z_2^2 + r^2}}{\cancel{z_2^2} - \cancel{z_2^2} + r^2}\frac{1}{\sqrt{z_2^2 + r^2}} - \left(\text{Idem p/ } z_1\right)\right]$$

$$B_\phi = -\frac{\mu_0 I r}{4\pi}\left(\frac{-1}{r^2}\right)\left[\frac{z_2}{\sqrt{z_2^2 + r^2}} - \cancel{1} - \frac{z_1}{\sqrt{z_1^2 + r^2}} + \cancel{1}\right]$$

$$B_\phi = \frac{\mu_0 I}{4\pi r}\left[\frac{z_2}{\sqrt{z_2^2 + r^2}} - \frac{z_1}{\sqrt{z_1^2 + r^2}}\right].$$

Como $\sin\theta_1 = \dfrac{z_1}{\sqrt{z_1^2 + r^2}} \therefore$

$$\boxed{\vec{B} = \frac{\mu_0 I}{4\pi r}\left(\sin\theta_2 - \sin\theta_1\right)\hat{\phi}}$$

Igual resultado do Exemplo 5.5

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Problema 5.23 Griffths.

Qual $\vec{J}$ produz $\vec{A} = k\hat{\phi}$ $k = cte$.

Sabemos que $\vec{B} = \nabla \times \vec{A} = \frac{1}{s}\begin{vmatrix} \hat{s} & s\hat{\phi} & \hat{z} \\ \frac{\partial}{\partial s} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ 0 & sA_\phi & 0 \end{vmatrix} = \frac{1}{s}\frac{\partial}{\partial s}(ks)\hat{z}$

$$\therefore \boxed{\vec{B} = \frac{k}{s}\hat{z}}$$

Vemos que $\left(\nabla \cdot \vec{A} = \frac{\partial k}{\partial \phi} = 0\right)$

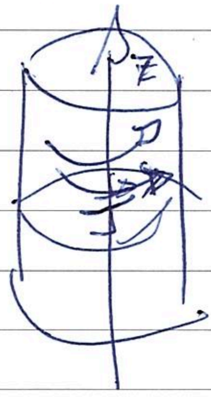
Como esperado!

Mas $\nabla \times \vec{B} = \mu_0 \vec{J}$

$$\therefore \vec{J} = \frac{1}{\mu_0 s}\begin{vmatrix} \hat{s} & s\hat{\phi} & \hat{z} \\ \frac{\partial}{\partial s} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ 0 & 0 & \frac{k}{s} \end{vmatrix} = \frac{-1}{\mu_0 s} s\hat{\phi}\frac{\partial}{\partial s}\left(\frac{k}{s}\right)$$

$$\boxed{\vec{J} = \frac{k}{\mu_0 s^2}\hat{\phi}}$$

Um caso que exemplificaria isto seria um ~~disco~~ cilindro girando com $\vec{\omega} = \omega\hat{z}$ e densidade de carga $\rho(s)$ nos dando um $\vec{J}$ na direção ϕ



$\vec{J} = \rho(\vec{s}) \cdot \vec{v}$

$\vec{J} = \rho(\vec{s})\, \vec{\omega} \times \vec{s} = \rho(s)\, \omega s\, \hat{\phi}$

Fazendo $\omega \rho(s) s = \frac{K}{\mu_0 s^2}$