Charged Particle Oscillating Near a Capacitor

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We study the oscillation of a charged particle near a capacitor in four different models: Classical mechanics, Weber's electrodynamics plus classical mechanics, relativistic mechanics, and Weber's electrodynamics plus the mechanics of Erwin Schrödinger. We show that only the third and fourth models yield physically reasonable results.

PACS numbers: 41.20.-q, 12.25.+e, 03.20.+I; **Key Words:**
Weber's electrodynamics, Schrödinger's mechanics, theory of relativity, potential dependent inertial mass.

Introduction

In this work, we analyze a charged particle oscillating near a capacitor. We shall show that, depending on the theoretical model used to analyze the situation, the frequency of oscillation may depend upon the difference of electrostatic potential in the capacitor. Other theoretical models do not predict this effect, so in principle it allows a possible experimental distinction between all the approaches. This problem is also important in the context of possible experimental tests of a potential-dependent inertial mass [1, p. 189]; [2], [3], and [4].

We analyze the problem in four different models to highlight the different approaches and predictions. The first one is based on classical mechanics (kinetic energy = $mv^2/2$) and Lorentz's force (which in this case reduces to Coulomb's law). It will be called the classical (C) model. The second one is based on classical mechanics and Weber's electrodynamics, [1]. It will be called the classical-Weber model (W). The third model is based on relativistic mechanics (relativistic kinetic energy) and Lorentz's force. It will be called the relativistic model (R). And the fourth model is based on Schrödinger's mechanics, [5] and [1, pp. 220-221], and Weber's electrodynamics. It will be called Schrödinger's model, (S). There is conservation of energy for this situation in all four models. This property will be employed in this work.

The geometry of the problem is presented in Figure 1. There is an ideal capacitor at rest relative to an inertial frame. Its infinite plates are parallel to the yz plane with the center of the capacitor on the origin of the coordinate system. The positive (negative) plate is located at x_0 $(-x_0)$ and has a uniform surface charge density σ_0 $(-\sigma_0)$. A charge q oscillates orthogonally to the plates of the capacitor along the x axis. It is outside the capacitor, on its left side. The oscillation is supposed to be due to an elastic force $\mathbf{F} = -k\mathbf{r}$. This force may be generated by a spring or by any other source not related with the capacitor. We want to analyze the velocity of the charge as a function of its energy, the velocity as a function of the difference of potential in the capacitor plates, and the period of oscillation as a function of this voltage.

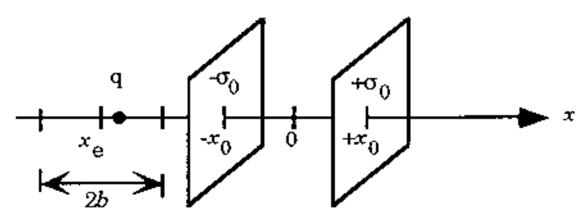

Figure 1: Geometry of the problem.

The goal of this work is to discuss conceptual differences between several theoretical models. An analysis fully taking into account the possible complications of electrostatic induction is beyond the scope of this paper. In order to avoid induction, we suppose the charged plates of the capacitor to be composed of a dielectric material. In this way, they are not influenced by the test charge, regardless of its motion. So even when the test charge is oscillating near a surface, the charge is distributed uniformly over it.

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The Classical Model

Classical electrodynamics is based on Lorentz's force $\mathbf{F} = q\mathbf{E} + q\mathbf{v}\times\mathbf{B}$. As the charges in the capacitor are at rest, they produce no magnetic field and this force reduces to Coulomb's law. For $x < -x_0$ or $x > x_0$ the capacitor generates only a constant potential, so that there is no electric field or force acting on the test charge in this model. In all models discussed here, the zero of the potential is defined at the negative plate of the capacitor. The energy E of the charge is given by

$$E = \frac{mv^2}{2} + \frac{k(x-x_e)^2}{2} \tag{1}$$

where m is the inertial mass of the particle, $v = |\mathbf{v}|$ is its velocity relative to the capacitor, k is the elastic constant, x its position, x_e the position of the charge when it feels no elastic force. It is easily seen that the maximal velocity, v_m, is obtained when $x = x_e$, so that: $v_m^2 / c^2 = 2E / mc^2$. The value of E can be varied according to the initial conditions. If E is bigger than $mc^2/2$ (if the test charge is an electron, this means $E \geq 0.5\text{Mev}/2 \approx 4\times 10^{-14}\,\text{J}$), then the maximal velocity will be greater than c. This has never been observed and indicates a limitation of the model.

With an initial condition given by $x - x_e = b$ and $v = 0$, the solution of Eq. (1) utilizing conservation of energy is given by

$$x = x_e + b\cos\omega_c t \tag{2}$$

where $\omega_c \equiv \sqrt{k/m}$ is the classical frequency of oscillation. The maximal velocity is given by $v_m = b\omega_c$. This can be bigger than c depending on the values of b and ω_c.

The period of oscillation will be calculated in all models as

$$T = 4\int_0^b \frac{dx}{v} \tag{3}$$

where b is the amplitude of oscillation: $x_e - b \leq x \leq x_e + b$. In this classical situation the period is easily found to be equal to $T \equiv 2\pi\sqrt{m/k}$.

The Classical-Weber Model

In this case the energy of the test charge moving orthogonally to the plates of the capacitor is given by [1, p. 189]; [2], [3] and [4]:

$$E = \frac{mv^2}{2} + \frac{k(x-x_e)^2}{2} - \frac{qV}{2c^2}\frac{v^2}{2} \tag{4}$$

Here $V = 2\sigma_0 x_0 / \varepsilon_0$ is the voltage or difference of potential between the plates of the capacitor.

The main difference relative to the previous result is that now the electromagnetic energy outside the ideal capacitor depends on the velocity of the test charge relative to the plates. This does not happen with classical electrodynamics based on Lorentz's force.

In classical Newtonian mechanics the kinetic energy of the particle is $T = mv^2/2$ or $T = m_{ei}v^2/2$. The coefficient of $v^2/2$ appears also in front of the acceleration in Newton's second law of motion, as in $m\mathbf{a}$ or $m_{ei}\mathbf{a}$ [Ref. 1, Sec. 7.3].

We can combine the coefficients of $v^2/2$ in (4) yielding $(m - qV/2c^2)v^2/2$. This indicates that we can define an effective inertial mass $m - qV/2c^2$. That is, Weber's electrodynamics combined with Newtonian mechanics is equivalent to classical electrodynamics with an effective inertial mass which depends on the electrostatic potential energy of the test charge. In the Weber model, the test charge will behave as if it had an effective inertial mass given by $m - qV/2c^2$.

Defining a dimensionless constant α by $qV/2mc^2$ (relating the electrostatic energy of the test charge to its rest energy) we obtain for $\alpha < 1$ that the maximal velocity when $x = x_e$ is given by

$$v_m^2 / c^2 = 2E / mc^2(1-\alpha) \tag{5}$$

When $\alpha = 0$ we recover the classical model. The classical-Weber model also predicts a maximal velocity for the test charge bigger than c. Now this happens when $E \geq mc^2(1-\alpha)/2$. If $\alpha \to 1$, this should be easily obtained for small values of E. Again there is no experimental indication of this effect.

For $\alpha < 1$ and an initial condition given by $x - x_e = b$ and $v = 0$ the solution of Eq. (4) utilizing conservation of energy is given by

$$x = x_e + b\cos\omega_w t \tag{6}$$

where $\omega_w \equiv \sqrt{k/m(1-\alpha)} = \omega_c/\sqrt{1-\alpha}$. The maximal velocity is given by $v_m = b\omega_w = b\omega_c/\sqrt{1-\alpha}$. Even for $b\omega_c < c$ the maximal velocity can become bigger than c for α close to 1.

The period of oscillation for $\alpha < 1$ is given by $T_w \equiv \sqrt{1-\alpha}T_c$. For $\alpha = 1$ Eq. (4) yields $E = k(x-x_e)^2/2 =$ constant. This means that x will be a constant at any time and $v = 0$. That is, with an effective inertial mass going to zero the test charge will not interact with the spring.

For $\alpha > 1$ and an initial condition given by $x - x_e = b$ and $v = 0$ the solution of Eq. (4) is given by

$$x = x_e + b\cosh(\omega_{w2}t) \tag{7}$$

where $\omega_{w2} \equiv \omega_c/\sqrt{\alpha - 1}$. The motion is not oscillatory anymore. Now x and v increase indefinitely with time. Once extended, the spring will continue extending itself to infinity in this model. Instead of a restoring force, the spring will enlarge all perturbations of the equilibrium position.

The Relativistic Model

We now have a relativistic kinetic energy $mc^2/\sqrt{1-v^2/c^2} - mc^2$ and Lorentz's or Coulomb's force. This means that the energy of the test charge will be given by

$$E=\frac{mc^2}{\sqrt{1-v^2/c^2}}-mc^2+\frac{k(x-x_e)^2}{2} \tag{8}$$

When $v^2 \ll c^2$ we recover the classical model. The maximal velocity when $x=x_e$ as a function of E is given by

$$\frac{v_m^2}{c^2}=1-\frac{1}{(1+E/mc^2)^2} \tag{9}$$

Even when E tends to infinity, $v_m \to c$. This indicates that the velocity of the test charge cannot increase beyond c in this model.

We can integrate Eq. (8) to obtain the period of oscillation in terms of elliptic integrals; namely,

$$T=\frac{2^{3/2}b}{c}\left[\frac{1}{\sqrt{\gamma}}\int_0^{\pi/2}\frac{d\phi}{\sqrt{1+\gamma\sin^2\phi}}+\sqrt{\gamma}\int_0^{\pi/2}\frac{\sin^2\phi d\phi}{\sqrt{1+\gamma\sin^2\phi}}\right] \tag{10}$$

where b is the amplitude of oscillation and $\gamma=kb^2/2mc^2$ is a dimensionless parameter relating the elastic energy to the rest energy of the test particle.

We now expand this result in powers of γ (that is, supposing $\gamma \ll 1$, which is equivalent to $v^2 \ll c^2$). Going until first order in γ (or until v^4/c^4) and utilizing $T_c=2\pi\sqrt{m/k}$ yields

$$T_r \approx T_c\left(1+\frac{3\gamma}{8}\right) \tag{11}$$

This means that for the same initial conditions the relativistic harmonic oscillator has a larger period than the classical one. This can be understood observing that the relativistic inertial mass increases with the velocity of the particle. This means that during a period of oscillation the particle will behave as if it had a larger inertial mass than the classical one. The average relativistic inertial mass during one period is then given by $<m>=m(1+3\gamma/4)$.

Schrödinger's Model

We now have a kinetic energy similar to the relativistic one, although derived in compliance with Mach's principle, [5], and a Weberian electromagnetic energy. Apart from an unimportant constant, the energy of the test particle is given by

$$E=\frac{mc^2}{\sqrt{1-v^2/c^2}}-mc^2+\frac{k(x-x_e)^2}{2}-\frac{qV}{2}\frac{v^2}{2c^2} \tag{12}$$

With $V=0$ we recover the relativistic situation. Once more we can see that when $E\to\infty$, then $v\to c$. This shows that with Schrödinger's mechanics and Weber's electrodynamics, we have the velocity of the test particle limited by c, as in the relativistic model.

We now utilize the conservation of energy and the fact that when $v=0$ we have $x-x_e=b$. This equation can then be written as

$$\frac{1}{\sqrt{1-v^2/c^2}}-\frac{\alpha v^2}{2c^2}=1+\gamma-\frac{k(x-x_e)^2}{2mc^2} \tag{13}$$

The maximal velocity v_m is obtained when $x=x_e$. This equation shows that even when $\alpha\to\infty$ we have $v_m\to c$, no matter the value of E. That is, the velocity is always limited by c, not only for high total energies of the particle, but also for high voltages in the capacitor.

In order to calculate the period of oscillation we expand this last result in powers of v^2/c^2. Retaining terms only up to v^4/c^4 yields

$$\frac{3v^4}{8c^4}+\frac{(1-\alpha)}{2}\frac{v^2}{c^2}-\gamma+\frac{k(x-x_e)^2}{2mc^2}=0 \tag{14}$$

With $\alpha \ll 1$, the period is found to be

$$T_s=T_c\sqrt{1-\alpha}\left[1+\frac{3}{8}\frac{\gamma}{(1-\alpha)^2}\right] \tag{15}$$

For $\alpha\geq 1$ the motion is not oscillatory anymore. The velocity will tend to the critical velocity $v_c\equiv\sqrt{1-\alpha^{2/3}}c$ which for $\alpha>1$ is smaller than c. This result is reasonable because now the effective inertial mass depends not only on the potential but also on the velocity. The equation of motion can be written as

$$m_{ei}a+k(x-x_e)=0 \tag{16}$$

where $a=d^2x/dt^2$ and m_{ei} is the effective inertial mass given by

$$m_{ei}\equiv m\left[\frac{1}{(1-v^2/c^2)^{3/2}}-\alpha\right] \tag{17}$$

Discussion and Conclusion

We have obtained that the classical and classical-Weber models do not correspond to reality. Both models predict that the velocity of the test charge can reach velocities larger than c, depending on initial conditions. The relativistic and Schrödinger's models do not have this problem. There is, however, a difference between these two last models: The period of oscillation does not depend on the voltage in the capacitor according to relativity, but should depend according to Schrödinger's model with Weber's electrodynamics. Only careful experiments can decide this question. To our knowledge there has never been any experiment designed to test this property.

All the results of this paper could be maintained if the test charge oscillated parallel to the plates of the capacitor, provided that we change the sign in front of $\alpha=qV/2mc^2$ in all formulas where it appears, [1]. This means that on average the result will be independent of α in the classical-Weber and Schrödinger's models if we have a particle moving in a circular orbit in a plane orthogonal to the plates of the capacitor. On the other hand if we have a circular orbit in a plane parallel to the plane of the capacitor the results of this paper might be maintained provided we changed the sign of α. The same would happen if the charge were oscillating orthogonally to the plates of the capacitor, but now on the right of the positive plate; see Figure 1.

It should be emphasized that in this study we are neglecting radiation losses, the currents induced in the plates of the capacitor due to the oscillation of the test charge, border effects, etc. Moreover, we are assuming an elastic constant k independent of the voltage in the capacitor. With Weber's theory this might be questioned due to the fact that the inertial mass depends on the potential. As the elastic force may be due to a microscopic electromagnetic force, it could also be influenced by the voltage in the capacitor. Only carefully designed experiments could decide this question.

Acknowledgments

J.J. Caluzi wishes to thank CNPq for financial support.

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Correspondence

Magnetic Field Outside a Current-Carrying Solenoid

Since the solenoid plays in magnetism a role identical to the role played by the capacitor in electricity, it deserves a thorough comprehension at the undergraduate level. Customary mathematical descriptions of the flux density of a current-carrying solenoid focus on the magnetic flux density, $\mathbf{B}_i$, inside the solenoid, using universally accepted principles to calculate it. Less commonly, it is noted that a linear solenoid also has an equally understandable *external* azimuthal magnetic field $\mathbf{B}_{ext}$. And it is almost never noted that with toroidal current-carrying coils, there can exist an external flux density $\mathbf{B}_{ext}$, resembling that due to a circular loop.

Our considerations can be valuable in dealing with the so-called *leakage* or *disperse flux*. It is common to resort to random or unknown causes in order to explain such leakage or disperse flux. But here we discuss a possible causal origin for the observable effect, an origin understandable within the framework of classical electrodynamics.

Analysis

Let us consider a long linear solenoid, centered on the z axis of an orthogonal frame, with n turns per unit length, carrying steady current of I ampères (Fig. 1). In the International (mksa) system [1], the inner magnetic flux on the axis has value

$$B_z = \mu_0 nI(\cos\theta_1 + \cos\theta_2)/2$$

Now, we must recognize that current, while traveling circularly along each loop, also moves upwards along the z axis (if the coil were infinitely long, from $z = -\infty$ to $z = +\infty$). Therefore, far enough from the axis, the above component of current behaves as a linear, upward current I. According to classical electrodynamics, the above current is responsible for an external azimuthal magnetic flux density $\mathbf{B}_{ext} = (\mu_0 I / 2\pi r)\mathbf{u}_\phi$, with $\mathbf{u}_\phi$ being the unit vector along ϕ (cylindrical coordinates) and r is measured from the z axis.

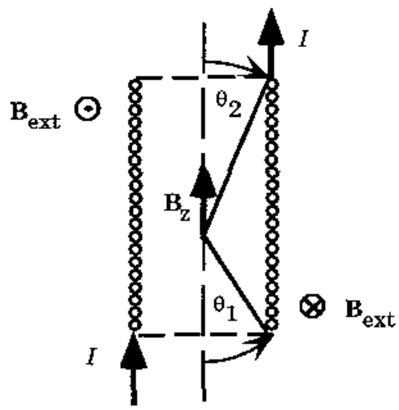

Figure 1.

The same argument remains valid when dealing with a toroidal current-carrying coil (Fig. 2). If the coil is centered on the z axis and has N turns, we get,

$$\mathbf{B}_{int} = (\mu_0 NI / 2\pi r)\mathbf{u}_\phi \quad (1)$$

Viewed from the outside, the coil behaves as a circular loop carrying a counterclockwise current I (Fig. 3). In mksa units, the r component of the external magnetic flux has value [2],

$$B_r = \frac{\mu_0 I a^2}{4}\cos\theta\frac{2a^2 + 2r^2 + ar\sin\theta}{(a^2 + r^2 + 2ar\sin\theta)^{5/2}}$$

continued on page 110