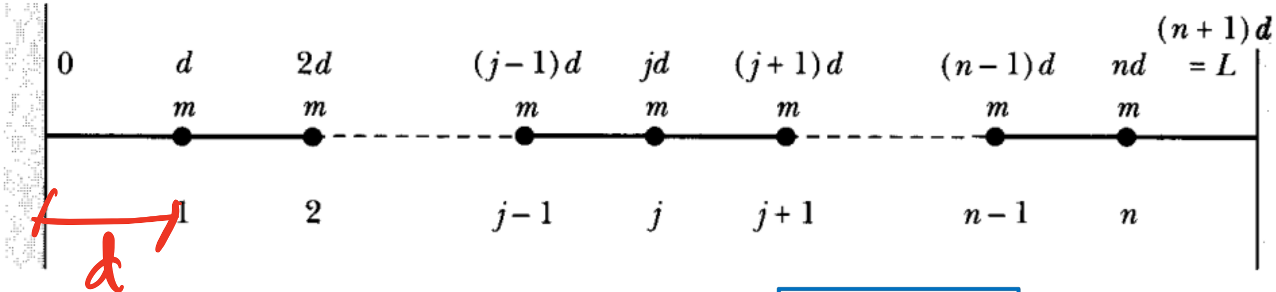
F 415 – Mecânica Geral II

1º semestre de 2024

06/06/2024

Aula 21

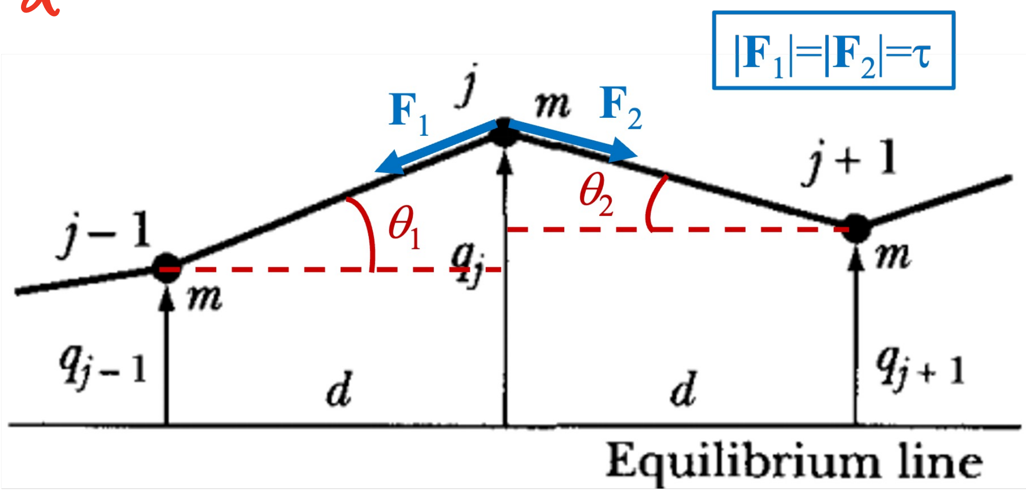
A corda com massas



n MASSAS m

CORDA SEM MASSA

COM UMA TENSÃO τ



POSIÇÃO VERTICAL DA j-ÉSIMA MASSA É q_j

$$L = (n+1)d$$

$$F_y(j-1 \to j) = -\tau \sin\theta_1$$

$$F_y(j+1 \to j) = -\tau \sin\theta_2$$

$$\sin\theta_1 \cong \frac{q_j - q_{j-1}}{d} \cong \tan\theta_1 \cong \theta_1$$

$$\sin\theta_2 \cong \frac{q_j - q_{j+1}}{d} \cong \tan\theta_2 \simeq \theta_2$$

$$F_y(j) = -\tau\left[\frac{q_j - q_{j-1}}{d} + \frac{q_j - q_{j+1}}{d}\right]$$

$$= \frac{\tau}{d}\left[q_{j+1} - 2q_j + q_{j-1}\right]$$

EQ. DE MOVIMENTO:

$$m\ddot{q}_j = \frac{\tau}{d}\left[q_{j+1} - 2q_j + q_{j-1}\right] \qquad (j = 1, \dots, n)$$

FIXAMOS: $q_0 = q_{n+1} = 0 \quad \forall t$

LAGRANGIANA: $T = \frac{1}{2}\sum_{j=1}^{n} m(\dot{q}_j)^2$

$$U = \frac{\tau}{2d}\sum_{j=1}^{n+1}(q_{j-1} - q_j)^2$$

FORÇA SOBRE A k-ÉSIMA MASSA É: $-\frac{\partial U}{\partial q_k}$

q_k APARECE EM: $\frac{\tau}{2d}\left[(q_{k-1} - q_k)^2 + (q_k - q_{k+1})^2\right]$

$$-\frac{\partial U}{\partial q_k} = -\frac{\tau}{\cancel{2}d}\left[-\cancel{2}(q_{k-1} - q_k) + \cancel{2}(q_k - q_{k+1})\right]$$

$$= \frac{\tau}{d}\left[q_{k+1} - 2q_k + q_{k-1}\right] = F_y(k)$$

$$\Rightarrow L = \frac{1}{2}\sum_{j=1}^{n} m(\dot{q}_j)^2 - \frac{\tau}{2d}\sum_{j=1}^{n+1}(q_{j-1} - q_j)^2$$

QUADRÁTICA
PEQUENAS
OSCILAÇÕES

Método dos modos normais

$q_j(t) = a_j e^{i\omega t}$ $\quad a_j \in \mathbb{C}$

$$\Rightarrow -m\omega^2 a_j \cancel{e^{i\omega t}} = \frac{\tau}{d}\left[a_{j+1} - 2a_j + a_{j-1}\right]\cancel{e^{i\omega t}} \qquad j = 1, \ldots, n$$

$$-\frac{\tau}{d}a_{j+1} + \underbrace{\left(\frac{2\tau}{d} - m\omega^2\right)}_{\text{ }} a_j - \frac{\tau}{d}a_{j+1} = 0$$

SISTEMA DE $\underline{n}$ EQS. LINEARES HOMOGÊNEO

PODEMOS ESCREVER A MATRIZ DE COEFICIENTES E IGUALAR SEU DETERMINANTE A ZERO

$$
\begin{pmatrix}
\lambda & -\tau/d & 0 & \cdots & 0 \\
-\tau/d & \lambda & -\tau/d & \cdots & 0 \\
0 & -\tau/d & \lambda & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & -\tau/d \\
0 & 0 & 0 & -\tau/d & \lambda
\end{pmatrix}
\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{pmatrix}
=
\begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}
$$

AO INVÉS DO MÉTODO USUAL, VAMOS CHUTAR A SOLUÇÃO: $a_j = a' e^{i\gamma j}$ $\quad j = 1, 2, \dots, n$

$$-\frac{\tau}{d} a_{j+1} + \lambda a_j - \frac{\tau}{d} a_{j-1} = 0$$

$$\Rightarrow -\frac{\tau}{d} \cancel{a'} e^{i\gamma(j+1)} + \lambda \cancel{a'} e^{i\gamma j} - \frac{\tau}{d} \cancel{a'} e^{i\gamma(j-1)} = 0$$

$$\Rightarrow -\frac{\tau}{d} e^{i\gamma} + \lambda - \frac{\tau}{d} e^{-i\gamma} = 0 \Rightarrow \lambda = \frac{\tau}{d}\left(e^{i\gamma} + e^{-i\gamma}\right) = \frac{2\tau}{d}\cos\gamma$$

$$\Rightarrow \frac{2\tau}{d} - \omega^2 m = \frac{2\tau}{d}\cos\gamma \Rightarrow m\omega^2 = \frac{2\tau}{d}(1 - \cos\gamma) = \frac{4\tau}{d}\sin^2\left(\frac{\gamma}{2}\right)$$

$$\Rightarrow \boxed{\omega = 2\sqrt{\frac{\tau}{md}}\,\sin\frac{\gamma}{2}}$$

$$a_j = a' e^{i\gamma j} = |a'| e^{i(\gamma j + \delta)}$$

$$a_j^{ph} = |a'| \cos(\gamma j + \delta)$$

IMPONHO AS CONDIÇÕES DE CONTORNO $j = 0, m+1$:

$$a_0 = |a'| \cos(\delta) = 0 \Rightarrow \delta = -\frac{\pi}{2}$$

$$\Rightarrow a_j = |a'| \cos\left(\gamma j - \frac{\pi}{2}\right) = |a'| \sin(\gamma j)$$

$$a_{m+1} = |a'| \sin[\gamma(m+1)] = 0 \Rightarrow \gamma_n = \frac{\pi n}{m+1} \quad (n = 1, 2, \dots, m)$$

$$\omega_n = 2\sqrt{\frac{\tau}{md}}\,\sin\left[\frac{\pi n}{2(m+1)}\right]$$

$$\Rightarrow \qquad (n = 1, 2, \dots, m)$$

$$a_j^{(n)} = a^{(n)} \sin\left[\frac{\pi n j}{m+1}\right]$$

MODOS NORMAIS:

$$x_j^{(n)}(t) = \alpha_n \sin\left[\frac{n\pi j}{N+1}\right] e^{i\omega_n t}$$

MODOS NORMAIS FÍSICOS:

$$x_j^{(n)}(t) = \sin\left[\frac{n\pi j}{N+1}\right]\left(\mu_n \cos \omega_n t + \nu_n \sin \omega_n t\right)$$

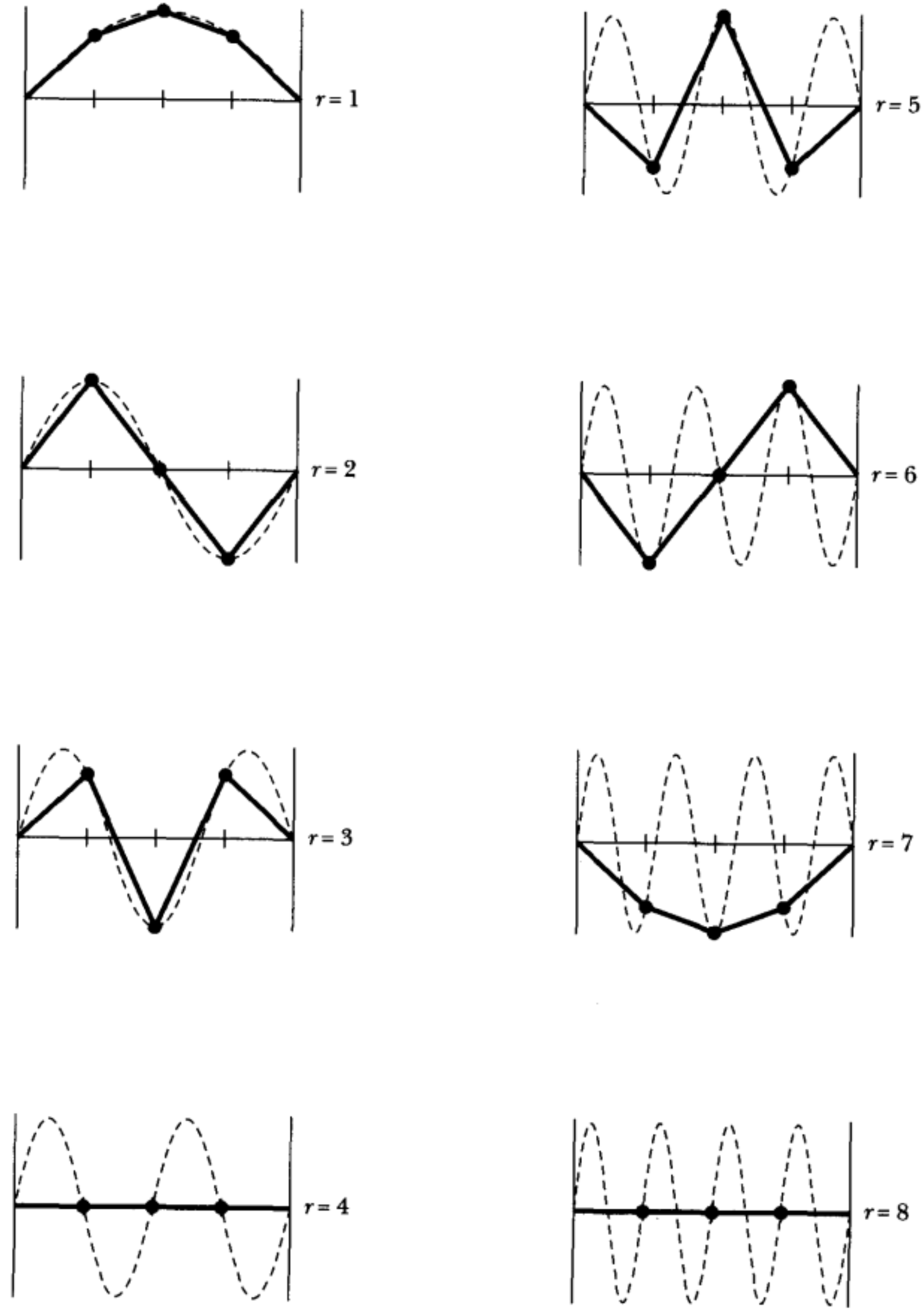
$\mu_n, \nu_n \Rightarrow$ CONSTANTES ARBITRÁRIAS A SEREM DETERMINADAS PELAS CONDIÇÕES INICIAIS

A SOLUÇÃO GERAL:

$$q_j(t) = \sum_{n=1}^{N} \sin\left(\frac{n\pi j}{N+1}\right)\left(\mu_n \cos\omega_n t + \nu_n \sin\omega_n t\right)$$

(μ_n, ν_n) $(n=1,\ldots,N) \Longrightarrow 2N$ CONSTANTES ARBITRÁRIAS

O caso $n = 3$



Condições iniciais

$$q_j(t) = \sum_{n=1}^{m} \sin\left(\frac{n\pi j}{m+1}\right)\left(\mu_n \cos\omega_n t + \nu_n \sin\omega_n t\right)$$

DADOS : $q_j(0)$, $\dot{q}_j(0)$ $\Rightarrow$ 2m PARÂMENTROS

$j = 1, \ldots, m$

$$q_j(0) = \sum_{n=1}^{m} \mu_n \sin\left(\frac{n\pi j}{m+1}\right) \quad (1)$$

$$\dot{q}_j(t) = \sum_{n=1}^{m} \sin\left(\frac{n\pi j}{m+1}\right)\left[-\omega_n \mu_n \sin\omega_n t + \omega_n \nu_n \cos\nu_n t\right]$$

$$\dot{q}_j(0) = \sum_{n=1}^{m} \nu_n \omega_n \sin\left(\frac{n\pi j}{m+1}\right) \quad (2)$$

QUERO (μ_n, ν_n) EM TERMOS DOS $q_j(0)$, $\dot{q}_j(0)$

$$q_j(0) = \sum_{n=1}^{m} \mu_n \sin\left(\frac{n\pi j}{m+1}\right) \quad (1)$$

MULTIPLICO POR $\sin\left(\frac{s\pi j}{m+1}\right)$ E SOMO EM $j=1,\dots,m$

$$\sum_{j=1}^{m} q_j(0)\sin\left(\frac{s\pi j}{m+1}\right) = \sum_{j=1}^{m}\sum_{n=1}^{m} \mu_n \sin\left(\frac{s\pi j}{m+1}\right)\sin\left(\frac{n\pi j}{m+1}\right)$$

$$\sum_{j=1}^{m} \sin\left(\frac{s\pi j}{m+1}\right)\sin\left(\frac{n\pi j}{m+1}\right) = \left(\frac{m+1}{2}\right)\delta_{n,s}$$

$$LD = \sum_{n=1}^{m} \mu_n \left(\frac{m+1}{2}\right)\delta_{n,s} = \left(\frac{m+1}{2}\right)\mu_s$$

$$\Rightarrow \mu_s = \frac{2}{m+1}\sum_{j=1}^{m} \sin\left(\frac{s\pi j}{m+1}\right) q_j(0) \quad (1)$$

ANALOGAMENTE:

$$V_s = \frac{2}{\omega_s(m+1)}\sum_{j=1}^{m} \sin\left(\frac{s\pi j}{m+1}\right) \dot{q}_j(0) \quad (2)$$

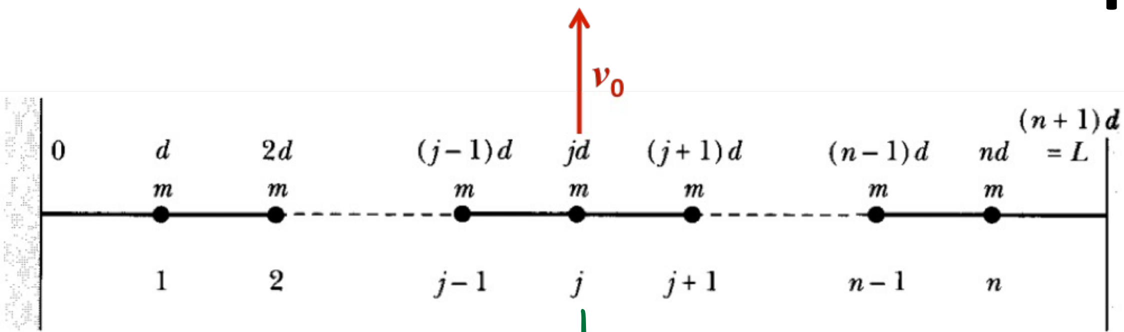
$$\omega_r = 2\sqrt{\frac{\tau}{md}}\sin\left[\frac{r\pi}{2(n+1)}\right], \; r = 1, \ldots, n$$

$$q_j(t) = \sum_{r=1}^{n} \sin\left(\frac{r\pi j}{n+1}\right)(\mu_r \cos\omega_r t + \nu_r \sin\omega_r t)$$

$$\mu_r = \frac{2}{n+1}\sum_{j=1}^{n} \sin\left(\frac{r\pi j}{n+1}\right) q_j(0),$$

$$\nu_r = \frac{2}{\omega_r(n+1)}\sum_{j=1}^{n} \sin\left(\frac{r\pi j}{n+1}\right) \dot{q}_j(0)$$

Exemplo



$\frac{n+1}{2}$

IMPRIMO v_0 NA MASSA CENTRAL: (n ÍMPAR)

$q_j(0) = 0$

$\dot{q}_j(0) = v_0 \, \delta_{j,\frac{n+1}{2}}$

COMO $q_j(0) = 0 \ \forall j$, $\boxed{\mu_r = 0}$

$$\nu_r = \frac{2}{\omega_r(n+1)} \sum_{j=1}^{n} \sin\left(\frac{r\pi j}{n+1}\right) \dot{q}_j(0) = \frac{2}{\omega_r(n+1)} \sum_{j=1}^{n} \sin\left(\frac{r\pi j}{n+1}\right) v_0 \, \delta_{j,\frac{n+1}{2}}$$

$$\nu_r = \frac{2v_0}{\omega_r(n+1)} \sin\left(\frac{r\pi}{2}\right) = \begin{cases} 0, & \text{SE } r \text{ FOR PAR} \\ \dfrac{2v_0}{\omega_r(n+1)}(-1)^{(r-1)/2}, & \text{SE } r \text{ FOR ÍMPAR} \end{cases}$$

FAZENDO: $r = 2p-1$ E $p = 1, \dots, \frac{n+1}{2}$

$\Rightarrow r = 1, 3, 5, \dots n$

$$\Rightarrow q_j(t) = \sum_{p=1}^{(m+1)/2} V_{2p-1} \sin\left[\frac{(2p-1)\pi j}{m+1}\right] \sin\left[\omega_{(2p-1)} t\right]$$

$$q_j(t) = \frac{2v_0}{(m+1)} \sum_{p=1}^{\frac{m+1}{2}} \frac{(-1)^{(p-1)}}{\omega_{(2p-1)}} \sin\left[\frac{(2p-1)\pi j}{m+1}\right] \sin\left[\omega_{(2p-1)} t\right]$$