F 415 – Mecânica Geral II

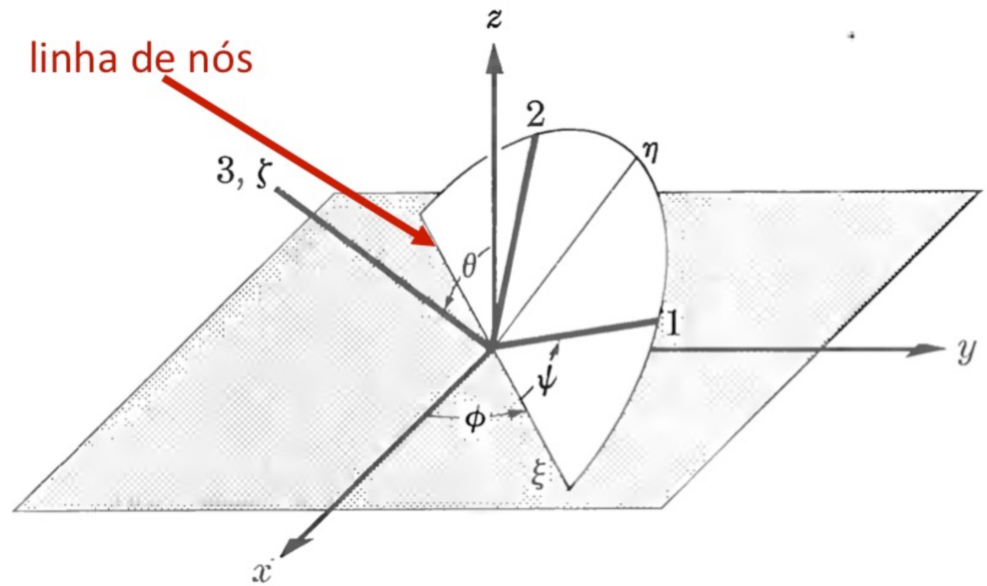
1º semestre de 2024

07/05/2024

Aula 16

Aula passada

Ângulos de Euler (ϕ, θ, ψ)



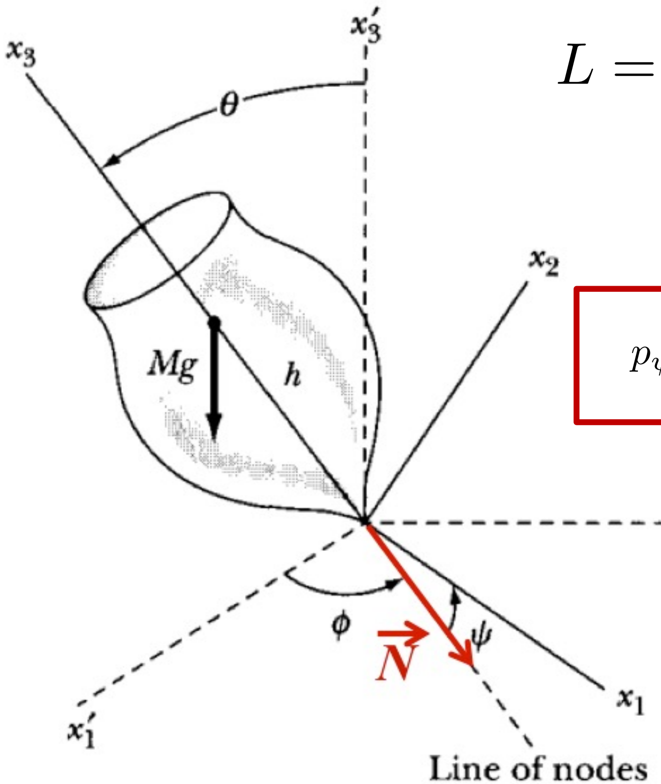
Velocidade angular instantânea de rotação em termos dos ângulos de Euler:

$$
\begin{aligned}
\omega_1 &= \dot{\theta}\cos\psi + \dot{\phi}\sin\theta\sin\psi \\
\omega_2 &= -\dot{\theta}\sin\psi + \dot{\phi}\sin\theta\cos\psi \\
\omega_3 &= \dot{\psi} + \dot{\phi}\cos\theta
\end{aligned}
$$

Energia cinética de um corpo simétrico $(I_1 = I_2)$:

$$L = \frac{1}{2}\boldsymbol{\omega}\cdot\mathsf{I}\cdot\boldsymbol{\omega} = \frac{I_1}{2}\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right) + \frac{I_3}{2}\left(\dot{\psi} + \dot{\phi}\cos\theta\right)^2$$

Aula passada



$$L = \frac{I_1}{2}\left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right) + \frac{I_3}{2}\left(\dot{\psi} + \dot{\phi}\cos\theta\right)^2 - Mgh\cos\theta$$

Três constantes do movimento

$p_\psi = \dfrac{\partial L}{\partial \dot{\psi}} = I_3\left(\dot{\psi} + \dot{\phi}\cos\theta\right) = I_3\omega_3$ componente do mom. angular ao longo de $\widehat{e}_3$

$p_\phi = \dfrac{\partial L}{\partial \dot{\phi}} = I_1\dot{\phi}\sin^2\theta + I_3\left(\dot{\psi} + \dot{\phi}\cos\theta\right)\cos\theta = I_1\dot{\phi}\sin^2\theta + p_\psi\cos\theta$

componente do mom. angular ao longo de $\widehat{z}\,(x_3')$

O torque do peso atua ao longo da linha de nós e é perpendicular a $\widehat{e}_3$ e $\widehat{z}$

$$E = \frac{I_1}{2}\left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right) + \frac{I_3}{2}\left(\dot{\psi} + \dot{\phi}\cos\theta\right)^2 + Mgh\cos\theta$$

energia mecânica conservada: sistema conservativo

Aula passada

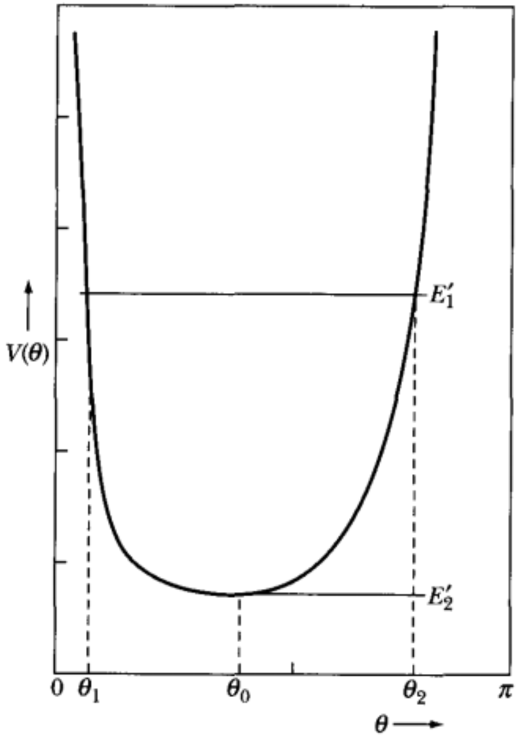
Eliminamos a dependência com $\dot{\phi}, \dot{\psi}$

$$\left(\dot{\psi} + \dot{\phi}\cos\theta\right) = \frac{p_\psi}{I_3} \qquad \dot{\phi} = \frac{p_\phi - p_\psi \cos\theta}{I_1 \sin^2\theta}$$

$$E - \frac{p_\psi^2}{2I_3} \equiv E' = \frac{I_1}{2}\dot{\theta}^2 + \frac{1}{2I_1}\left(\frac{p_\phi - p_\psi\cos\theta}{\sin\theta}\right)^2 + Mgh\cos\theta$$

Potencial efetivo:

$$\boxed{V'(\theta) = \frac{1}{2I_1}\left(\frac{p_\phi - p_\psi\cos\theta}{\sin\theta}\right)^2 + Mgh\cos\theta}$$



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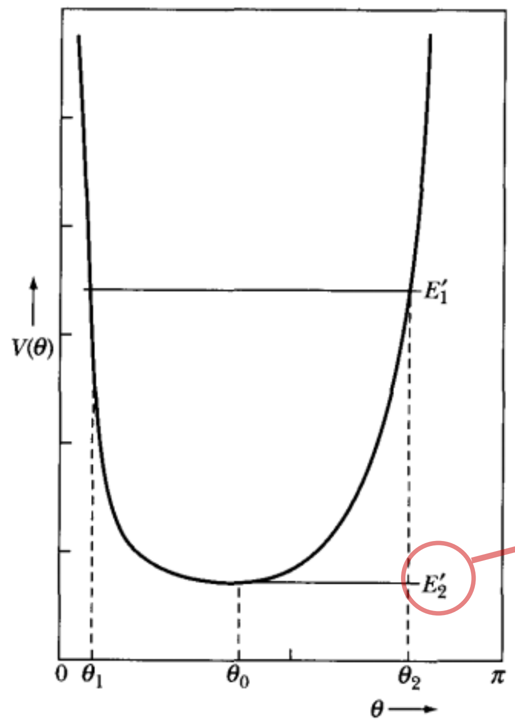
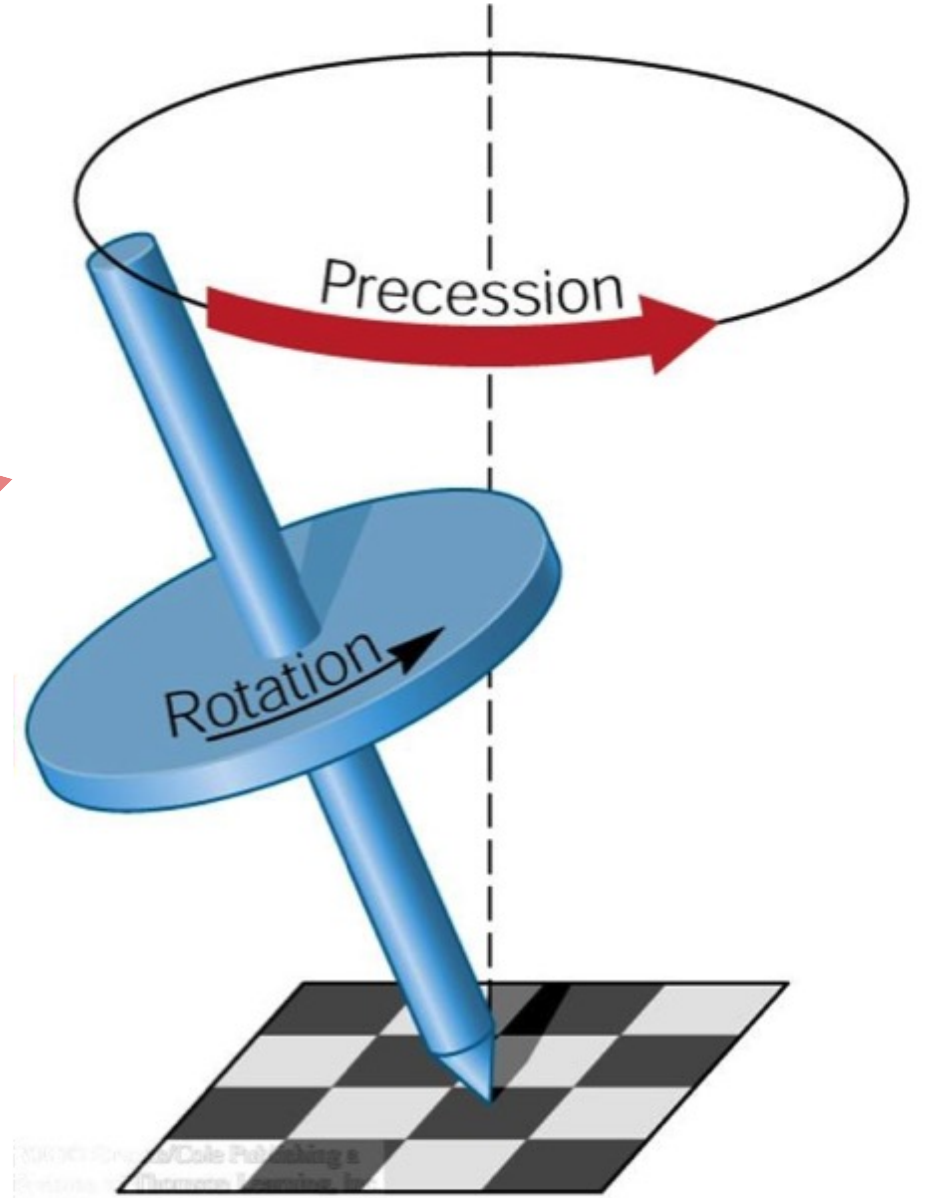


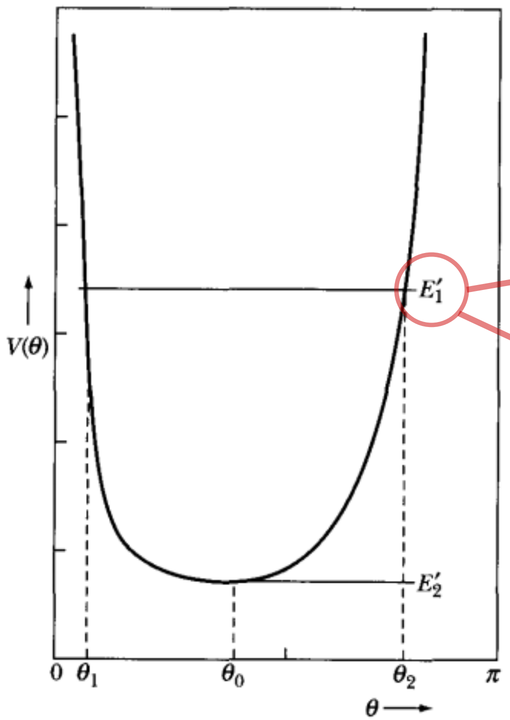


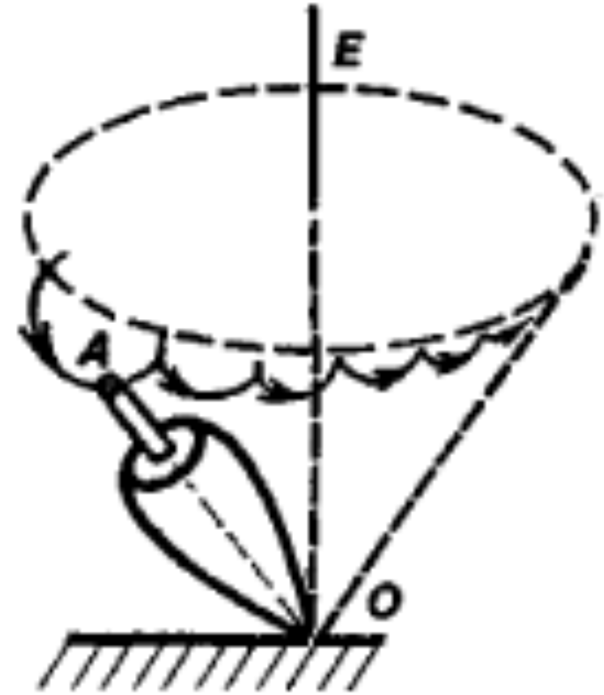
Precessão regular

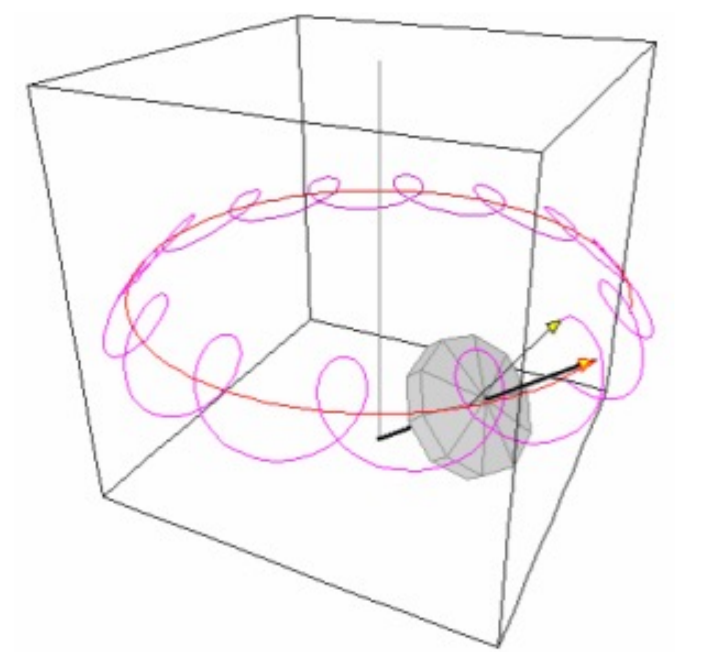
$$\dot{\phi} = \frac{p_\phi - p_\psi \cos\theta_0}{I_1 \sin^2\theta_0}$$

$$\dot{\psi} = \frac{p_\psi}{I_3} - \dot{\phi}\cos\theta_0$$

Aula passada





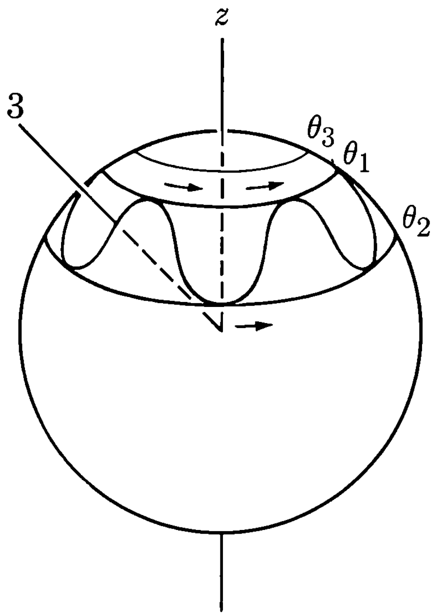


Nutação

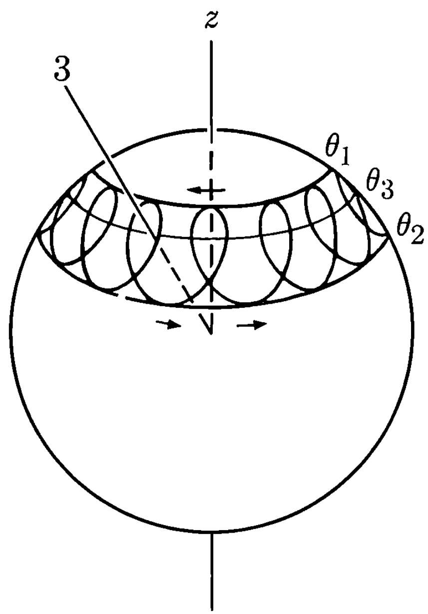
$$\dot{\phi}(t) = \frac{p_\phi - p_\psi \cos\theta(t)}{I_1 \sin^2\theta(t)}$$

$$\dot{\psi}(t) = \frac{p_\psi}{I_3} - \dot{\phi}(t)\cos\theta(t)$$

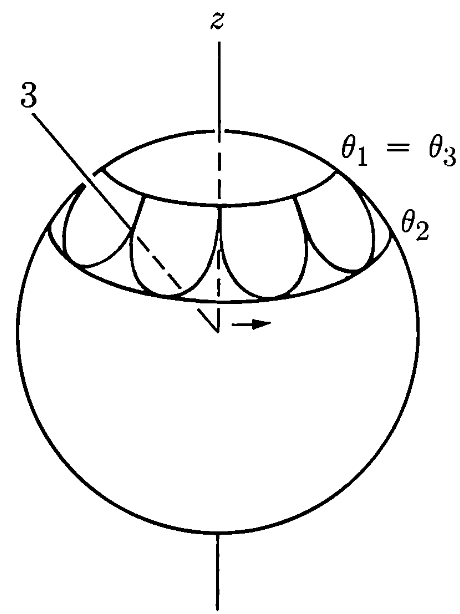
Aula passada



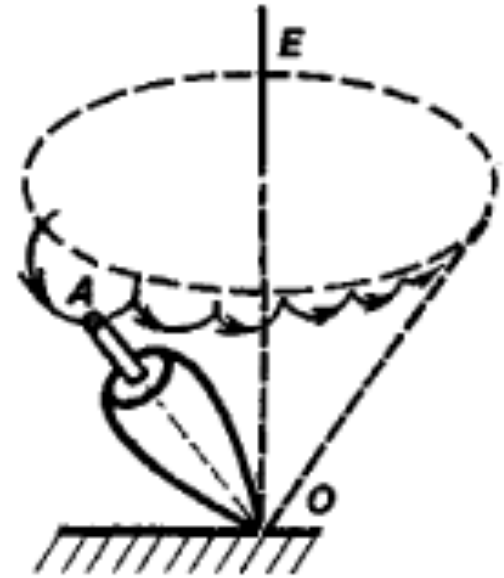
(a)



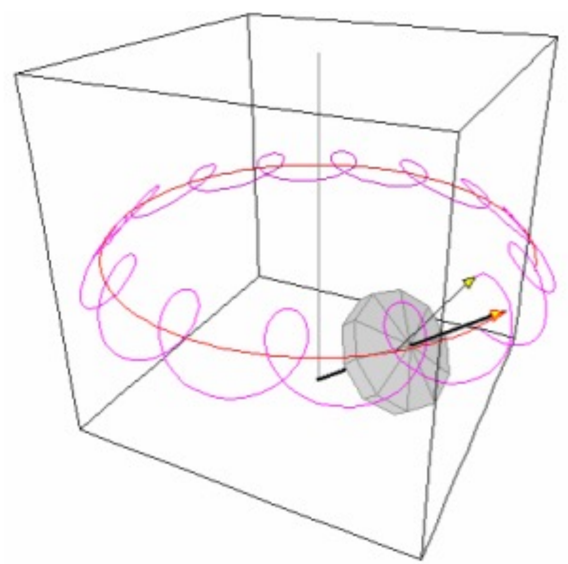
(b)



(c)



b

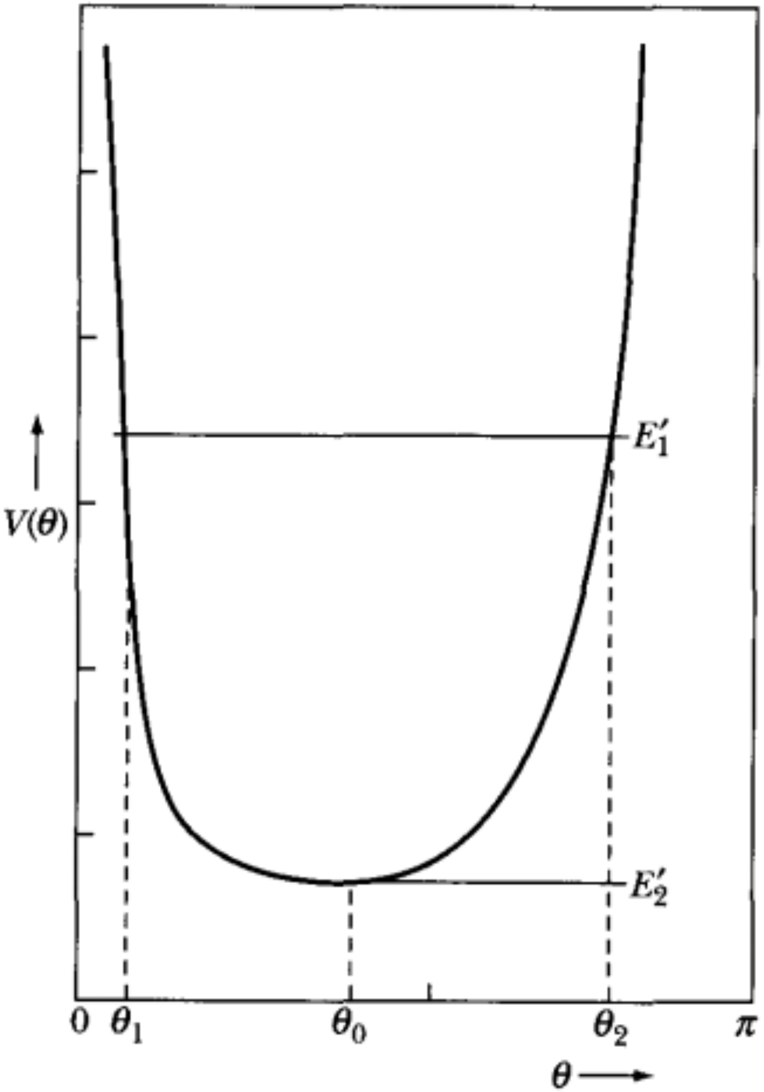


Nutação

$$\dot{\phi}(t) = \frac{p_\phi - p_\psi \cos\theta(t)}{I_1 \sin^2\theta(t)}$$

$$\dot{\psi}(t) = \frac{p_\psi}{I_3} - \dot{\phi}(t)\cos\theta(t)$$

O mínimo do potencial efetivo



$$\boxed{V'(\theta) = \frac{1}{2I_1}\left(\frac{p_\phi - p_\psi\cos\theta}{\sin\theta}\right)^2 + Mgh\cos\theta}$$

MÍNIMO: $\left.\frac{dV'(\theta)}{d\theta}\right|_{\theta=\theta_0} = 0$

$$\left.\frac{dV'}{d\theta}\right|_{\theta=\theta_0} = \frac{(p_\phi - p_\psi\cos\theta_0)(p_\psi - p_\phi\cos\theta_0)}{I_1\sin^3\theta_0} - Mgh\sin\theta_0 = 0$$

DEFINIMOS: $\beta = p_\phi - p_\psi\cos\theta_0 \Rightarrow \dot{\phi} = \frac{\beta}{I_1\sin^2\theta_0}$

EM TERMOS DE β AO INVÉS DE p_ϕ:

$$\cos\theta_0\,\beta^2 - p_\psi\sin^2\theta_0\,\beta + MghI_1\sin^4\theta_0 = 0 \quad (1)$$

(1) TEM DUAS SOLUÇÕES:

$$\beta_\pm = \frac{p_\psi \sin^2\theta_0}{2\cos\theta_0}\left[1 \pm \left(1 - \frac{4MghI_1}{p_\psi^2}\cos\theta_0\right)^{1/2}\right]$$

CASO (a): $\theta_0 \in [0, \pi/2]$. SÓ HÁ RAÍZES REAIS SE:

$$1 \geq \frac{4MghI_1}{p_\psi^2}\cos\theta_0 \Rightarrow p_\psi = I_3\omega_3 \geq 4MghI_1\cos\theta_0$$

$$\Rightarrow \omega_3 \geq \frac{2}{I_3}\sqrt{MghI_1\cos\theta_0} \equiv \omega_{min}$$

SÓ HÁ PRECESSÃO REGULAR SE $\omega_3 \geq \omega_{min}$

SE $\omega_3 > \omega_{min}$, HÁ DUAS VELOCIDADES POSSÍVEIS DE PRECESSÃO (DADAS POR $\dot\phi = \beta / I_1 \sin^2\theta_0$): SE $\omega_3 \gg \omega_{min}$

$$\beta_+ = \beta_{rap} \simeq \frac{I_3\omega_3 \sin^2\theta_0}{\cos\theta_0} \Rightarrow \dot\phi_r \simeq \frac{I_3\omega_3}{I_1\cos\theta_0}$$

(PRECESSÃO RÁPIDA)

A precessão regular

$$\beta_- = \beta_\ell \cong \frac{I_1}{I_3}\frac{Mgh}{\omega_3}\sin^2\theta_0 \Rightarrow \dot{\phi}_\ell \cong \frac{Mgh}{I_3\omega_3}$$

(PRECESSÃO LENTA)

USUALMENTE, OBSERVA-SE A PRECESSÃO LENTA.

CASO (b): $\theta_0 \in \left[\frac{\pi}{2}, \pi\right] \Rightarrow \cos\theta_0 < 0$ E NÃO HÁ ω_3 MÍNIMA PARA QUE HAJA PRECESSÃO.

TAMBÉM HÁ $\dot{\phi}_\ell$ E $\dot{\phi}_a$ COMO ANTES.

A nutação

$$\dot{\phi}(t) = \frac{p_\phi - p_\psi \cos\theta(t)}{I_1 \sin^2\theta(t)}$$

À MEDIDA QUE θ VARIA $\dot{\phi}(t)$ TAMBÉM VARIA.

EM PARTICULAR, $\dot{\phi}(t)$ PODE ZERAR OU TROCAR DE SINAL DURANTE O MOVIMENTO.

PARA QUE $\dot{\phi}(t)$ SE ANULE, EU DEVO TER:

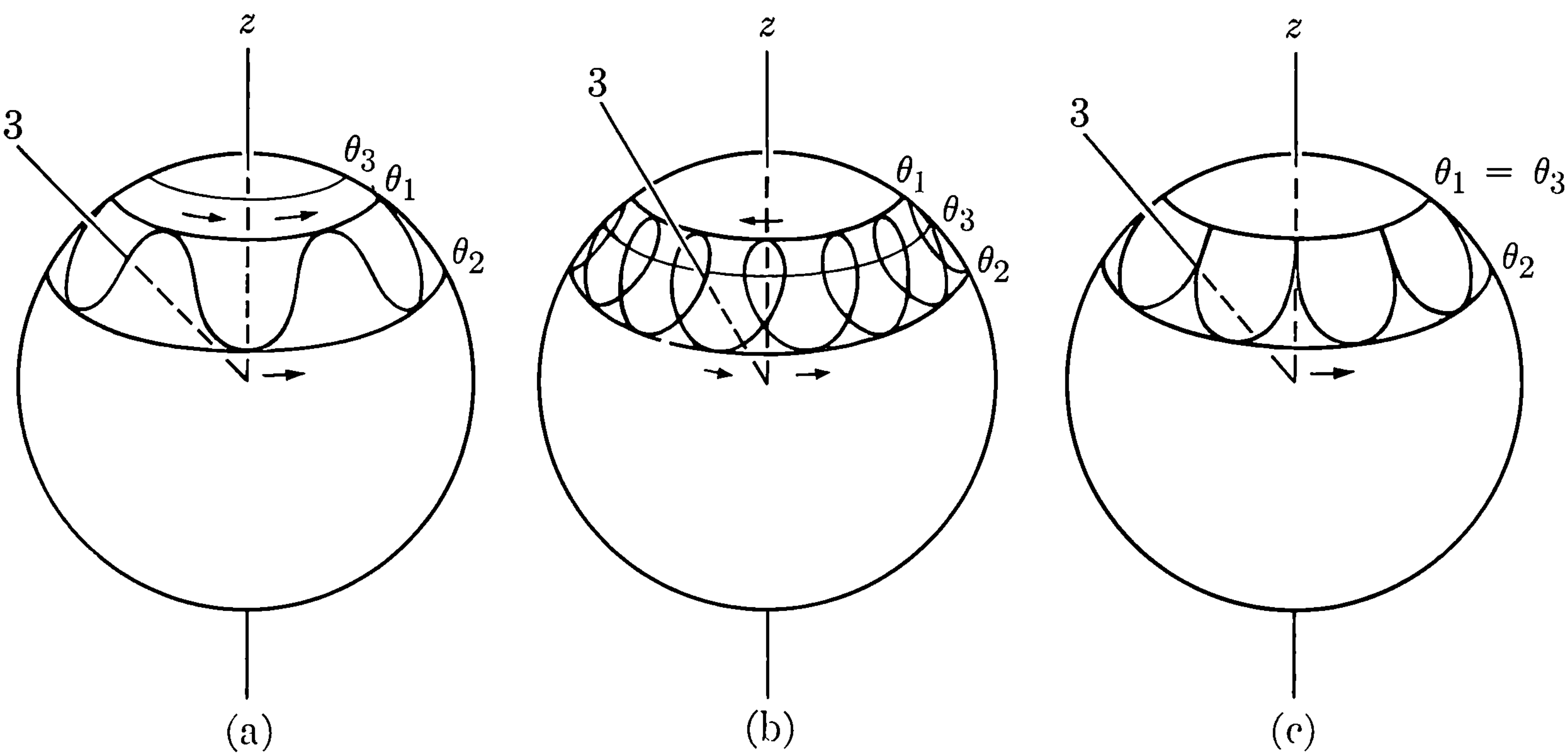
$p_\phi = p_\psi \cos\theta_3$ PARA ALGUM VALOR DE θ

$\Rightarrow \cos\theta_3 = \dfrac{p_\phi}{p_\psi}$. POSSIBILIDADES:

1) NÃO EXISTE $\theta_3 \in [\theta_1, \theta_2]$ E $\dot{\phi}$ NUNCA TROCA DE SINAL (FIGURA (a))

2) EXISTE $\theta_3 \in [\theta_1, \theta_2]$ E $\dot{\phi}$ TROCA DE SINAL DURANTE O MOVIMENTO (CASO (b))

3) EXISTE $\theta_3 = \theta_1$ E O MOVIMENTO TEM CÚSPIDES (CASO (C))

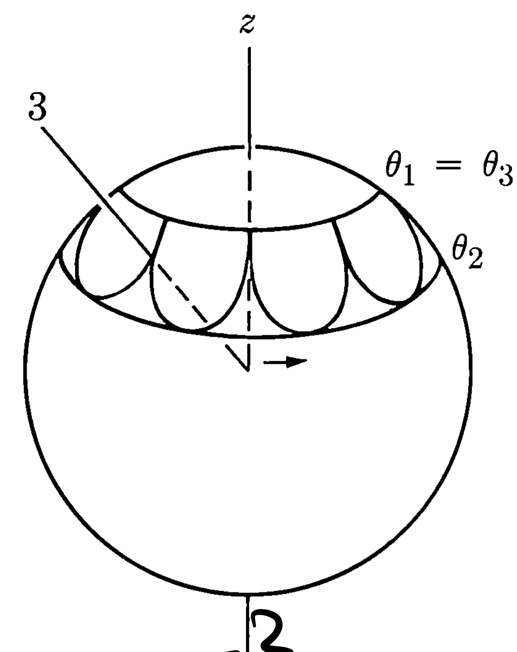


$$\dot{\phi}(t) = \frac{p_\phi - p_\psi \cos\theta(t)}{I_1 \sin^2\theta(t)}$$

Condições iniciais que realizam o caso (c)

$\theta(0) = \theta_1 \qquad \phi(0) = \phi_0 \qquad \dot{\psi}(0) = \Omega = \omega_3$

$\dot{\theta}(0) = 0 \qquad \dot{\phi}(0) = 0$



(c)

Constantes do movimento:

$$p_\psi = I_3\omega_3 = I_3\left(\dot{\psi} + \dot{\phi}\cos\theta\right) = I_3\Omega$$

$$p_\phi = I_1\dot{\phi}\sin^2\theta + I_3\left(\dot{\psi} + \dot{\phi}\cos\theta\right)\cos\theta = I_3\Omega\cos\theta_1$$

$$E' = \frac{I_1}{2}\dot{\theta}^2 + \frac{1}{2I_1}\left(\frac{p_\phi - p_\psi\cos\theta}{\sin\theta}\right)^2 + Mgh\cos\theta = \frac{1}{2I_1}\left[\cancelto{0}{\frac{I_3\Omega\cos\theta_1 - I_3\Omega\cos\theta_1}{\sin\theta_1}}\right]^2 + Mgh\cos\theta_1$$

$$V'(\theta) = \frac{1}{2I_1}\left(\frac{p_\phi - p_\psi\cos\theta}{\sin\theta}\right)^2 + Mgh\cos\theta$$

$$V'(\theta) = \frac{1}{2I_1}\left(\frac{I_3\Omega\cos\theta_1 - I_3\Omega\cos\theta}{\sin\theta}\right)^2 + Mgh\cos\theta = \frac{I_3^2\Omega^2}{2I_1}\left(\frac{\cos\theta_1 - \cos\theta}{\sin\theta}\right)^2 + Mgh\cos\theta$$

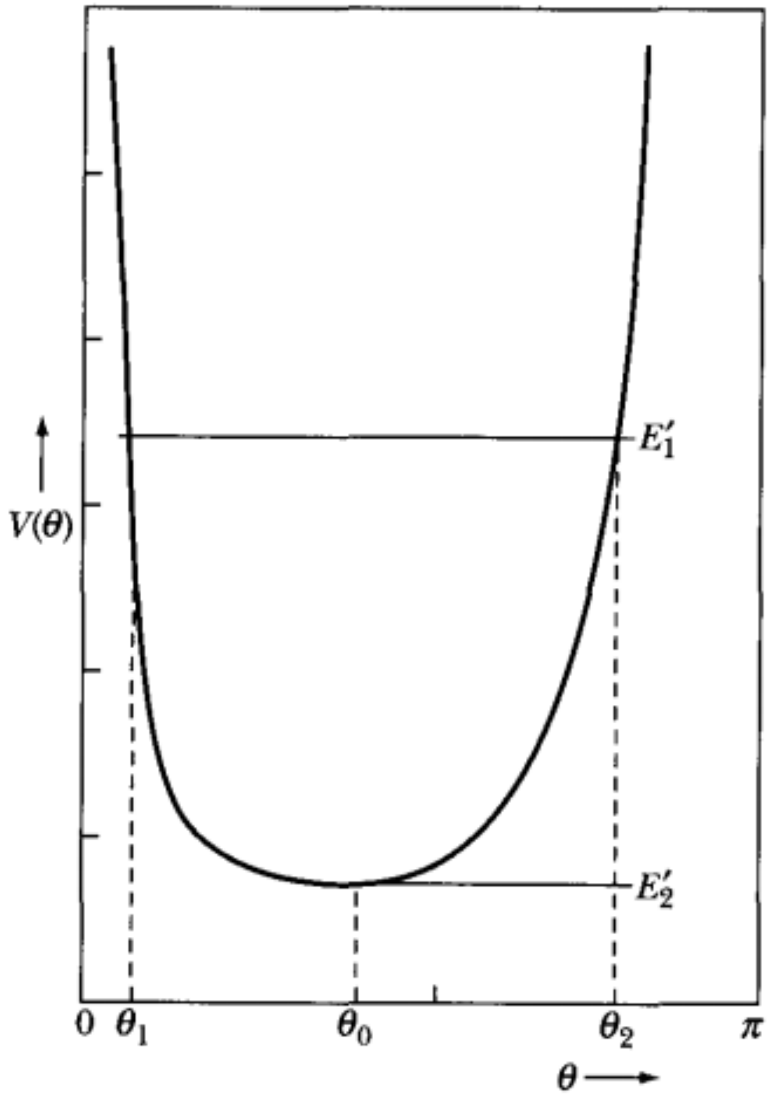
DE FATO, $\theta(0) = \theta_1$ É UM DOS PONTOS DE RETORNO:

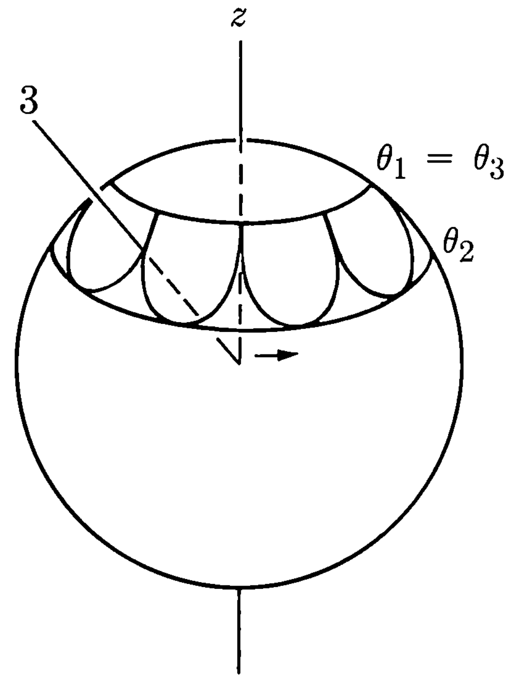
$$V'(\theta_1) = Mgh\cos\theta_1 = E'$$

ALÉM DISSO, $\dfrac{p_\phi}{p_\psi} = \dfrac{I_3 \Omega \cos\theta_1}{I_3 \Omega} = \cos\theta_1 = \cos\theta_3$

$$\Rightarrow \boxed{\theta_3 = \theta_1}$$

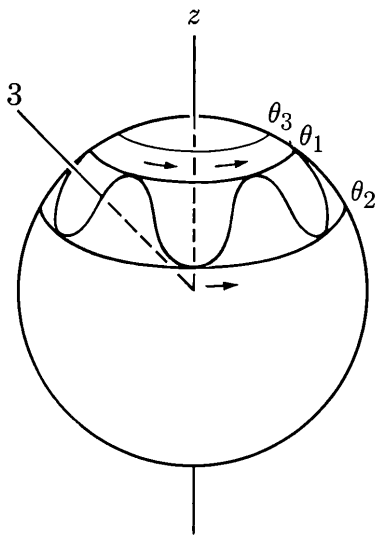
Condições iniciais que realizam o caso (c)



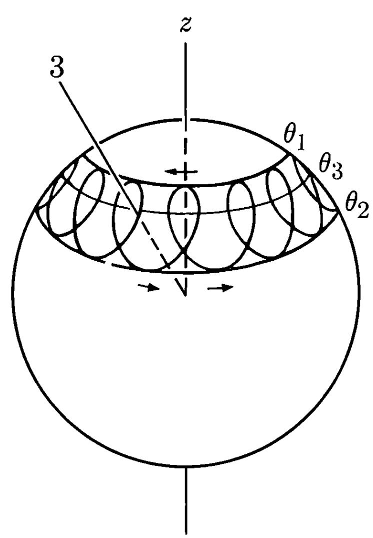


(c)

Condições iniciais que realizam os casos (a) e (b)



(a)



(b)

SE $\omega_3 > 0$, ENTÃO:

(a) $\Rightarrow \dot{\phi}(0) > 0$

$\nexists\, \theta_3 \in [\theta_1, \theta_2]$

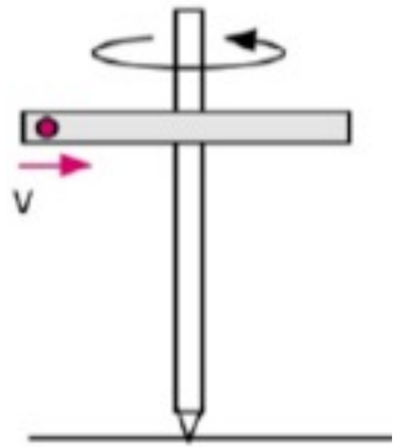
(b) $\Rightarrow \dot{\phi}(0) < 0$

$\exists\, \theta_3 \in [\theta_1, \theta_2]$

$\theta_3 \neq \theta_1$

O pião dormente

Condições iniciais: o pião é posto a girar na vertical



$$\begin{aligned} \theta(0) = \dot\theta(0) &= 0; \\ \dot\phi(0) + \dot\psi(0) &= \Omega. \end{aligned}$$

Apenas $\phi + \psi$ faz sentido quando $\theta = 0$.

$$p_\psi = I_3\omega_3 = I_3\left(\dot\psi + \dot\phi\cos\theta\right) = I_3(\dot\psi + \dot\phi) = I_3\Omega$$

$$p_\phi = I_1\dot\phi\sin^2\theta + I_3\left(\dot\psi + \dot\phi\cos\theta\right)\cos\theta = I_3\Omega$$

$$E' = E - \frac{p_\psi^2}{2I_3} = \frac{I_1}{2}\left(\dot\theta^2 + \dot\phi^2\sin^2\theta\right) + Mgh\cos\theta = Mgh$$

$$V'(\theta) = \frac{1}{2I_1}\left(\frac{p_\phi - p_\psi\cos\theta}{\sin\theta}\right)^2 + Mgh\cos\theta = \frac{1}{2I_1}\left(\frac{I_3\Omega - I_3\Omega\cos\theta}{\sin\theta}\right)^2 + Mgh\cos\theta$$

$$V'(\theta) = \frac{I_3^2\Omega^2}{2I_1}\left(\frac{1-\cos\theta}{\sin\theta}\right)^2 + Mgh\cos\theta = \frac{I_3^2\Omega^2}{2I_1}\tan^2\frac{\theta}{2} + Mgh\cos\theta$$

$$= \frac{I_3^2\Omega^2}{2I_1}\left[\overbrace{\tan^2\frac{\theta}{2} + \alpha\cos\theta}^{f(\theta)}\right] \qquad \alpha = \frac{2MghI_1}{I_3^2\omega_3^2}$$

ANALISANDO $V'(\theta)$ EM TORNO DE $\theta = 0$:

$$f(\theta=0) = \alpha$$

$$f'(\theta=0) = 0$$

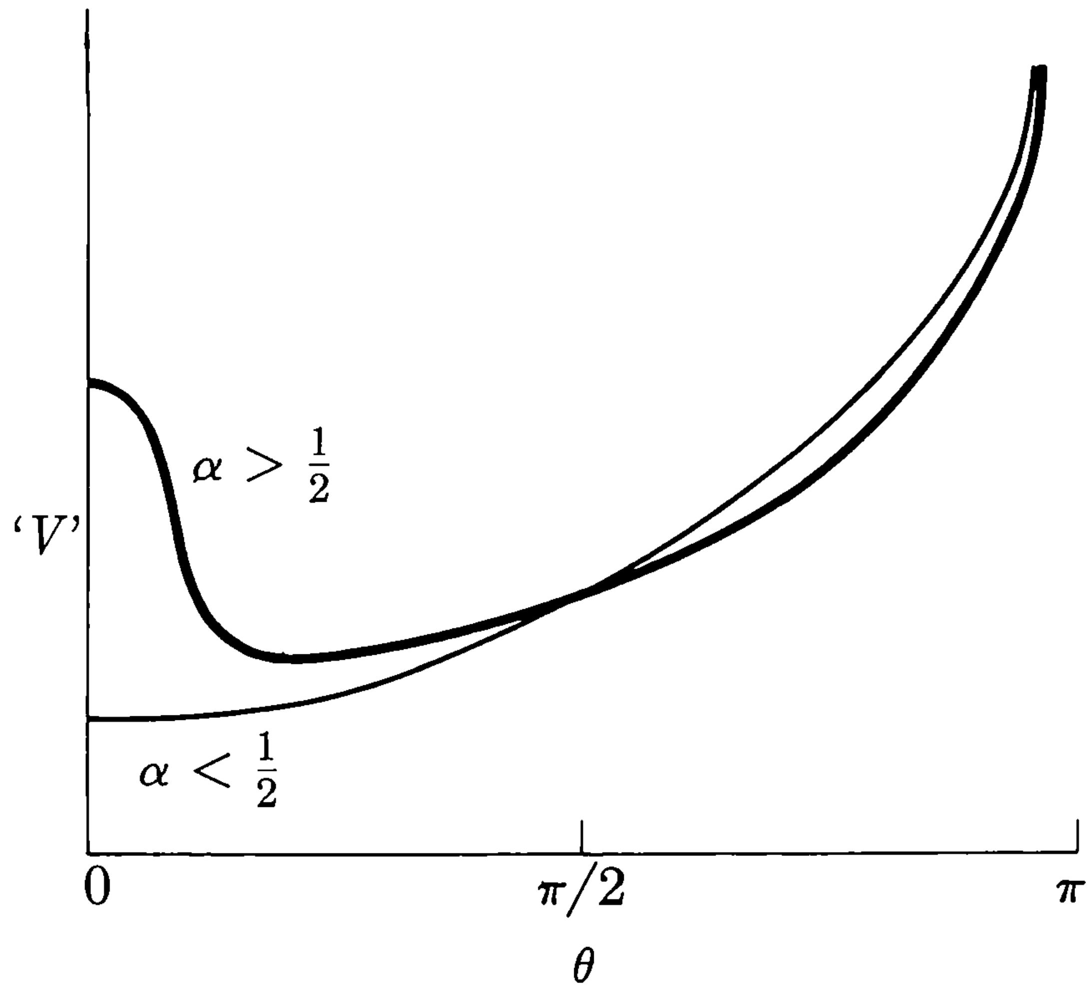
$$f''(\theta=0) = \frac{1}{2} - \alpha \Rightarrow \begin{cases} \text{SE } \alpha < \frac{1}{2} : \text{ MÍNIMO} \\ \text{SE } \alpha > \frac{1}{2} : \text{ MÁXIMO} \end{cases}$$

$\alpha < \frac{1}{2}$: EQ. ESTÁVEL

$\alpha > \frac{1}{2}$: " INSTÁVEL

INSTABILIDADE: $\alpha = \frac{1}{2} \Rightarrow \omega_3^c = \frac{2}{I_3}\sqrt{MghI_1}$

SE $\omega_3 < \omega_3^c \Rightarrow$ INSTÁVEL



'V'
$\alpha > \frac{1}{2}$
$\alpha < \frac{1}{2}$
0
$\pi/2$
π
θ