F 415 – Mecânica Geral II

1º semestre de 2024

23/05/2024

Aula 20

Aula passada

$$\boxed{L = \frac{1}{2}\sum_{j,k} m_{jk}\dot{q}_j\dot{q}_k - \frac{1}{2}\sum_{j,k} A_{jk}q_jq_k}$$ ou $$L = \frac{1}{2}\dot{\bar{q}}^T \cdot \bar{\bar{m}} \cdot \dot{\bar{q}} - \frac{1}{2}\bar{q}^T \cdot \bar{\bar{A}} \cdot \bar{q}$$

Equações de Euler-Lagrange: $\bar{\bar{m}} \cdot \ddot{\bar{q}} + \bar{\bar{A}} \cdot \bar{q} = 0$

Supondo dependência harmônica com uma única frequência: $\boxed{\bar{q}\,(t) = \bar{a}e^{i\omega t}}$

$$\left(-\omega^2\bar{\bar{m}} + \bar{\bar{A}}\right) \cdot \bar{a} = 0 \qquad \boxed{\Rightarrow \det\left(-\omega^2\bar{\bar{m}} + \bar{\bar{A}}\right) = 0}$$

Equação polinomial de grau n em ω^2 com n soluções positivas: $\omega_r \quad (r = 1, 2, \ldots, n)$

Frequências normais

Aula passada

Há n soluções possíveis, indexadas pelo índice r $\left(-\omega_r^2 \bar{\bar{m}} + \bar{\bar{A}}\right) \cdot \bar{a}^{(r)} = 0 \; (r = 1, 2, \ldots, n)$

As soluções diferentes são ortogonais entre si num sentido generalizado:

$$\bar{a}^{(s)} \cdot \bar{\bar{m}} \cdot \bar{a}^{(r)} = \delta_{r,s}$$

Definindo a matriz: $\boxed{\Lambda_{rk} \equiv \bar{a}_k^{(r)}}$

$$\Rightarrow \bar{\bar{\Lambda}} \cdot \bar{\bar{m}} \cdot \bar{\bar{\Lambda}}^T = \bar{\bar{1}}$$

**Notem que a matriz Λ não é ortogonal!

$$\Rightarrow \boxed{\bar{\bar{\Lambda}} \cdot \bar{\bar{A}} \cdot \bar{\bar{\Lambda}}^T = \begin{pmatrix} \omega_1^2 & 0 & \cdots & 0 \\ 0 & \omega_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \omega_n^2 \end{pmatrix}}$$

Aula passada

A solução geral, com $\underline{2n}$ parâmetros arbitrários determinados pelas condições iniciais é:

$$\bar{q}(t) = \sum_r \alpha_r \bar{a}^{(r)} \cos(\omega t + \delta_r)$$

$$q_j(t) = \sum_r \alpha_r \Lambda_{rj} \cos(\omega_r t + \delta_r) \equiv \sum_r \Lambda_{rj} \eta_r(t)$$

$$\eta_r(t) \equiv \alpha_r \cos(\omega_r t + \delta_r) \longrightarrow \text{coordenadas normais}$$

Matricialmente: $\bar{q} = \bar{\bar{\Lambda}}^T \cdot \bar{\eta}$ e $\bar{\eta} = \bar{\bar{\Lambda}} \cdot \bar{\bar{m}} \cdot \bar{q}$

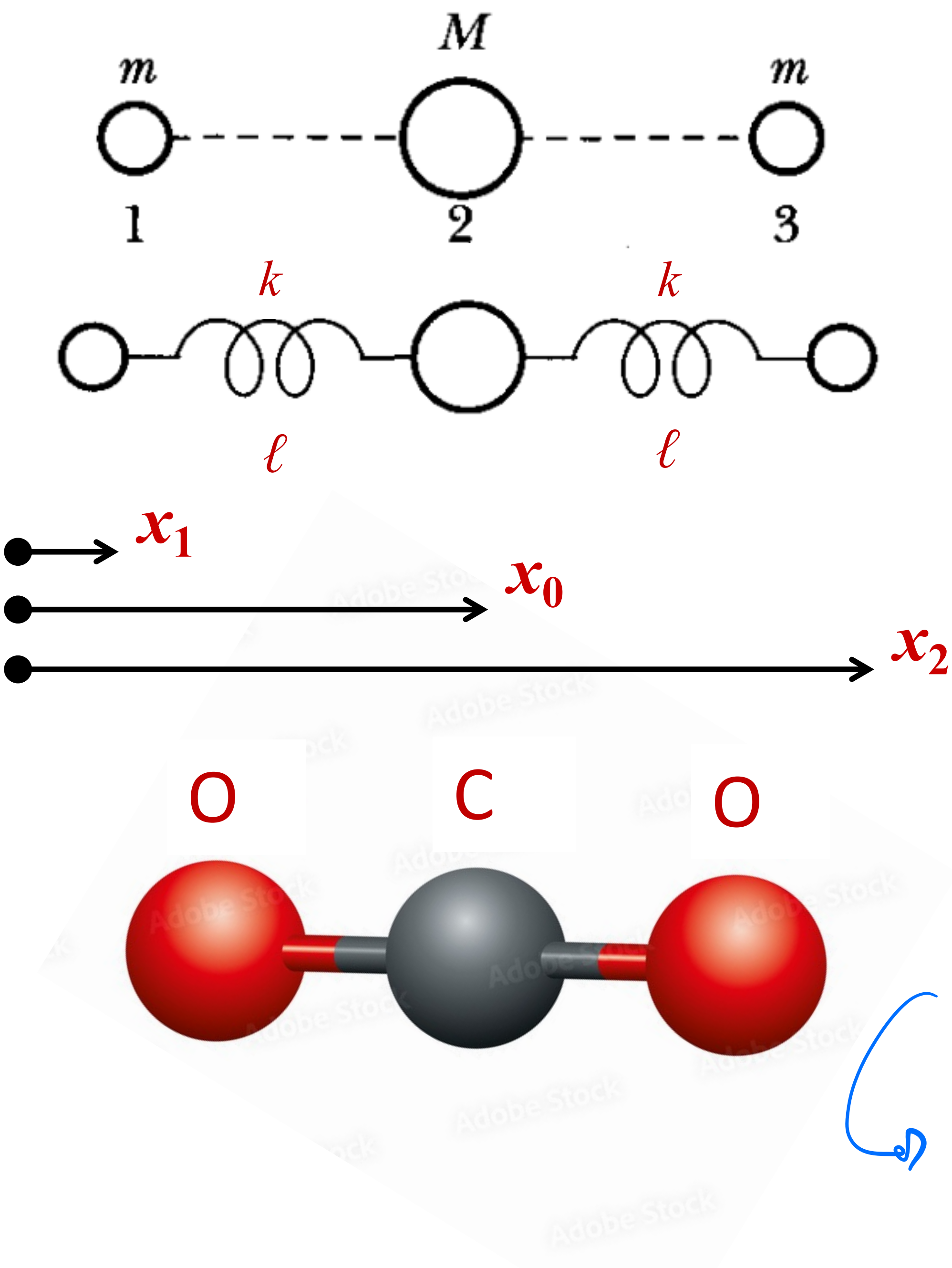
$$L = \frac{1}{2}\dot{\bar{q}} \cdot \bar{\bar{m}} \cdot \dot{\bar{q}} - \frac{1}{2}\bar{q} \cdot \bar{\bar{A}} \cdot \bar{q}$$

$$L = \frac{1}{2}\dot{\bar{\eta}} \cdot \bar{\bar{\Lambda}} \cdot \bar{\bar{m}} \cdot \bar{\bar{\Lambda}}^T \cdot \dot{\bar{\eta}} - \frac{1}{2}\bar{\eta} \cdot \bar{\bar{\Lambda}} \cdot \bar{\bar{A}} \cdot \bar{\bar{\Lambda}}^T \cdot \bar{\eta}$$

$$L = \frac{1}{2}\dot{\bar{\eta}} \cdot \dot{\bar{\eta}} - \frac{1}{2}\bar{\eta} \cdot \mathrm{diag}\left(\omega_1^2, \ldots, \omega_n^2\right) \cdot \bar{\eta} = \frac{1}{2}\sum_r \left(\dot{\eta}_r^2 - \omega_r^2 \eta_r^2\right)$$

$$\ddot{\eta}_r + \omega_r^2 \eta_r = 0 \Rightarrow \eta_r(t) = \alpha_r \cos(\omega_r t + \delta_r)$$

Molécula triatômica linear



$$T = \frac{m}{2}\left(\dot{x}_1^2 + \dot{x}_2^2\right) + \frac{M}{2}\left(\dot{x}_0\right)^2$$

$$U = \frac{k}{2}\left(x_0 - x_1 - \ell\right)^2 + \frac{k}{2}\left(x_2 - x_0 - \ell\right)^2$$

PONTOS DE EQUILÍBRIO:

$$\left.\frac{\partial U}{\partial x_i}\right|_{\{x_i^{(0)}\}} = 0$$

$$\left.\frac{\partial U}{\partial x_1}\right|_0 = -k\left(x_0^{(0)} - x_1^{(0)} - \ell\right) = 0 \quad (1)$$

$$\left.\frac{\partial U}{\partial x_2}\right|_0 = k\left(x_2^{(0)} - x_0^{(0)} - \ell\right) = 0 \quad (2)$$

$$\left.\frac{\partial U}{\partial x_0}\right|_0 = k\left(x_0^{(0)} - x_1^{(0)} - \cancel{\ell}\right) - k\left(x_2^{(0)} - x_0^{(0)} - \cancel{\ell}\right) = 0 \quad (3)$$

$$2x_0^{(0)} = x_1^{(0)} + x_2^{(0)}$$

$$(1) \Rightarrow x_1^{(0)} = x_0^{(0)} - \ell \qquad (2) \Rightarrow x_2^{(0)} = x_0^{(0)} + \ell$$

$$(1)+(2) \Rightarrow x_1^{(0)} + x_2^{(0)} = 2x_0^{(0)}$$

ESCOLHO OS PONTOS: $\begin{cases} x_0^{(0)} = 0 \\ x_1^{(0)} = -\ell \\ x_2^{(0)} = \ell \end{cases}$

ESSA LIBERDADE DE ESCOLHA DO PONTO DE EQUILÍBRIO REFLETE A AUSÊNCIA DE FORÇAS EXTERNAS E A RESULTANTE CONSERVAÇÃO DO MOMENTO LINEAR TOTAL.

DEFINO ~~AGORA~~ OS DESVIOS DO EQUILÍBRIO:

$$q_0 = x_0 - x_0^{(0)} = x_0$$

$$q_1 = x_1 - x_1^{(0)} = x_1 + \ell$$

$$q_2 = x_2 - x_2^{(0)} = x_2 - \ell$$

$$U = \frac{k}{2}(x_0 - x_1 - \ell)^2 + \frac{k}{2}(x_2 - x_0 - \ell)^2$$

$$= \frac{k}{2}\left[q_0 - q_1 + \cancel{\ell} - \cancel{\ell}\right]^2 + \frac{k}{2}(q_2 + \cancel{\ell} - q_0 - \cancel{\ell})^2$$

$$= \frac{k}{2}(q_0 - q_1)^2 + \frac{k}{2}(q_2 - q_0)^2$$

$$= \frac{k}{2}\left[q_1^2 + q_2^2 + 2q_0^2 - 2q_0 q_1 - 2q_0 q_2\right]$$

$$\overline{\overline{A}} = k\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

$$T = \frac{m}{2}(\dot{q}_1^2 + \dot{q}_2^2) + \frac{M}{2}\dot{q}_0^2 \longrightarrow \overline{\overline{m}} = \begin{pmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & m \end{pmatrix}$$

$$\frac{k}{2}(q_1 \ q_0 \ q_2)\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} q_1 \\ q_0 \\ q_2 \end{pmatrix} = \frac{k}{2}(q_1 \ q_0 \ q_2)\begin{pmatrix} q_1 - q_0 \\ -q_1 + 2q_0 - q_2 \\ -q_0 + q_2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} q_1 \\ q_0 \\ q_2 \end{pmatrix} = \begin{pmatrix} q_1 - q_0 \\ -q_1 + 2q_0 - q_2 \\ -q_0 + q_2 \end{pmatrix}$$

$$= \frac{k}{2} \begin{pmatrix} q_1 & q_0 & q_2 \end{pmatrix} \begin{pmatrix} q_1 - q_0 \\ -q_1 + 2q_0 - q_2 \\ -q_0 + q_2 \end{pmatrix} = \frac{k}{2}\Big[q_1(q_1 - q_0) + q_0(-q_1 + 2q_0 - q_2) + q_2(-q_0 + q_2) \Big] =$$

$$= \frac{k}{2}\Big[q_1^2 - q_0 q_1 - q_0 q_1 + 2q_0^2 - q_0 q_2 - q_0 q_2 + q_2^2 \Big]$$

$$= \frac{k}{2}\Big[q_1^2 + q_2^2 + 2q_0^2 - 2q_0 q_1 - 2q_0 q_2 \Big]$$

$$\det\Big[-\omega^2 \bar{\bar{m}} + \bar{\bar{A}} \Big] = 0$$

$$\begin{vmatrix} -\omega^2 m + k & -k & 0 \\ -k & -\omega^2 M + 2k & -k \\ 0 & -k & -\omega^2 m + k \end{vmatrix} = 0$$

$$\Rightarrow \omega^2 (k - \omega^2 m)\left[\omega^2 m M - k(M+2m)\right] = 0$$

$$\omega_0 = 0$$

$$\omega_1 = \sqrt{\frac{k}{m}}$$

$$\omega_2 = \sqrt{\frac{k}{m}\left(1 + \frac{2m}{M}\right)}$$

$\omega_0 = 0$:

$$\begin{pmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{pmatrix} \begin{pmatrix} a_1^{(0)} \\ a_0^{(0)} \\ a_2^{(0)} \end{pmatrix} = 0$$

$$\Rightarrow a_1^{(0)} = a_0^{(0)}$$

$$a_0^{(0)} = a_2^{(0)}$$

$$\Rightarrow a_0^{(0)} = a_1^{(0)} = a_2^{(0)}$$

$$\Rightarrow \bar{a}^{(0)} = K \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \Rightarrow \bar{a}^{(0)T} \cdot \bar{\bar{m}} \cdot \bar{a}^{(0)} = 1$$

$$K^2 \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & m \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 1$$

$$\Rightarrow K^2 \left[m + M + m \right] = K^2 (M + 2m) = 1$$

$$K = \frac{1}{\sqrt{M+2m}} \Rightarrow \boxed{\vec{a}^{(0)} = \frac{1}{\sqrt{M+2m}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}$$

$$\omega_1 = \sqrt{\frac{k}{m}} : \begin{pmatrix} 0 & -k & 0 \\ -k & k\left(2 - \frac{M}{m}\right) & -k \\ 0 & -k & 0 \end{pmatrix} \begin{pmatrix} a_1^{(1)} \\ a_0^{(1)} \\ a_2^{(1)} \end{pmatrix} = 0$$

$$\Rightarrow a_0^{(1)} = 0$$

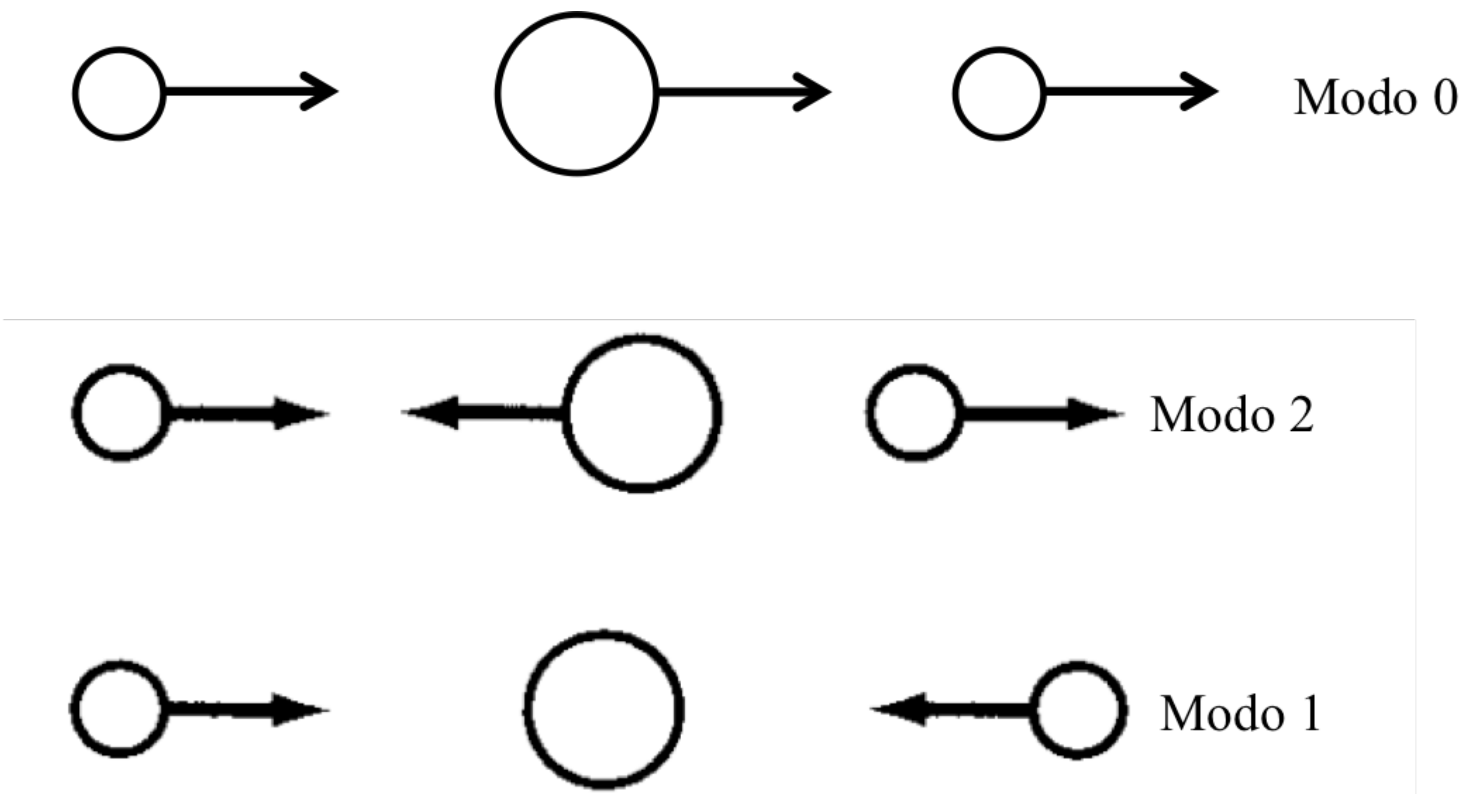
$$\Rightarrow -k \left(a_1^{(1)} + a_2^{(1)} \right) = 0 \Rightarrow a_1^{(1)} = -a_2^{(1)} \qquad \vec{a}^{(1)} = K' \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\boxed{\bar{a}^{(1)} = \frac{1}{\sqrt{2m}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}$$

FINALMENTE, PARA $\omega_2 = \sqrt{\frac{k}{m}\left(1 + \frac{2m}{M}\right)}$

$$\begin{pmatrix} -k\left(\frac{2m}{M}\right) & -k & 0 \\ -k & -k\left(\frac{M}{m}\right) & -k \\ 0 & -k & -k\left(\frac{2m}{M}\right) \end{pmatrix} \begin{pmatrix} a_1^{(2)} \\ a_0^{(2)} \\ a_2^{(2)} \end{pmatrix} = 0$$

$$\boxed{\bar{a}^{(2)} = \frac{1}{\sqrt{2m}\sqrt{1 + \frac{2m}{M}}} \begin{pmatrix} 1 \\ -2m/M \\ 1 \end{pmatrix}}$$

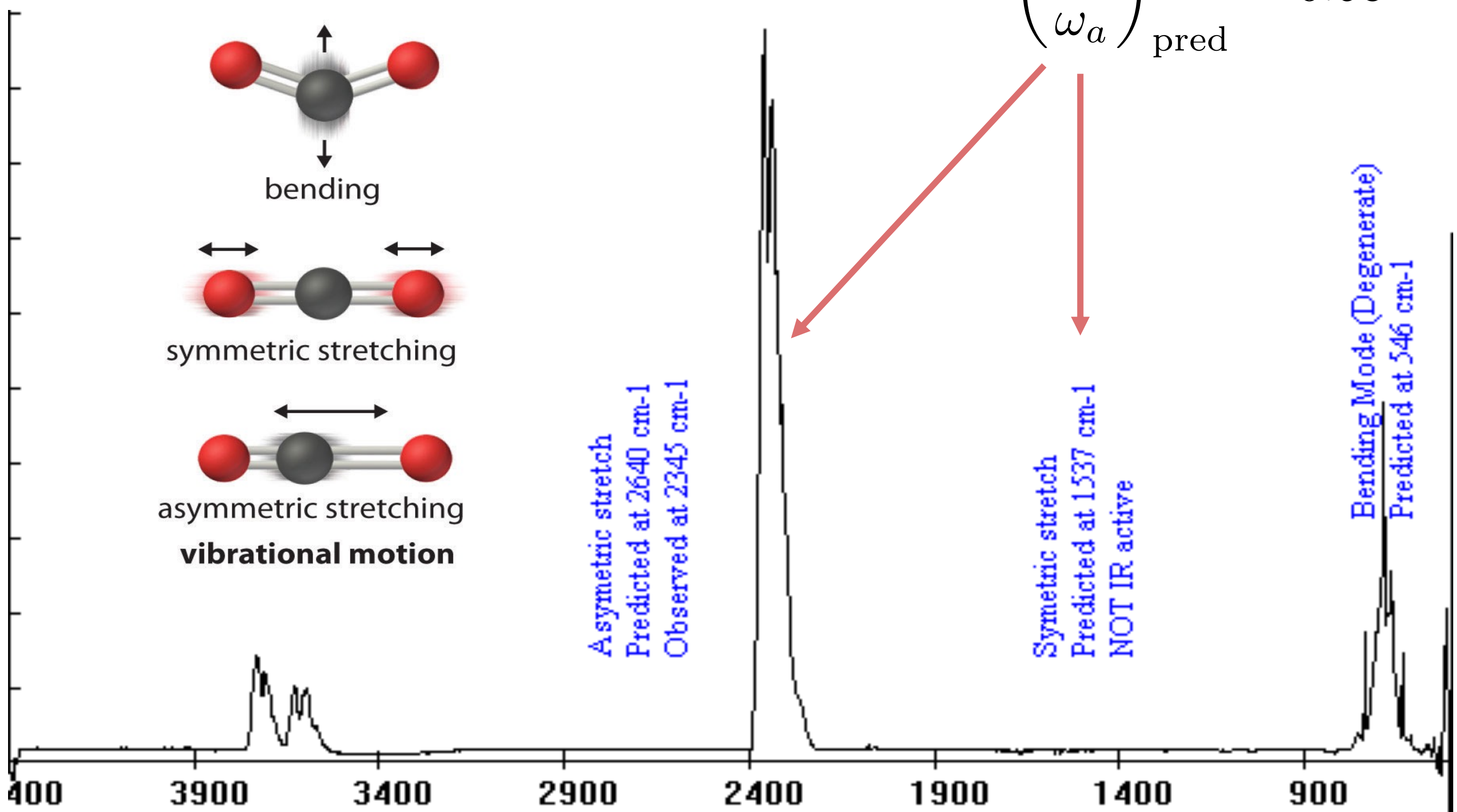

Modo 0
Modo 2
Modo 1

Espectro do CO_2

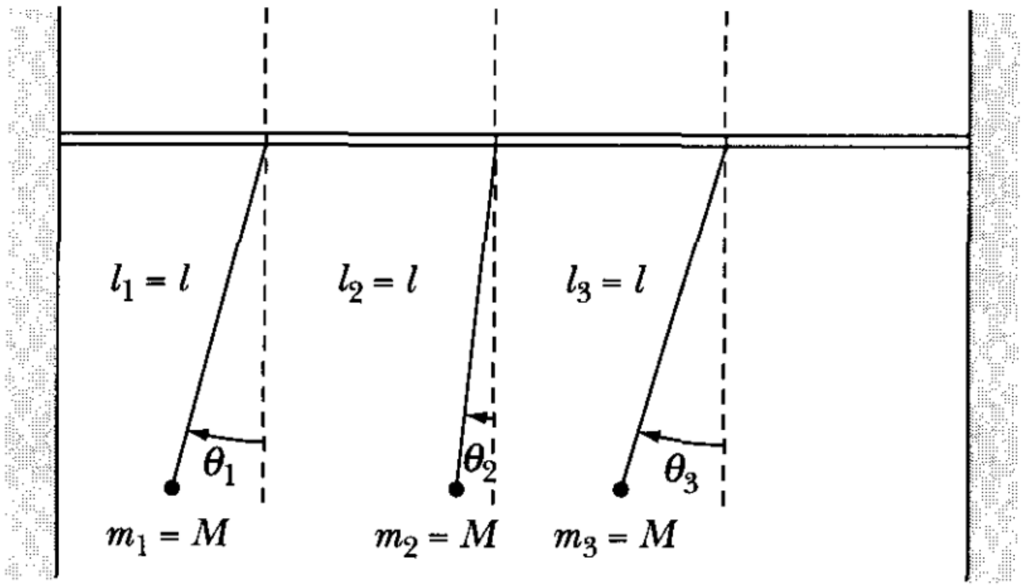
Note que a frequência do modo simétrico é menor que a do assimétrico. Pelos nossos cálculos:

$$\frac{\omega_s}{\omega_a} = \left(1 + \frac{2m_O}{m_C}\right)^{-1/2} = \left(1 + \frac{2 \times 16}{12}\right)^{-1/2} = \sqrt{\frac{3}{11}} \approx 0.52$$

$$\left(\frac{\omega_s}{\omega_a}\right)_{\text{pred}} \approx 0.58$$



Pêndulo triplo acoplado simetricamente



$$T = \frac{1}{2}\left(\dot{\theta}_1^2 + \dot{\theta}_2^2 + \dot{\theta}_3^2\right)$$

$m = l = g = 1$

$$U = \frac{1}{2}\left[\theta_1^2 + \theta_2^2 + \theta_3^2 - 2\epsilon\left(\theta_1\theta_2 + \theta_2\theta_3 + \theta_1\theta_3\right)\right]$$

$$\bar{\bar{m}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \bar{\bar{A}} = \begin{pmatrix} 1 & -\epsilon & -\epsilon \\ -\epsilon & 1 & -\epsilon \\ -\epsilon & -\epsilon & 1 \end{pmatrix}$$

$$\begin{vmatrix} -\omega^2+1 & -\epsilon & -\epsilon \\ -\epsilon & -\omega^2+1 & -\epsilon \\ -\epsilon & -\epsilon & -\omega^2+1 \end{vmatrix} = \left(\omega^2 - 1 - \epsilon\right)^2\left(\omega^2 - 1 + 2\epsilon\right) = 0$$

$\omega_1 = \omega_2 = \sqrt{1+\epsilon} \qquad \omega_3 = \sqrt{1-2\epsilon}$

$$\omega_3 = \sqrt{1-2\epsilon}$$

$$\begin{pmatrix} 2\epsilon & -\epsilon & -\epsilon \\ -\epsilon & 2\epsilon & -\epsilon \\ -\epsilon & -\epsilon & 2\epsilon \end{pmatrix} \begin{pmatrix} a_1^{(3)} \\ a_2^{(3)} \\ a_3^{(3)} \end{pmatrix} = 0$$

$$\left.\begin{aligned} 2a_1^{(3)} - a_2^{(3)} - a_3^{(3)} &= 0 \\ 2a_2^{(3)} - a_1^{(3)} - a_3^{(3)} &= 0 \end{aligned}\right\} \Rightarrow \quad a_1^{(3)} = a_2^{(3)} = a_3^{(3)}$$

$$\bar{a}^{(3)} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\omega_1 = \omega_2 = \sqrt{1+\epsilon}$$

$$\begin{pmatrix} -\epsilon & -\epsilon & -\epsilon \\ -\epsilon & -\epsilon & -\epsilon \\ -\epsilon & -\epsilon & -\epsilon \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = 0 \qquad \boxed{a_1 + a_2 + a_3 = 0}$$

$$\bar{a}^{(1)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \qquad \bar{a}^{(2)} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$