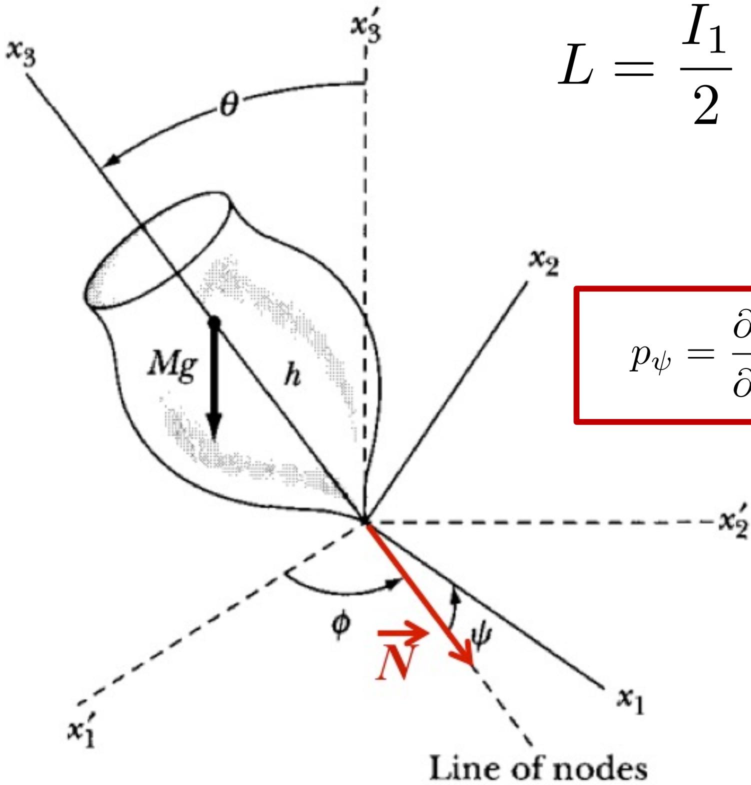
F 415 – Mecânica Geral II

1º semestre de 2024

09/05/2024

Aula 17

Aulas passadas



$$L = \frac{I_1}{2}\left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right) + \frac{I_3}{2}\left(\dot{\psi} + \dot{\phi}\cos\theta\right)^2 - Mgh\cos\theta$$

Três constantes do movimento

$p_\psi = \dfrac{\partial L}{\partial \dot{\psi}} = I_3\left(\dot{\psi} + \dot{\phi}\cos\theta\right) = I_3\omega_3$ componente do mom. angular ao longo de $\widehat{e}_3$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = I_1\dot{\phi}\sin^2\theta + I_3\left(\dot{\psi} + \dot{\phi}\cos\theta\right)\cos\theta = I_1\dot{\phi}\sin^2\theta + p_\psi\cos\theta$$

componente do mom. angular ao longo de $\widehat{z}\ (x_3')$

O torque do peso atua ao longo da linha de nós e é perpendicular a $\widehat{e}_3$ e $\widehat{z}$

$$E = \frac{I_1}{2}\left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right) + \frac{I_3}{2}\left(\dot{\psi} + \dot{\phi}\cos\theta\right)^2 + Mgh\cos\theta$$

energia mecânica conservada: sistema conservativo

Aulas passadas

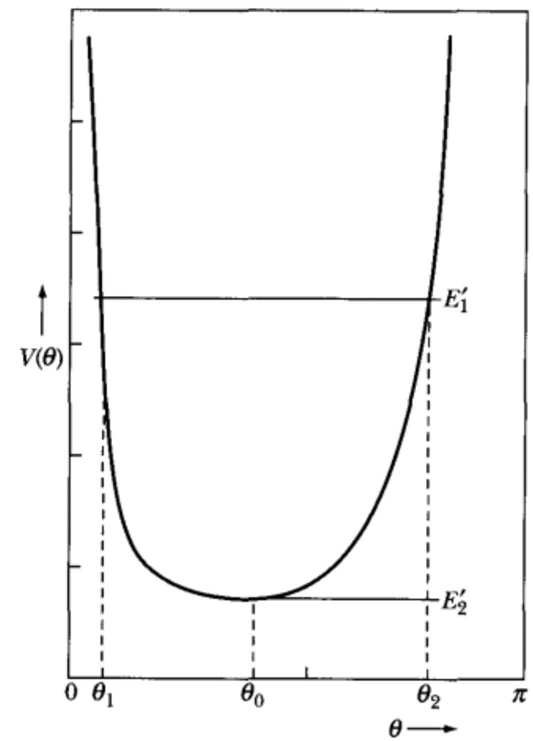
Eliminamos a dependência com $\dot{\phi}, \dot{\psi}$

$$\left(\dot{\psi} + \dot{\phi}\cos\theta\right) = \frac{p_\psi}{I_3} \qquad \dot{\phi} = \frac{p_\phi - p_\psi\cos\theta}{I_1\sin^2\theta}$$

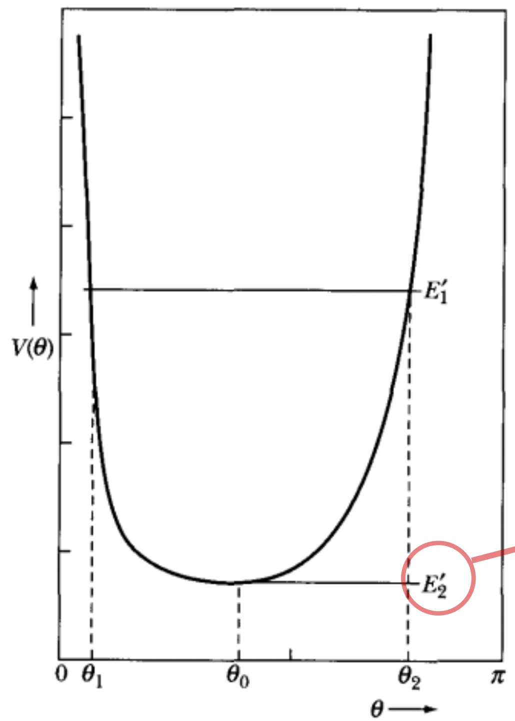
$$E - \frac{p_\psi^2}{2I_3} \equiv E' = \frac{I_1}{2}\dot{\theta}^2 + \frac{1}{2I_1}\left(\frac{p_\phi - p_\psi\cos\theta}{\sin\theta}\right)^2 + Mgh\cos\theta$$

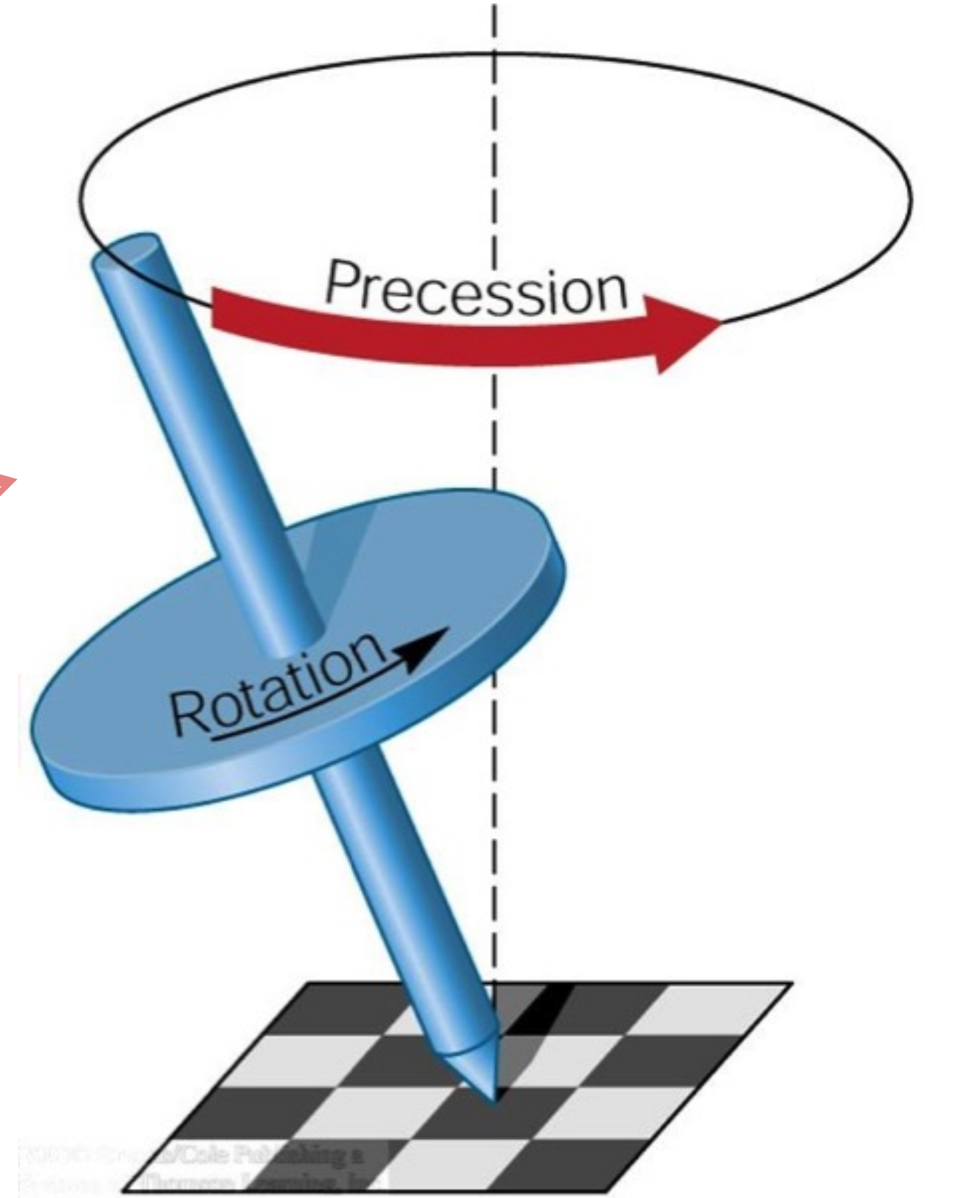
Potencial efetivo:

$$\boxed{V'\left(\theta\right) = \frac{1}{2I_1}\left(\frac{p_\phi - p_\psi\cos\theta}{\sin\theta}\right)^2 + Mgh\cos\theta}$$



Aulas passadas



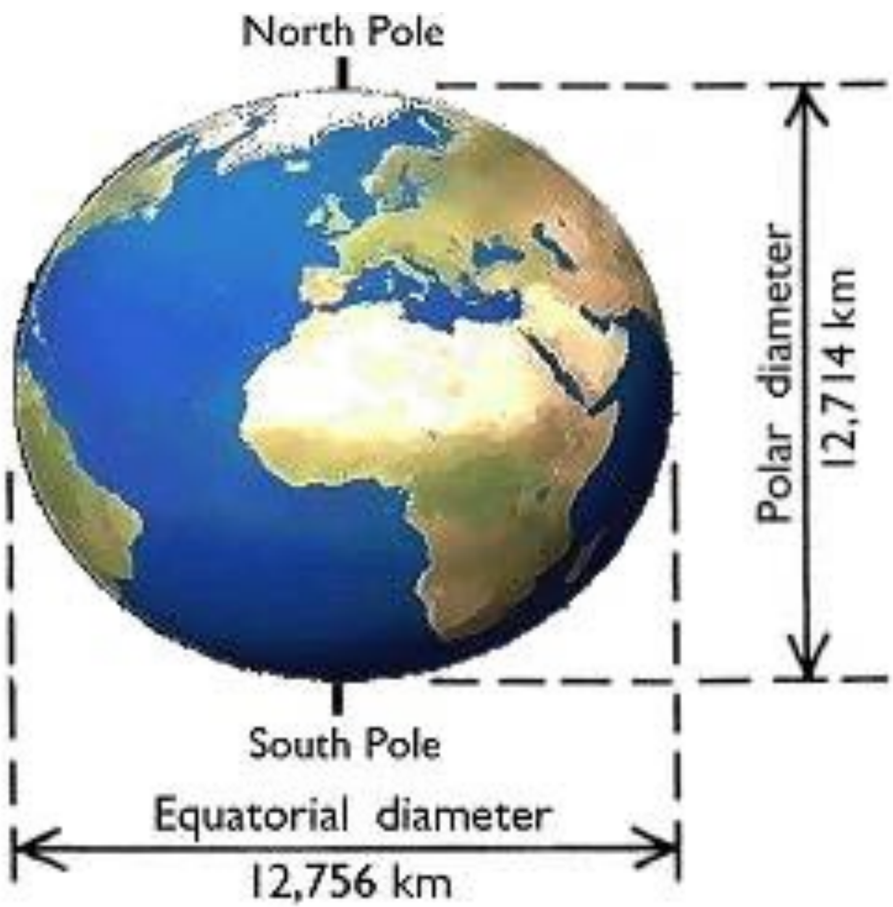


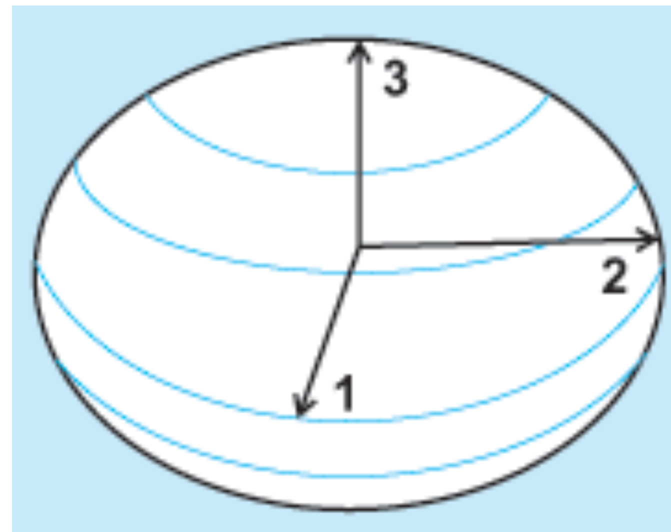
Precessão regular

$$\dot{\phi} = \frac{p_\phi - p_\psi \cos\theta_0}{I_1 \sin^2\theta_0}$$

$$\dot{\psi} = \frac{p_\psi}{I_3} - \dot{\phi}\cos\theta_0$$

A Terra: um esferóide oblato

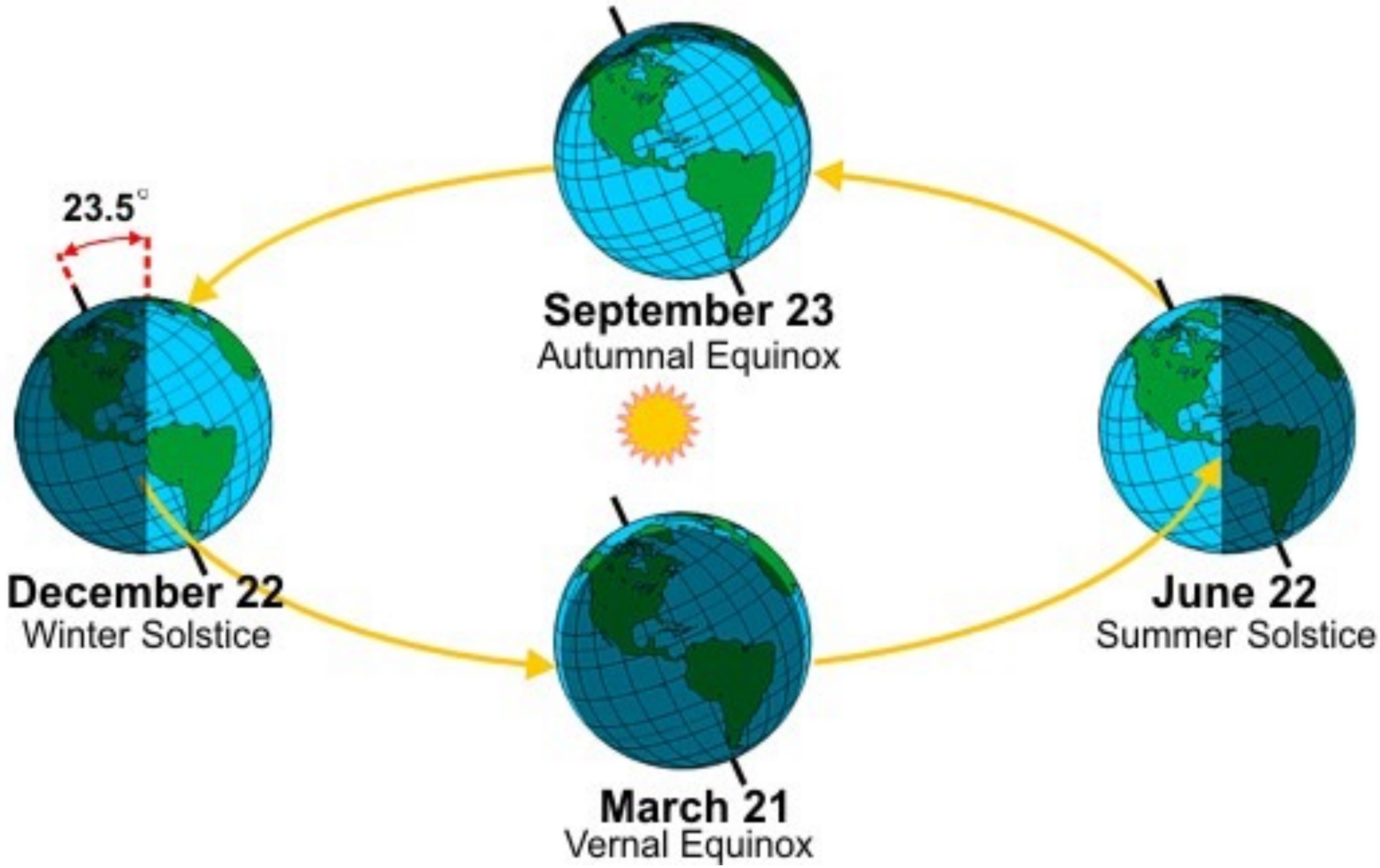




$$I_3 > I_1 = I_2$$

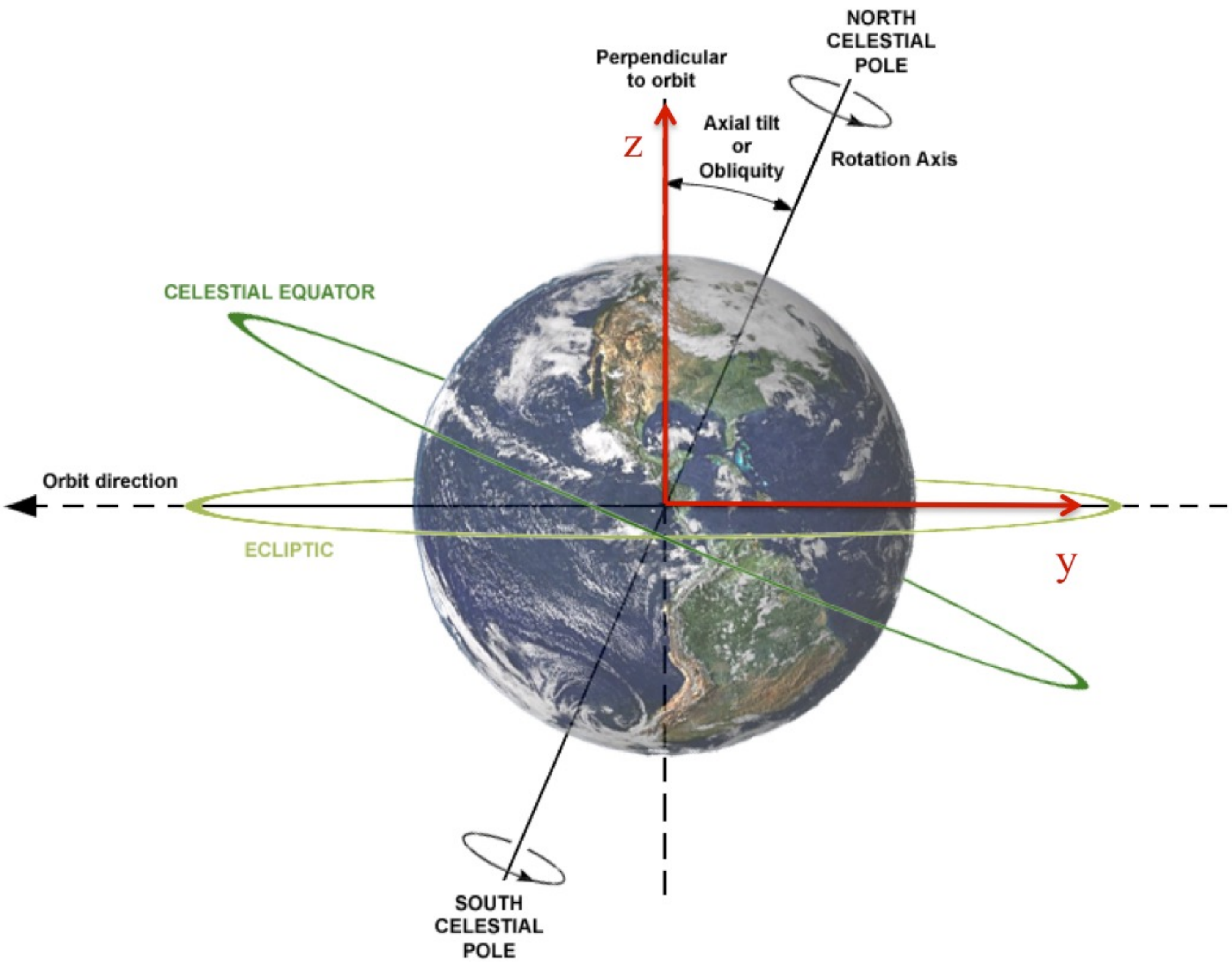
$$\frac{I_3 - I_1}{I_3} \approx \frac{1}{306}$$

Inclinação em relação à eclíptica

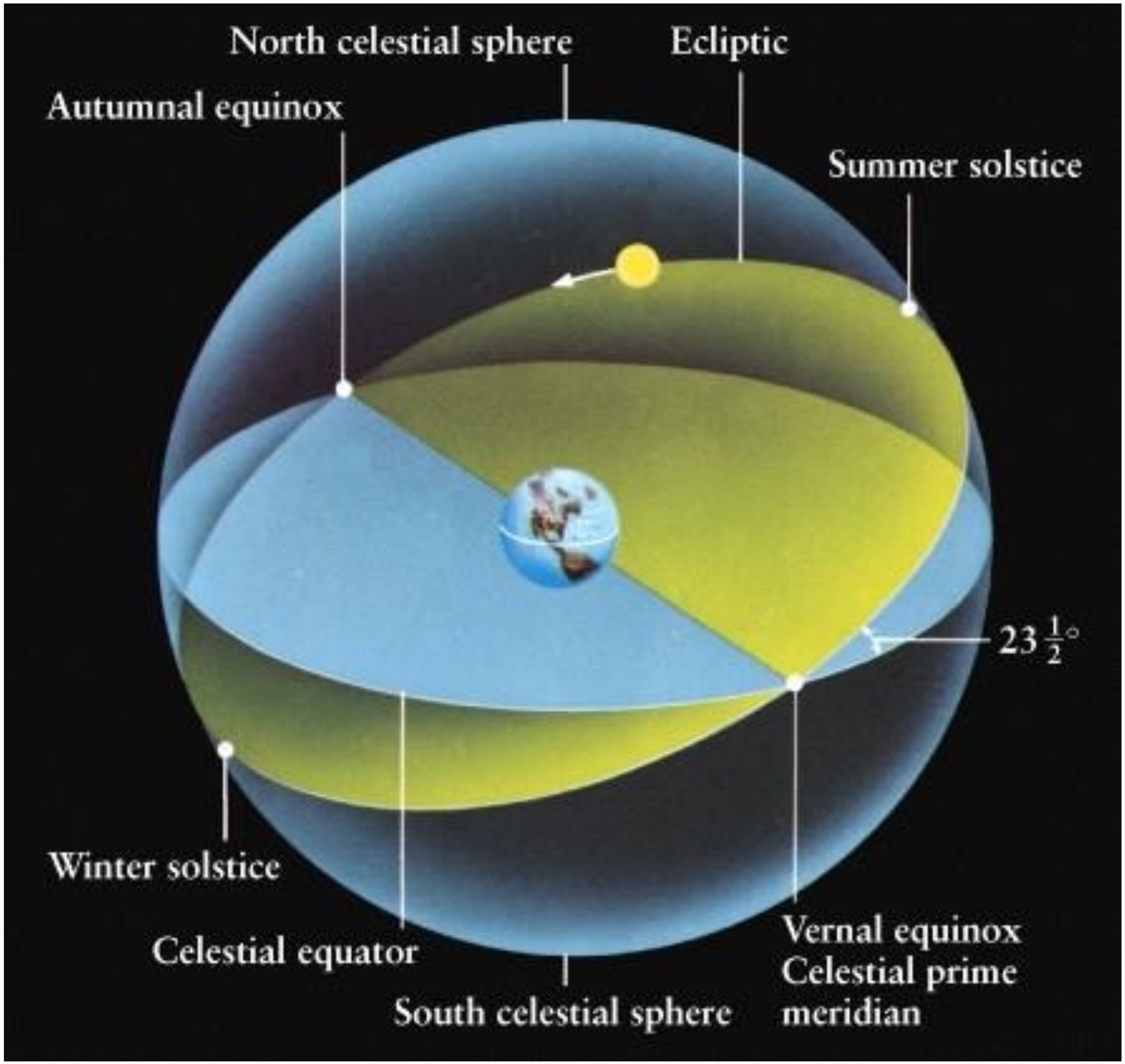


A velocidade angular ω é quase coincidente com o eixo principal 3. Entretanto, ambos estão inclinados de **23º27'** em relação à eclíptica (plano da órbita em torno do Sol): estações do ano.

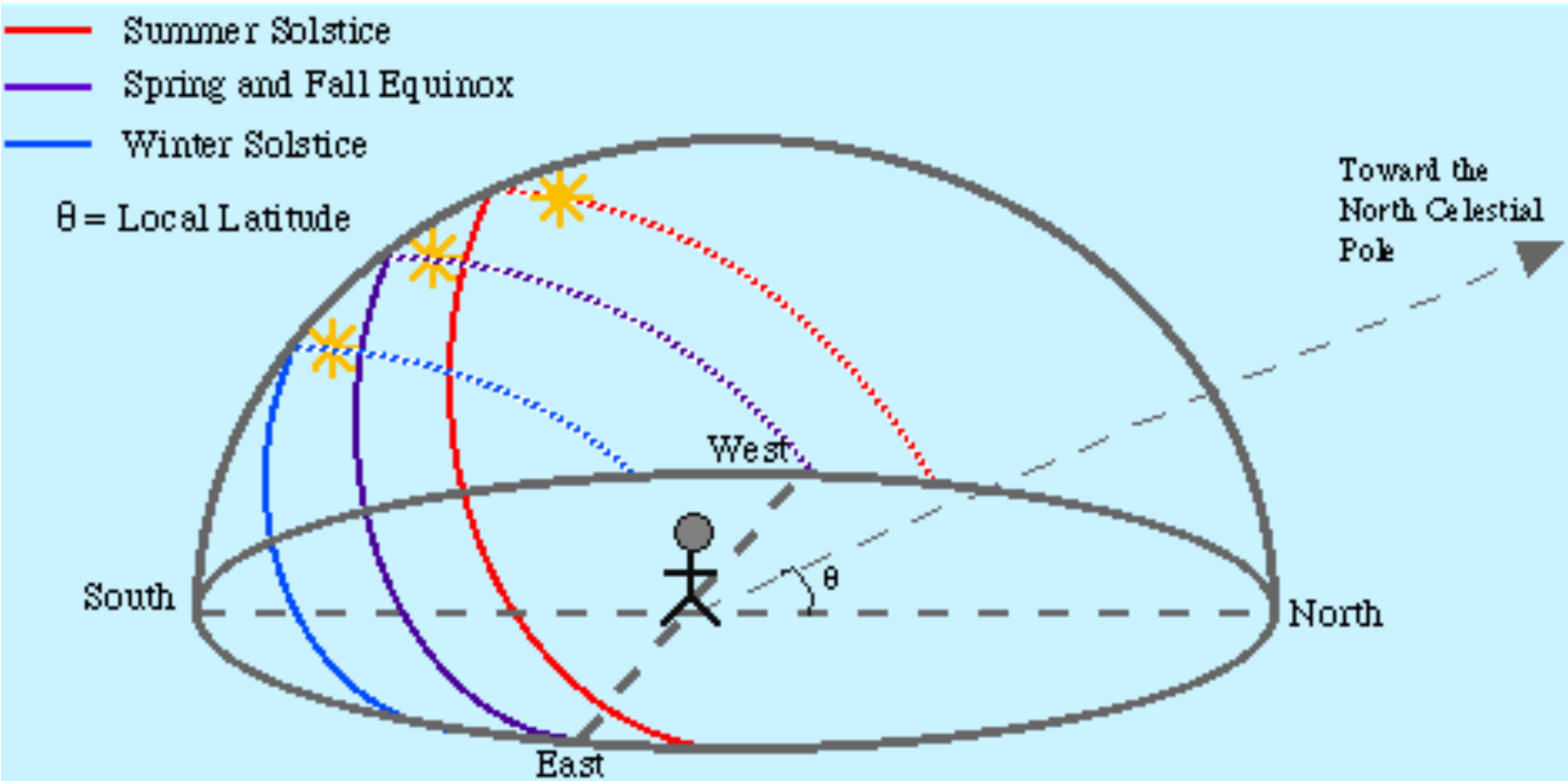
Inclinação em relação à eclíptica



O ponto de vista da Terra

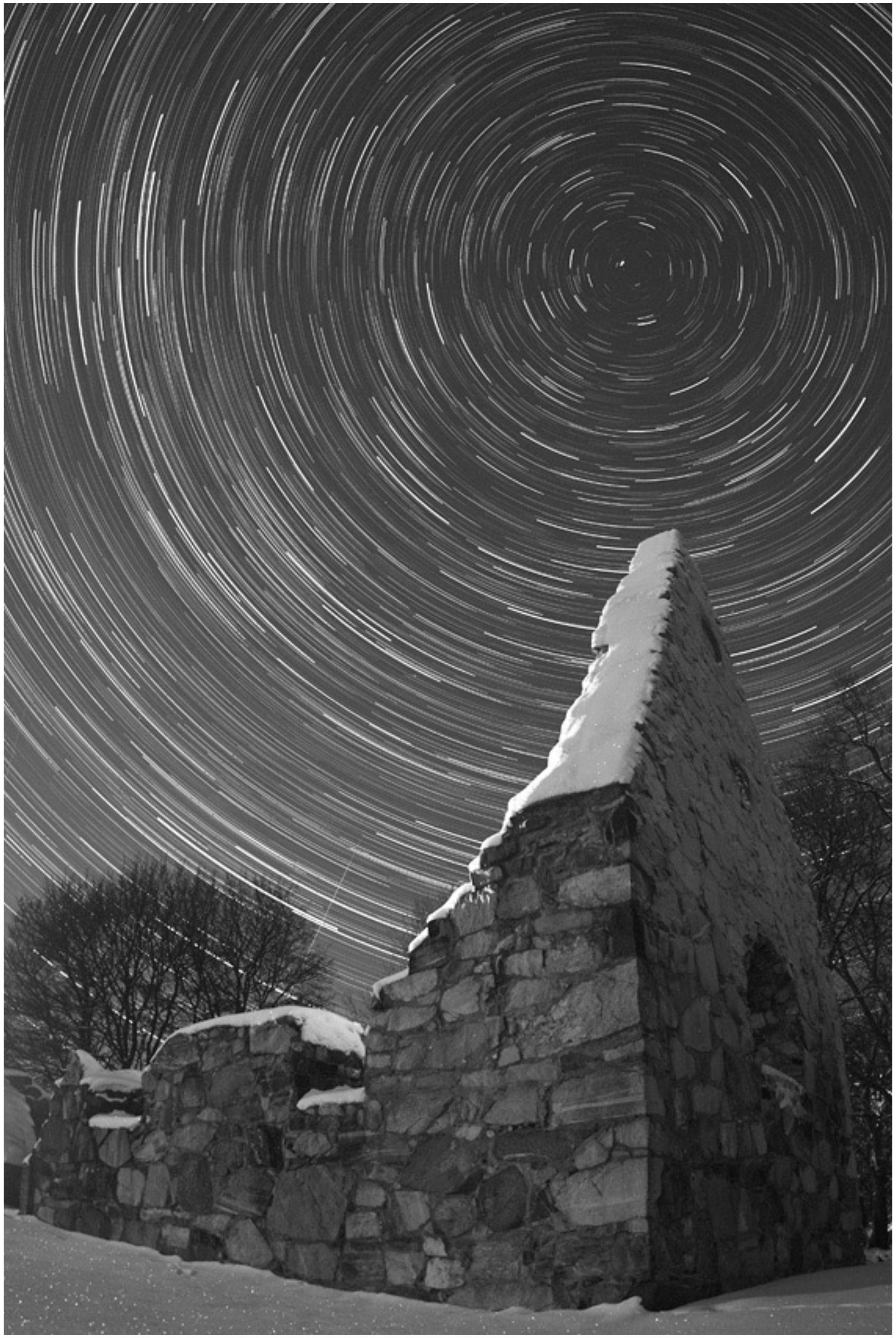


O arco solar em uma certa latitude



Polo celestial Norte:

Polaris, ou Estrela Polar
ou Estrela do Norte



Do arquivo do “Astronomy Picture of the Day”

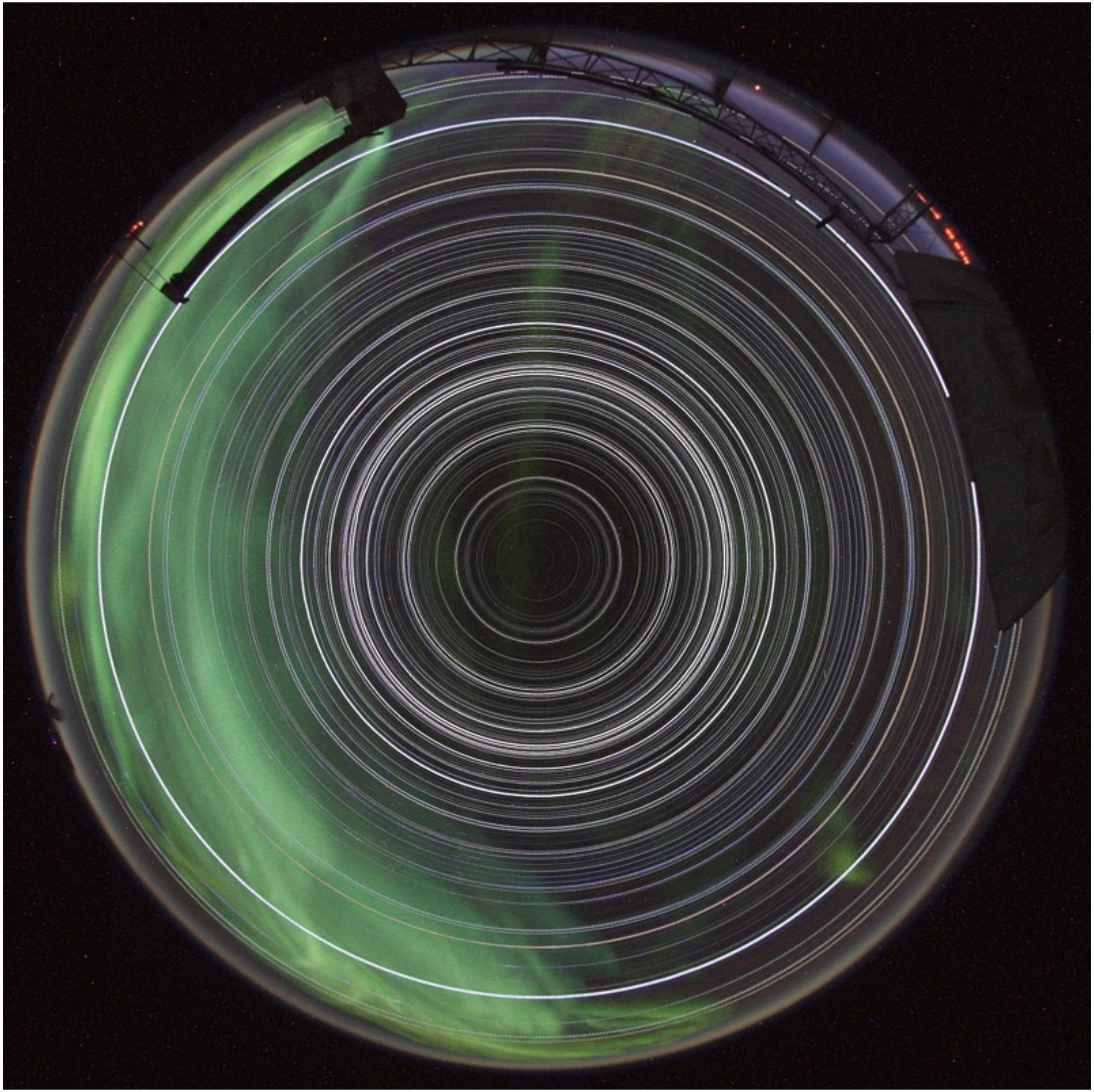
Polo celestial Norte: Polaris, ou Estrela Polar ou Estrela do Norte

Do arquivo do “Astronomy Picture of the Day”

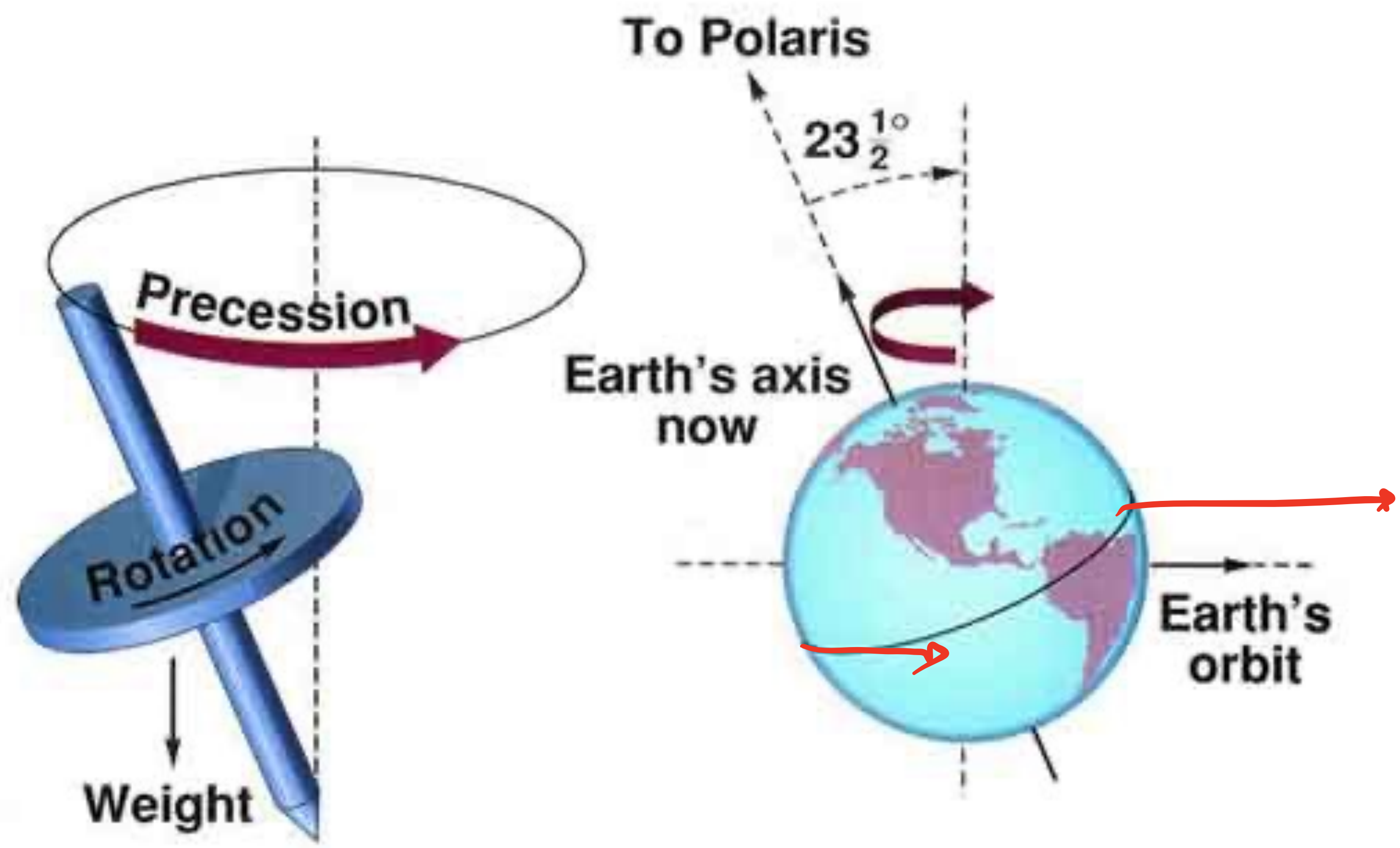
Polo celestial Sul:
Sigma Octantis

A mancha verde é
uma Aurora Australis

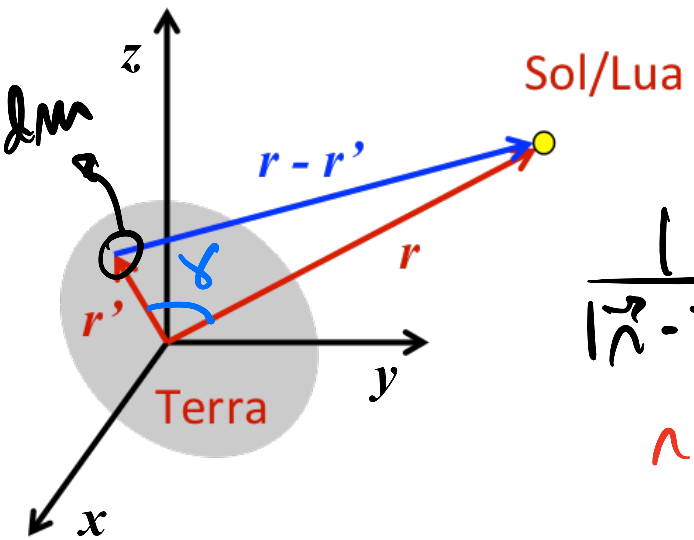
Do arquivo do
"Astronomy Picture
of the Day"



A precessão dos equinócios



A energia potencial gravitacional de interação Sol-Terra (ou Lua-Terra)



$$V(\vec{r}) = -G\int \frac{dm}{|\vec{r}-\vec{r}\,'|} = -G\int \frac{\rho(\vec{r}\,')d^3r'}{|\vec{r}-\vec{r}\,'|}$$

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \frac{1}{r}\sum_{l=0}^{\infty}\left(\frac{r'}{r}\right)^l P_l(\cos\gamma) \qquad (r'<r)$$

$r = |\vec{r}|$; $r' = |\vec{r}\,'|$, γ = ÂNGULO ENTRE $\vec{r}$ E $\vec{r}\,'$

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \frac{1}{[r^2+r'^2-2\vec{r}\cdot\vec{r}\,']^{1/2}} = \frac{1}{[r^2+r'^2-2rr'\cos\gamma]^{1/2}}$$

$$l=0: \ \frac{1}{r} \Rightarrow V_0(\vec{r}) = -\frac{G}{r}\int \rho(\vec{r}\,')d^3r' = -\frac{GM}{r}$$

Expansão multipolar do potencial gravitacional

$$
\begin{aligned}
P_0(x) &= 1, \\
P_1(x) &= x, \\
P_2(x) &= \frac{1}{2}\left(3x^2 - 1\right).
\end{aligned}
$$

$\ell = 1: \quad \frac{1}{r}\left(\frac{r'}{r}\right)\cos\gamma$

$$V_1(\vec{r}) = -\frac{G}{r^2}\int d^3r'\, r' \cos\gamma\, \rho(\vec{r}\,')$$

$$\cos\gamma = \frac{\vec{r}\cdot\vec{r}\,'}{r\, r'}$$

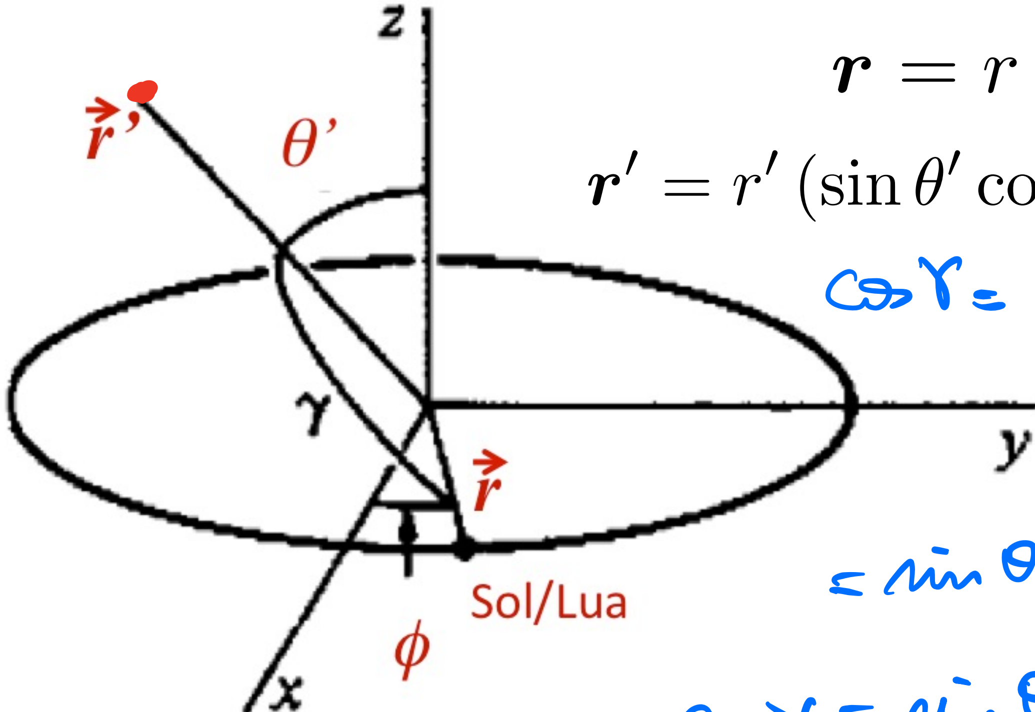
$$V_1(\vec{r}) = -\frac{G}{r^3}\int d^3r'\, \vec{r}\cdot\vec{r}\,'\, \rho(\vec{r}\,')$$

$$= -\frac{G}{r^3}\,\vec{r}\cdot\underbrace{\left[\int d^3r'\, \vec{r}\,'\, \rho(\vec{r}\,')\right]}_{m\vec{R}}$$

ESCOLHENDO A ORIGEM DO SISTEMA NO CM: $\vec{R} = 0 \Rightarrow V_1(\vec{r}) = 0$

$$\ell = 2: \quad \frac{1}{r}\left(\frac{r'}{r}\right)^2 P_2(\cos\gamma) = \frac{1}{2r}\frac{r'^2}{r^2}\left(3\cos^2\gamma - 1\right)$$

$$V_2(\vec{r}) = -\frac{G}{2r^3}\underbrace{\int d^3r'\, r'^2\left(3\cos^2\gamma - 1\right)\rho(\vec{r}\,')}_{A}$$



$$\boldsymbol{r} = r\left(\cos\phi\,\hat{\boldsymbol{x}} + \sin\phi\,\hat{\boldsymbol{y}}\right)$$

$$\boldsymbol{r}' = r'\left(\sin\theta'\cos\phi'\,\hat{\boldsymbol{x}} + \sin\theta'\sin\phi'\,\hat{\boldsymbol{y}} + \cos\theta'\,\hat{\boldsymbol{z}}\right)$$

$$\cos\gamma = \frac{\vec{r}\cdot\vec{r}'}{r r'} = (\sin\theta'\cos\phi'\cos\phi + \sin\theta'\sin\phi'\sin\phi) =$$

$$= \sin\theta'\left[\cos\phi\cos\phi' + \sin\phi\sin\phi'\right]$$

$$\cos\gamma = \sin\theta'\cos(\phi-\phi')$$

$$A = \int r'^2 dr' \sin\theta' d\theta' d\phi'\, r'^2 \rho(\vec{r}')\left[3\sin^2\theta'\cos^2(\phi-\phi') - 1\right]$$

O SOL REALIZA A TRAJETÓRIA CIRCULAR EM UM ANO E $\vec{\omega}$ PRECESSIONA ($\dot{\phi}$) NA ESCALA DE $\sim 26\times10^3$ ANOS

É RAZOÁVEL, PROMEDIAR SOBRE A POSIÇÃO DO SOL $\underline{\phi}$

$$\langle f(\phi)\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(\phi)\,d\phi \qquad \langle\cos^2(\phi-\phi')\rangle = \frac{1}{2}$$

Média sobre a posição do Sol/Lua

$$\langle A\rangle = \int r'^4\, dr'\, \sin\theta'\, d\theta'\, d\phi'\, \rho(\vec{r}\,')\left[\frac{3}{2}\sin^2\theta' - 1\right]$$

$$= \int d^3r'\, r'^2 \rho(\vec{r}\,')\left[\frac{3}{2}\sin^2\theta' - 1\right] \qquad r'^2\sin^2\theta' = x'^2 + y'^2$$

$$= \int d^3r'\, \rho(\vec{r}\,')\left[\frac{3}{2}(x'^2+y'^2) - (x'^2+y'^2+z'^2)\right]$$

$$\int d^3r'\, \rho(\vec{r}\,')\,(x'^2+y'^2) = I_{33}$$

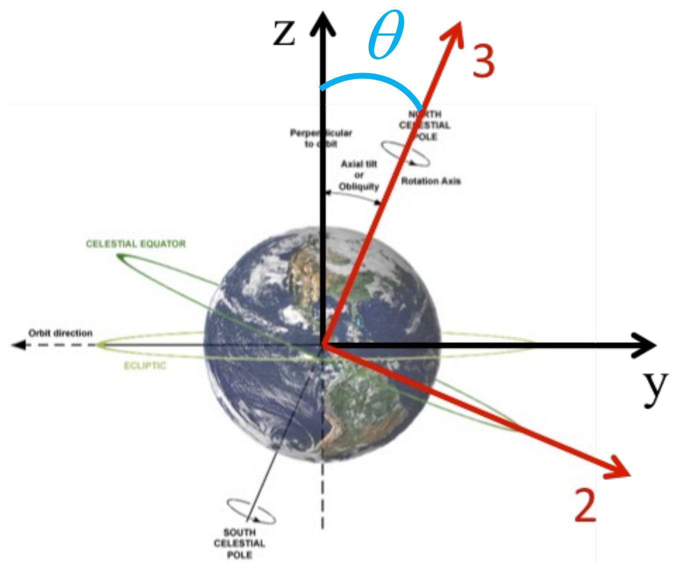
$$\int d^3r'\, \frac{\rho(\vec{r}\,')}{2}\left[(x'^2+y'^2)+(y'^2+z'^2)+(x'^2+z'^2)\right] = \frac{1}{2}(I_{11}+I_{22}+I_{33})$$

$$= \frac{\mathrm{Tr}\,[\,\underline{\underline{I}}\,]}{2} = \frac{1}{2}(I_1+I_2+I_3) = I_1 + \frac{I_3}{2}$$

$$\langle A\rangle = \frac{3}{2} I_{33} - I_1 - \frac{1}{2} I_3$$

Rodando os eixos

$23{,}5^\circ$



$$\mathsf{I}_{xyz} = \boldsymbol{\lambda} \cdot \mathsf{I}_{123} \cdot \boldsymbol{\lambda}^T$$

λ é uma matriz de rotação de um ângulo θ em torno de 1.

$$\boldsymbol{\lambda} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$$

$S = \sin\theta$

$C = \cos\theta$

I_{123} é o tensor de inércia no sistema de eixos principais:

$$\mathsf{I}_{123} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_1 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$$

FAZENDO A MULTIPLICAÇÃO MATRICIAL:

$$I_{xyz} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & C^2 I_1 + S^2 I_3 & (I_3 - I_1) SC \\ 0 & (I_3 - I_1) SC & \overset{I_{33}}{S^2 I_1 + C^2 I_3} \end{pmatrix}$$

$I_{33} = \sin^2\theta \, I_1 + \cos^2\theta \, I_3 =$

$= I_1 + \cos^2\theta \, (I_3 - I_1)$

$$V_2(\vec{r}) = -\frac{G}{4r^3}(I_3 - I_1)(3\cos^2\theta - 1)$$

$$V(\vec{r}) = V_0(\vec{r}) + V_2(\vec{r}) = -\frac{GM}{r} - \frac{G}{4r^3}(I_3 - I_1)(3\cos^2\theta - 1)$$

$$U(\vec{r}) = mV(\vec{r}) = -\frac{GMm}{r} - \frac{Gm}{4r^3}(I_3 - I_1)(3\cos^2\theta - 1)$$

DINÂMICA ROTACIONAL

MOVIMENTO DE REVOLUÇÃO (3 LEIS DE KEPLER, CAP. 8)

$$U_{rot}(\theta) = -\frac{3GM(I_3 - I_1)}{4r^3}\cos^2\theta$$

$$L = \frac{I_1}{2}\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right) + \frac{I_3}{2}\left(\dot{\psi} + \dot{\phi}\cos\theta\right)^2 - U_{rot}(\theta)$$

EQ. DE EULER-LAGRANGE PARA $\underline{\theta}$:

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{\theta}}\right] - \frac{\partial L}{\partial \theta} = 0$$

$$I_1\ddot{\theta} = I_1\dot{\phi}^2\sin\theta\cos\theta - I_3\overbrace{\left(\dot{\psi} + \dot{\phi}\cos\theta\right)}^{\omega_3}\dot{\phi}\sin\theta - \frac{3GM(I_3 - I_1)}{2r^3}\cos\theta\sin\theta$$

PRECESSÃO REGULAR: $\theta(t) = \theta_0$

$$0 = \underbrace{I_1\dot{\phi}^2\cancel{\sin\theta_0}\cos\theta_0}_{\sim\dot{\phi}^2} - \underbrace{I_3\omega_3\dot{\phi}}_{\sim\omega_3\dot{\phi}}\cancel{\sin\theta_0} - \frac{3GM}{2r^3}(I_3 - I_1)\cos\theta_0\cancel{\sin\theta_0}$$

RAZÃO: $\frac{\dot{\phi}}{\omega_3} \lll 1 \Rightarrow$ IGNORO $I_1\dot{\phi}^2$

A velocidade angular de precessão

$$\dot{\phi}_{\text{prec}} = -\frac{3}{2\omega_3}\frac{GM}{r^3}\overbrace{\left(\frac{I_3 - I_1}{I_3}\right)}^{\sim \frac{1}{306}}\cos\theta_0$$

$23{,}5°$

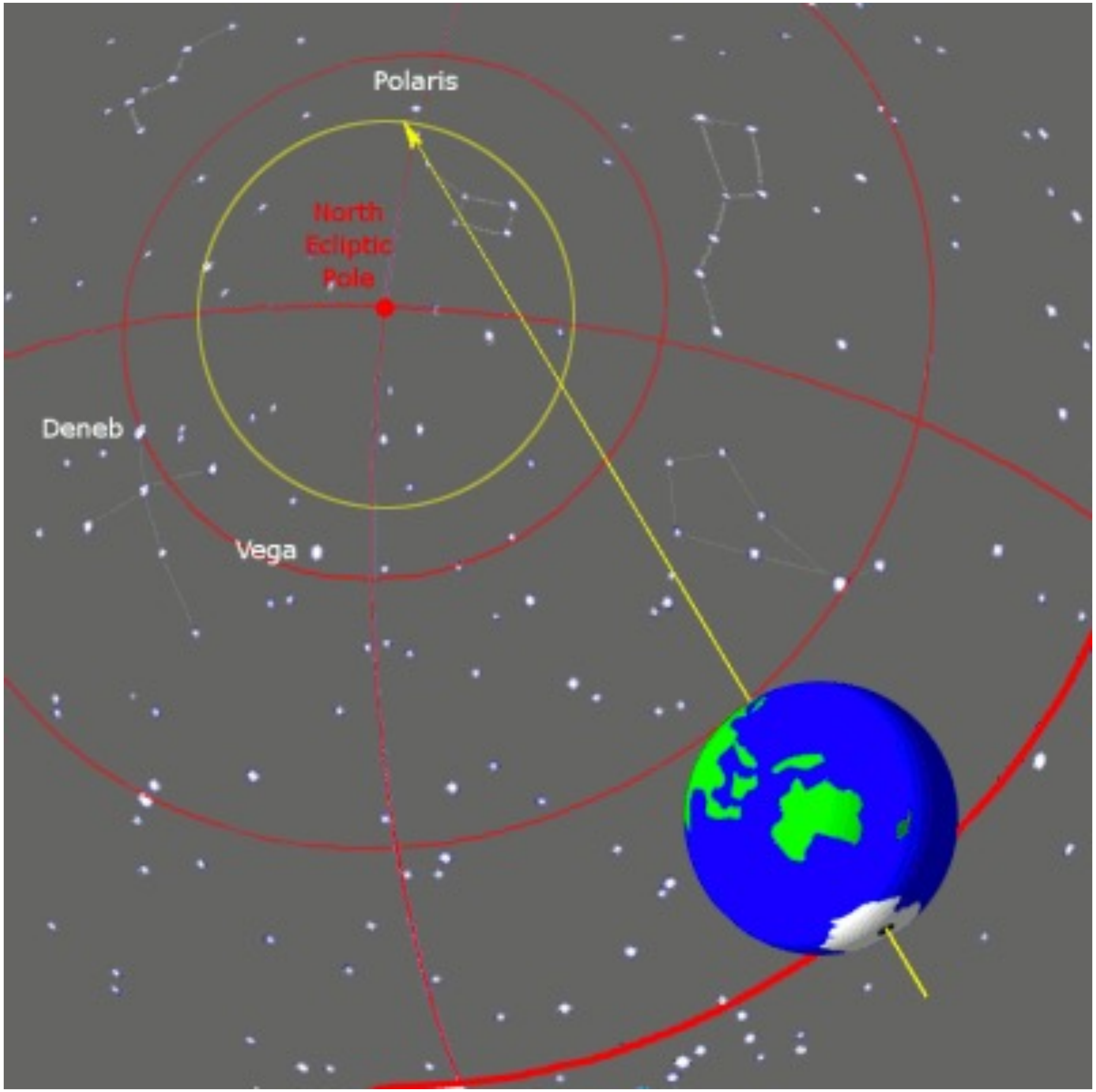
$$\omega_3 = \frac{2\pi}{1\,\text{dia}}$$

$$2.2\,\frac{M_{SOL}}{r_{SOL}^3} \simeq \frac{M_{LUA}}{r_{LUA}^3}$$

$$\Rightarrow T_{PREC} = 25.3 \times 10^3 \text{ ANOS}$$

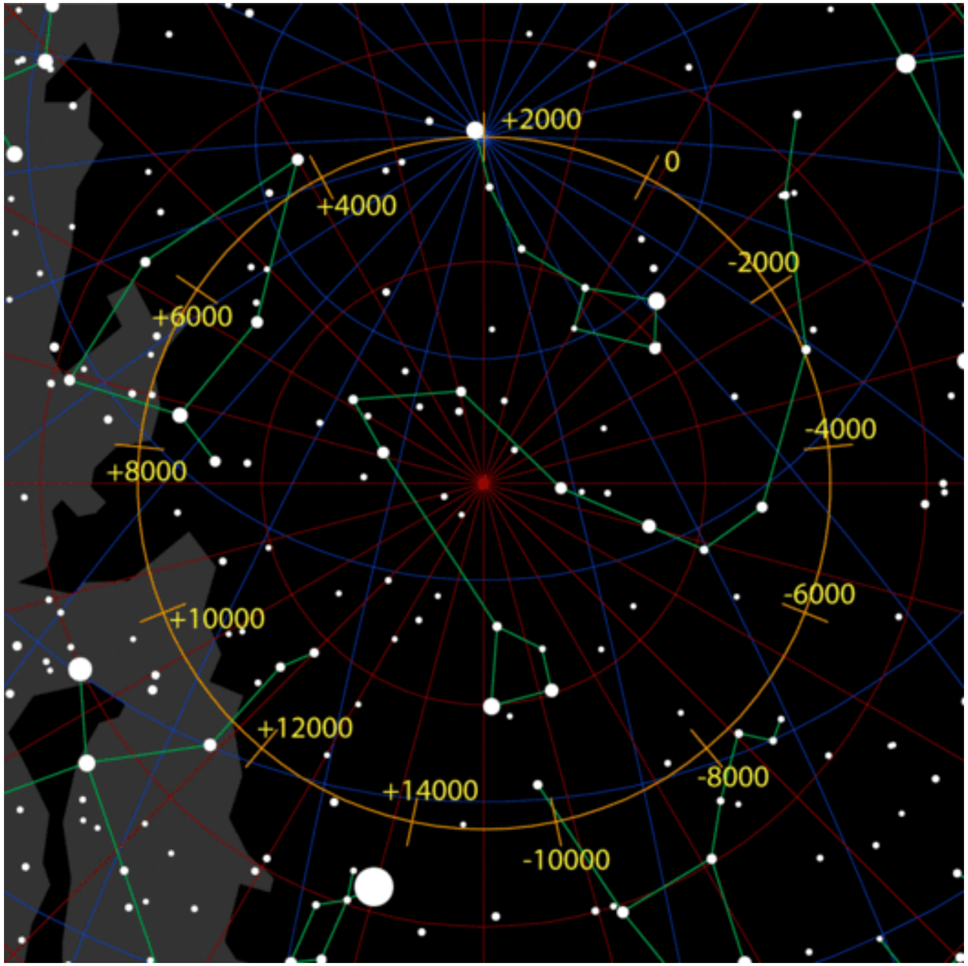
Polo celestial Norte:

Polaris, ou Estrela Polar ou Estrela do Norte

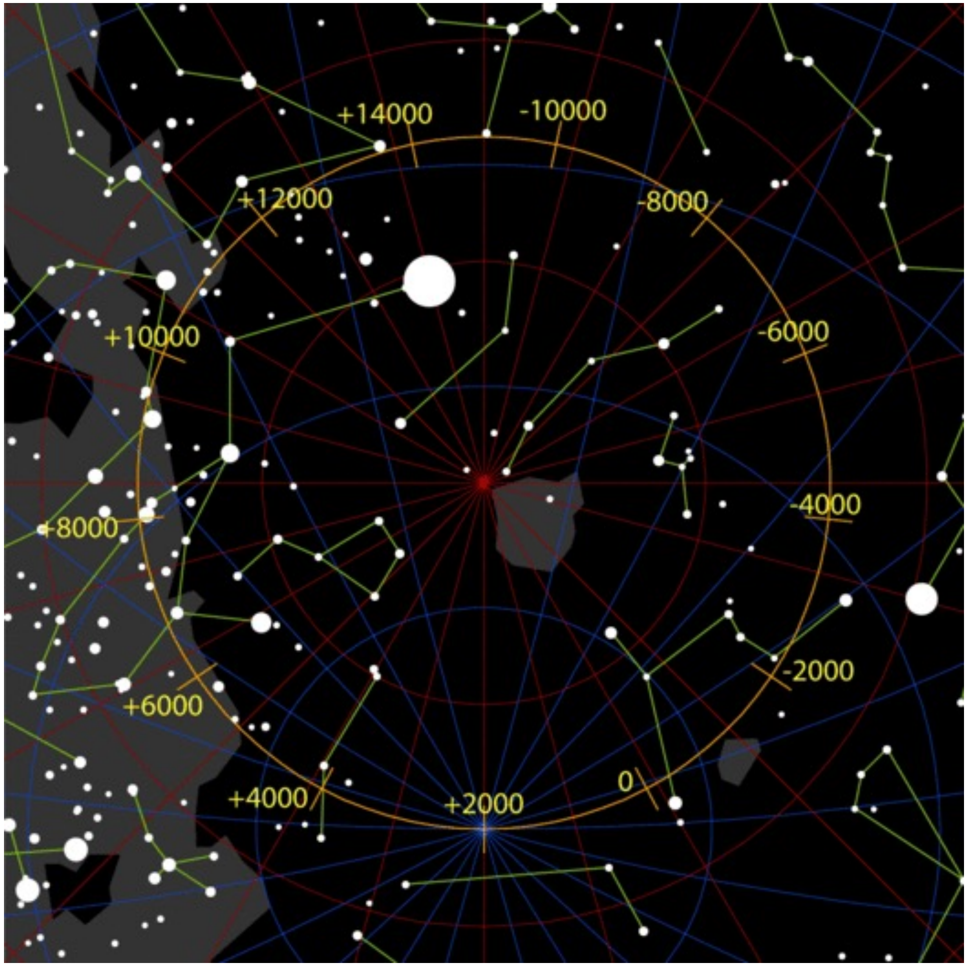


Precessão do eixo de rotação da Terra em torno do polo eclíptico

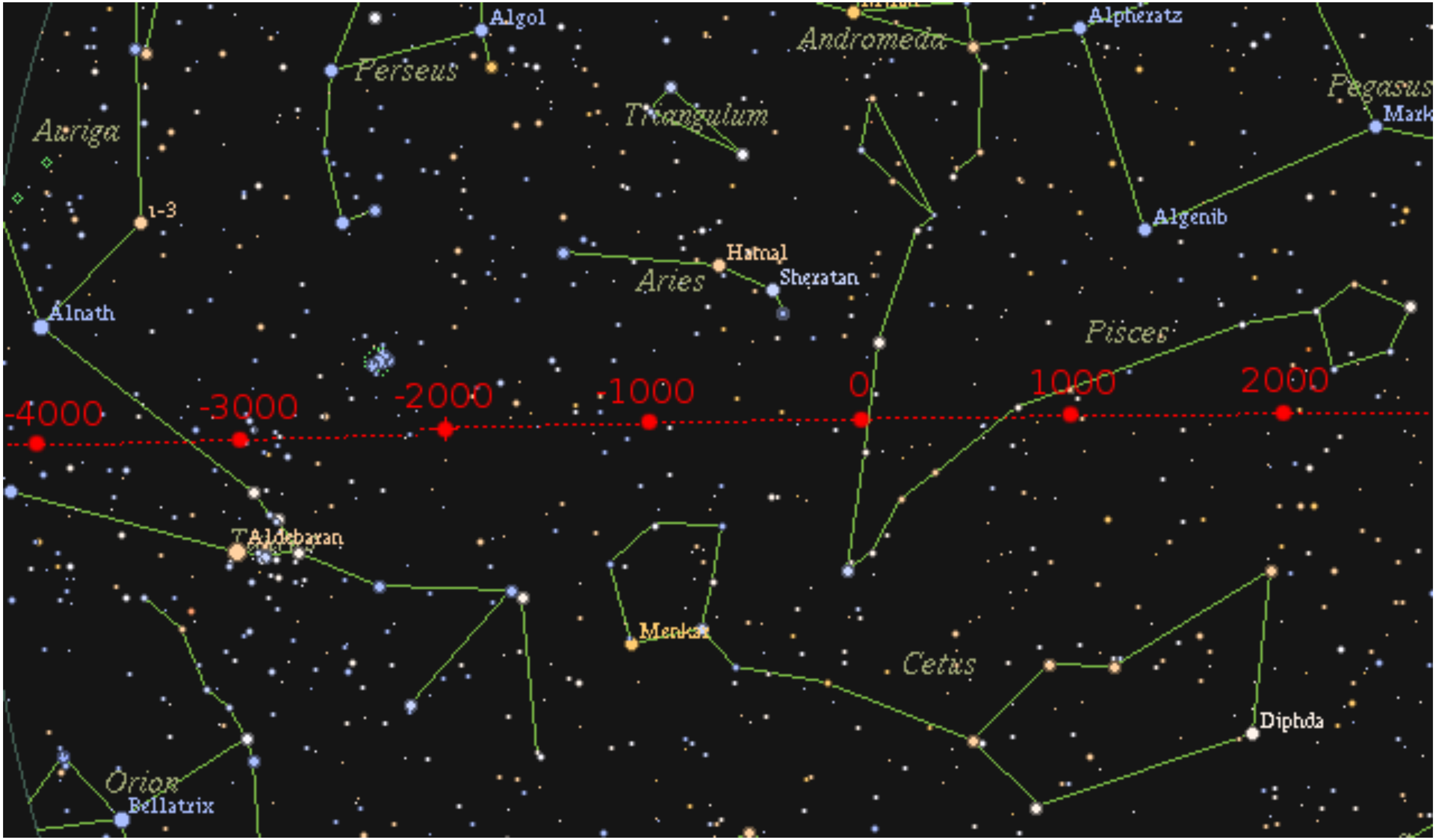
Norte



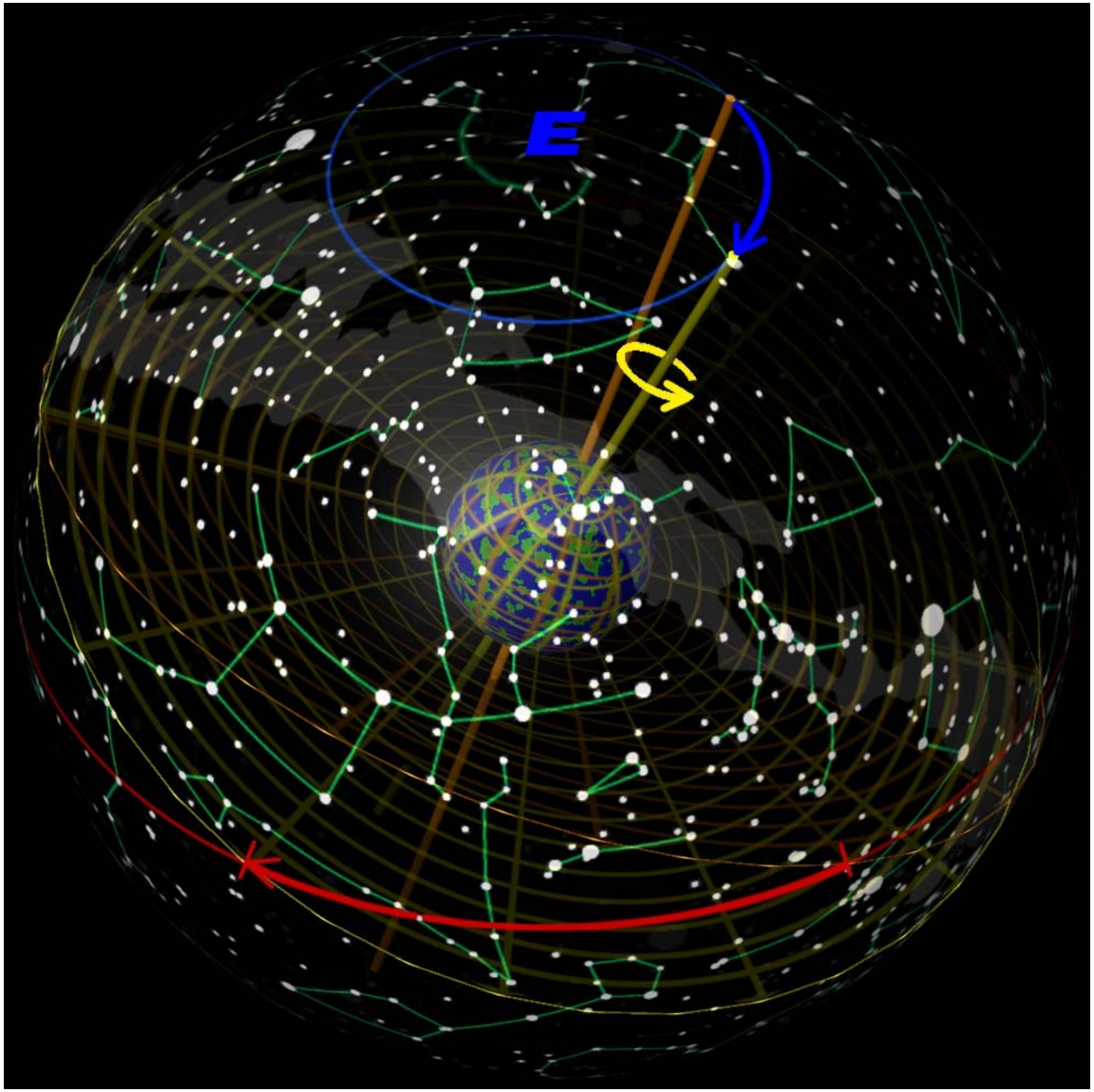
Sul



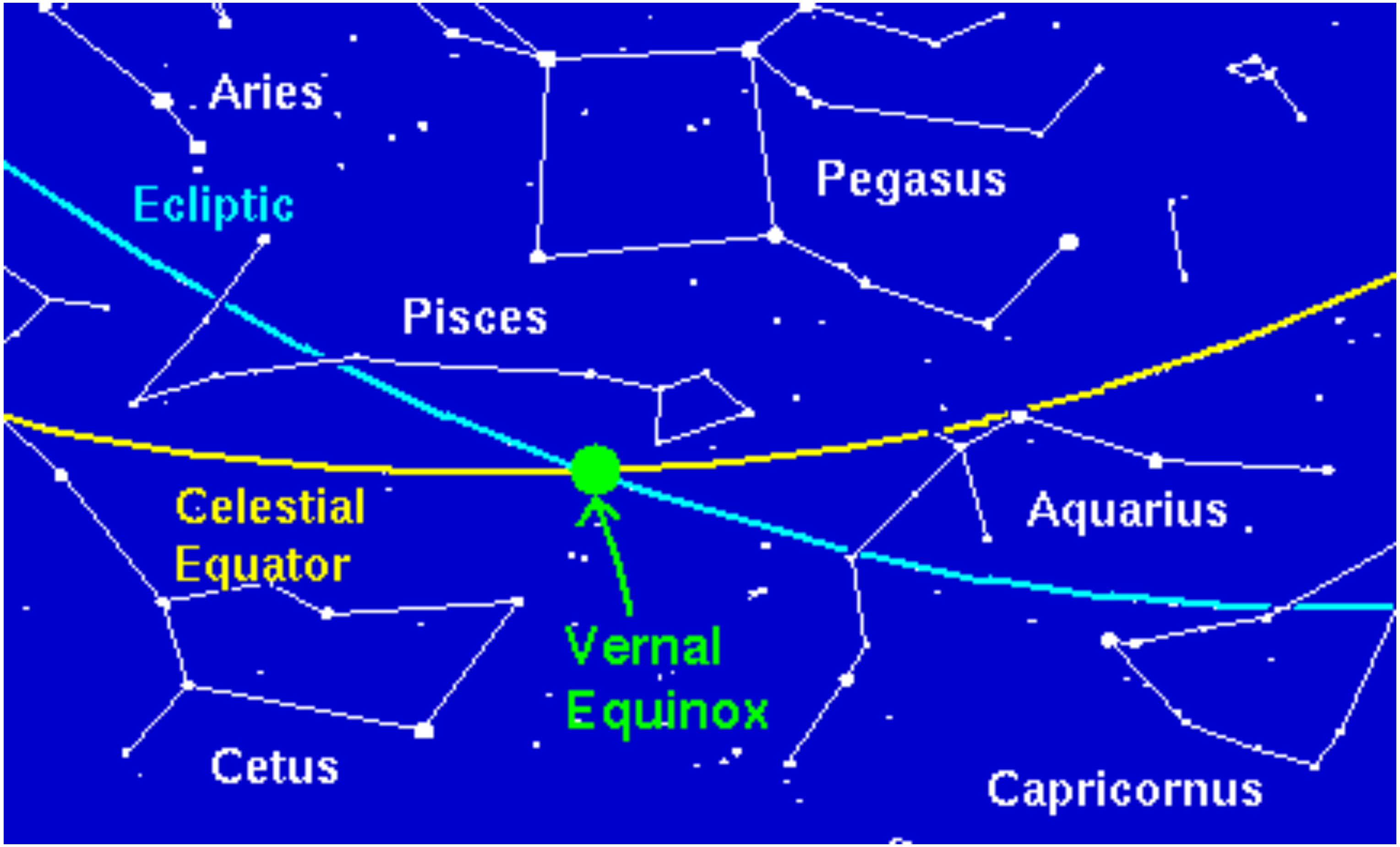
Precessão do equinócio (norte)



Visão de “fora” da abóbada celeste



A “era de Aquarius”



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