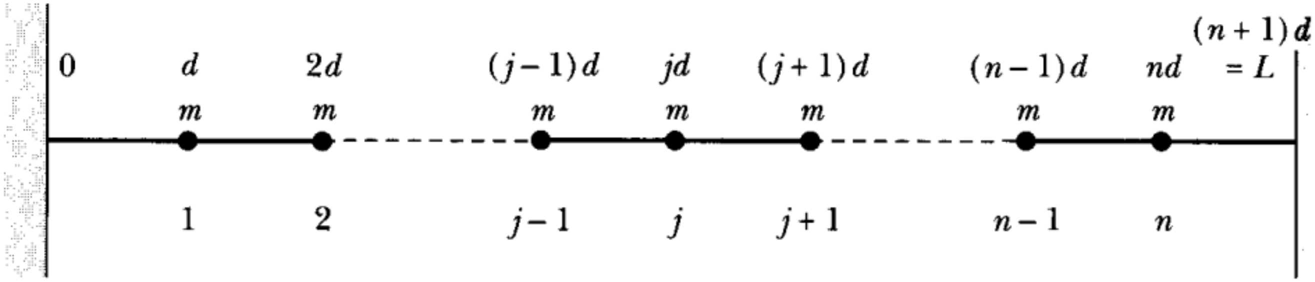
F 415 – Mecânica Geral II

1º semestre de 2024

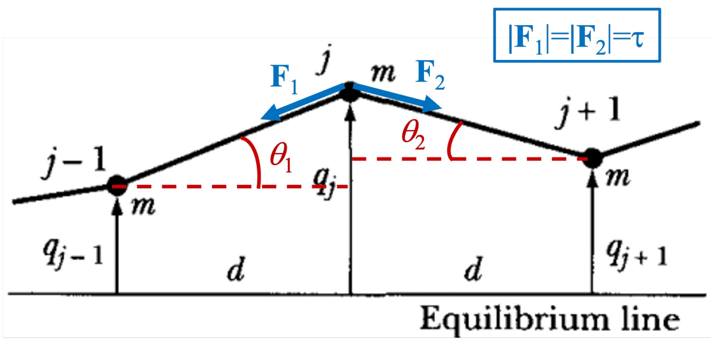
11/06/2024

Aula 22

Aula passada



- n massas m
- distância d entre as massas
- $L = (n+1)\, d$
- tração τ na corda



$$F_y\,(j) = -\tau \sin\theta_1 - \tau \sin\theta_2$$

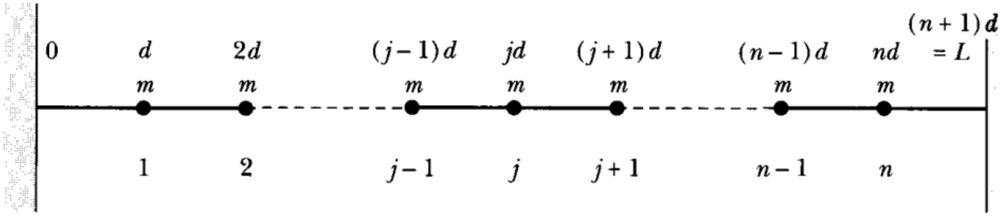
$$\sin\theta_1 \approx \tan\theta_1 = \frac{q_j - q_{j-1}}{d}$$

$$\sin\theta_2 \approx \tan\theta_2 = \frac{q_j - q_{j+1}}{d}$$

$$\Rightarrow \quad F_y\,(j) = \frac{\tau}{d}\,(q_{j-1} - 2q_j + q_{j+1})$$

$$\boxed{m\ddot{q}_j = \frac{\tau}{d}\,(q_{j-1} - 2q_j + q_{j+1}) \ \ (j = 1, \ldots, n)} \qquad q_0 = q_{n+1} = 0$$

Aula passada



Equações de movimento

$$m\ddot{q}_j = \frac{\tau}{d}\left(q_{j-1} - 2q_j + q_{j+1}\right) \quad (j = 1, \ldots, n)$$

Modos normais: $q_j(t) = a_j e^{i\omega t}$ $\quad \left(\frac{2\tau}{d} - \omega^2 m\right) a_j - \frac{\tau}{d}\left(a_{j-1} + a_{j+1}\right) = 0$

Periodicidade: $a_j = a' e^{i\gamma j} \Rightarrow a_j = |a'| \cos(\gamma j + \delta) \Rightarrow \omega = 2\sqrt{\frac{\tau}{md}} \sin\frac{\gamma}{2}$

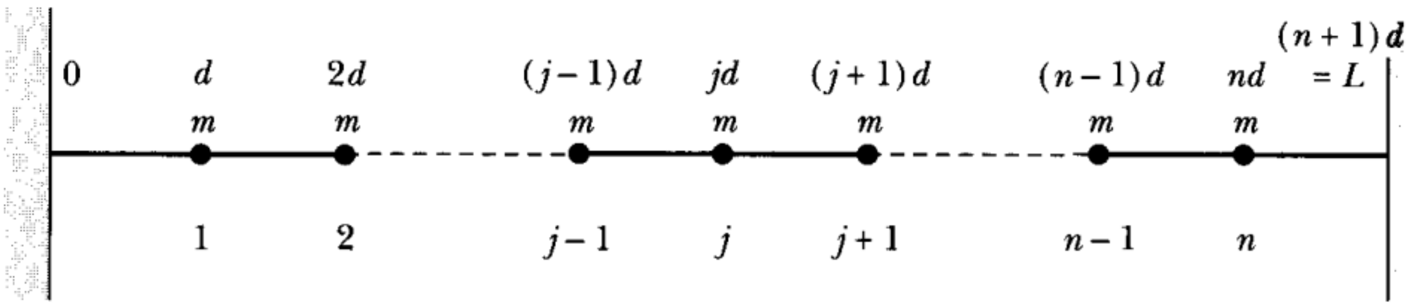
Condições de contorno:
$$\begin{aligned} a_0 &= a_{n+1} = 0 \Rightarrow a_j^{(r)} = \alpha_r \sin\left(\frac{r\pi j}{n+1}\right) \\ \omega_r &= 2\sqrt{\frac{\tau}{md}} \sin\left[\frac{r\pi}{2(n+1)}\right], \; r = 1, \ldots, n \end{aligned}$$

Solução geral:

$$q_j(t) = \sum_{r=1}^{n} \sin\left(\frac{r\pi j}{n+1}\right)\left(\mu_r \cos\omega_r t + \nu_r \sin\omega_r t\right)$$

$$\mu_r = \frac{2}{n+1}\sum_{j=1}^{n} \sin\left(\frac{r\pi j}{n+1}\right) q_j(0), \qquad \nu_r = \frac{2}{\omega_r(n+1)}\sum_{j=1}^{n} \sin\left(\frac{r\pi j}{n+1}\right) \dot{q}_j(0)$$

A corda como limite de um sistema discreto

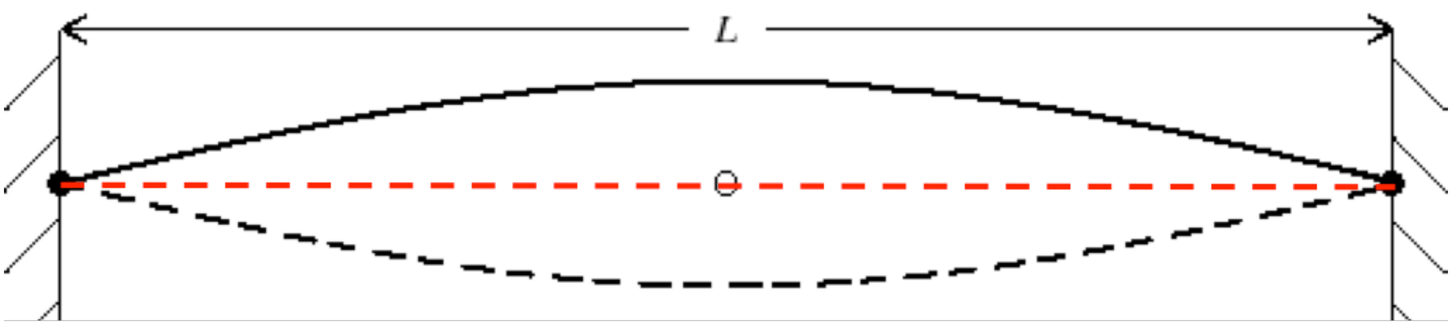


$n \to \infty \,, \; d \to 0$

$L = (n+1)d = \mathrm{CONST.}$

$M = n\, m = \mathrm{CONST.}$

$m \to 0$



$$\frac{m}{d} = \frac{M/n}{L/(n+1)} =$$

$$= \frac{n+1}{n} \frac{M}{L} \to \frac{M}{L}$$

$$\frac{M}{L} \equiv \rho$$

$jd \to x \in [0, L]$

$q_j(t) \to q(x,t)$

Equações de movimento

$$m\ddot{q}_j(t) = \frac{\tau}{d}\left(q_{j+1} - 2q_j + q_{j-1}\right) = \frac{\tau}{d}\left[(q_{j+1} - q_j) - (q_j - q_{j-1})\right]$$

$$= \tau\left[\left(\frac{q_{j+1} - q_j}{d}\right) - \left(\frac{q_j - q_{j-1}}{d}\right)\right]$$

$$\cong \tau d\left[\left.\frac{\partial q}{\partial x}\right|_{x=(j+\frac{1}{2})d} - \left.\frac{\partial q}{\partial x}\right|_{x=(j-\frac{1}{2})d}\right]\frac{1}{d}$$

$$m\frac{\partial^2 q(x,t)}{\partial t^2} = \tau d \left.\frac{\partial^2 q}{\partial x^2}\right|_{x=jd} \Rightarrow \frac{\partial^2 q}{\partial t^2} = \frac{\tau}{\underbrace{m/d}_{\rho}}\frac{\partial^2 q}{\partial x^2}$$

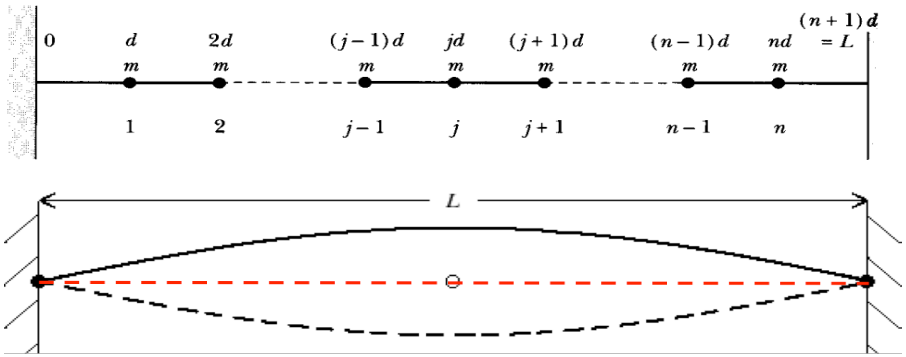
$$\boxed{\frac{\partial^2 q}{\partial x^2} - \frac{1}{(\tau/\rho)}\frac{\partial^2 q}{\partial t^2} = 0}$$

EQ. DE ONDA

$$c = \sqrt{\frac{\tau}{\rho}}$$

$$\frac{\partial^2 q}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 q}{\partial t^2} = 0 \quad ; \quad \text{EM 3D: } \frac{\partial^2 q}{\partial x^2} + \frac{\partial^2 q}{\partial y^2} + \frac{\partial^2 q}{\partial z^2} = \nabla^2 q = \frac{1}{c^2}\frac{\partial^2 q}{\partial t^2}$$

A equação de onda



Limite do contínuo:

$$\begin{aligned} n &\to \infty, d \to 0, m \to 0 \\ L &= (n+1)\, d = \text{const.} \\ \rho &= \frac{m}{d} = \text{const.} \end{aligned}$$

$$\begin{aligned} jd \in [0, d, 2d, \ldots, (n+1)\, d = L] &\to x \in [0, L] \\ q_j(t) &\to q(x,t) \end{aligned}$$

$$m\ddot{q}_j = \frac{\tau}{d}\left(q_{j-1} - 2q_j + q_{j+1}\right) \to \frac{\partial^2 q(x,t)}{\partial t^2} = \frac{\tau}{\rho}\frac{\partial^2 q(x,t)}{\partial x^2}$$

$$\boxed{\frac{1}{c^2}\frac{\partial^2 q(x,t)}{\partial t^2} - \frac{\partial^2 q(x,t)}{\partial x^2} = 0} \quad c \equiv \sqrt{\frac{\tau}{\rho}}$$

EQUAÇÃO DE ONDA

A corda presa nas extremidades

$$q_0(t) = q_{n+1}(t) = 0 \Rightarrow q(x=0,t) = q(x=L,t) = 0$$

DA AULA PASSADA:

olução geral:

$$q_j(t) = \sum_{r=1}^{n} \sin\left(\frac{r\pi j}{n+1}\right)\left(\mu_r \cos\omega_r t + \nu_r \sin\omega_r t\right)$$

$$\mu_r = \frac{2}{n+1}\sum_{j=1}^{n}\sin\left(\frac{r\pi j}{n+1}\right) q_j(0), \qquad \nu_r = \frac{2}{\omega_r(n+1)}\sum_{j=1}^{n}\sin\left(\frac{r\pi j}{n+1}\right)\dot{q}_j(0)$$

$$\sin\left(\frac{r\pi j}{n+1}\right) = \sin\left(\frac{r\pi jd}{(n+1)d}\right) \to \sin\left(\frac{r\pi x}{L}\right)$$

$$\omega_r = 2\sqrt{\frac{\tau}{md}}\sin\left[\frac{r\pi}{2(n+1)}\right] = 2\sqrt{\frac{\tau}{md}}\sin\left[\frac{r\pi d}{2(n+1)d}\right]$$

$$= 2\sqrt{\frac{\tau}{md}}\sin\left[\frac{r\pi d}{2L}\right] \simeq \cancel{2}\sqrt{\frac{\tau}{md}}\left(\frac{r\pi d}{\cancel{2}L}\right) = \sqrt{\frac{\tau d}{m}}\,\frac{r\pi}{L} = \sqrt{\frac{\tau}{\rho}}\,\frac{r\pi}{L}$$

$$\boxed{\omega_r = \frac{r\pi c}{L}}$$

$$\mu_r = \frac{2}{n+1}\sum_{j=1}^{n} \sin\left(\frac{r\pi j}{n+1}\right) q_j(0) = \frac{2}{(n+1)d}\sum_{j=1}^{n} \sin\left[\frac{r\pi j d}{(n+1)d}\right] q_j(t=0)\, d$$

$$\rightarrow \frac{2}{L}\int_0^L q(x,0)\sin\left(\frac{r\pi x}{L}\right)dx$$

$$\nu_r = \frac{2}{\omega_r L}\int_0^L \sin\left(\frac{r\pi x}{L}\right)\left.\frac{\partial q(x,t)}{\partial t}\right|_{t=0} dx$$

$$\nu_r = \frac{2}{r\pi c}\int_0^L \sin\left(\frac{r\pi x}{L}\right)\left.\frac{\partial q(x,t)}{\partial t}\right|_{t=0} dx$$

A equação de onda

$$\boxed{\frac{1}{c^2}\frac{\partial^2 q(x,t)}{\partial t^2} - \frac{\partial^2 q(x,t)}{\partial x^2} = 0} \quad c \equiv \sqrt{\frac{\tau}{\rho}}$$

A solução por modos normais da corda presa nas extremidades

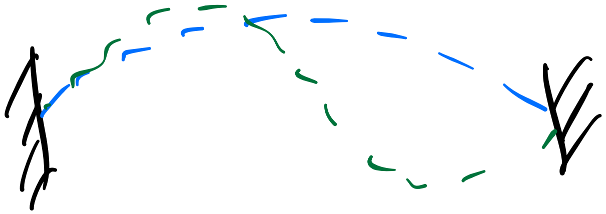
$$q_j(t) = \sum_{r=1}^{n} \sin\left(\frac{r\pi j}{n+1}\right)(\mu_r \cos\omega_r t + \nu_r \sin\omega_r t) \rightarrow q(x,t) = \sum_{r=1}^{\infty} \sin\left(\frac{r\pi x}{L}\right)\left[\mu_r \cos\left(\frac{r\pi ct}{L}\right) + \nu_r \sin\left(\frac{r\pi ct}{L}\right)\right]$$

$$\mu_r = \frac{2}{n+1}\sum_{j=1}^{n} \sin\left(\frac{r\pi j}{n+1}\right) q_j(0) \qquad \rightarrow \mu_r = \frac{2}{L}\int_0^L \sin\left(\frac{r\pi x}{L}\right) q(x,0)\,dx$$

$$\nu_r = \frac{2}{\omega_r(n+1)}\sum_{j=1}^{n} \sin\left(\frac{r\pi j}{n+1}\right) \dot{q}_j(0) \qquad \rightarrow \nu_r = \frac{2}{r\pi c}\int_0^L \sin\left(\frac{r\pi x}{L}\right) \left.\frac{\partial q(x,t)}{\partial t}\right|_{t=0} dx$$

$$\boxed{\omega_r = 2\sqrt{\frac{\tau}{md}}\sin\left[\frac{r\pi}{2(n+1)}\right]} \longrightarrow \boxed{\omega_r = \frac{r\pi c}{L}}$$

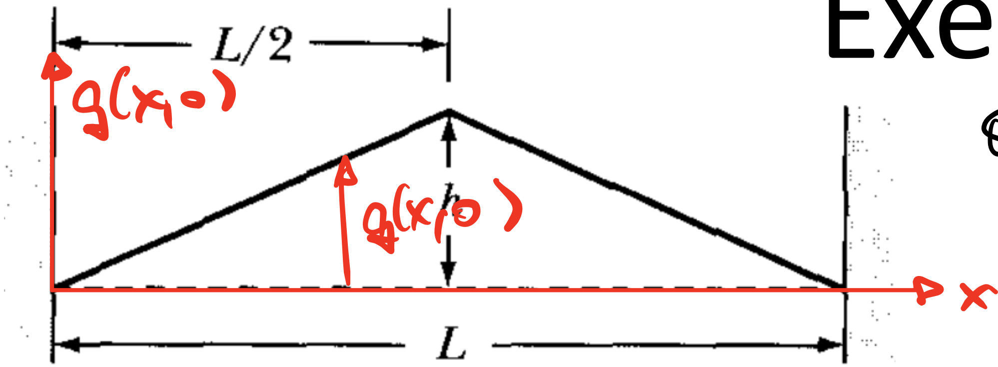
$$\omega_n = n\frac{\pi c}{L} = c\underbrace{\left(\frac{n\pi}{L}\right)}_{k_n} \qquad k_n = \frac{n\pi}{L}$$



$$\Rightarrow L = n\frac{\lambda_n}{2} \Rightarrow \lambda_n = \frac{2L}{n}$$

$$k_n = \frac{2\pi}{\lambda_n} = \frac{\cancel{2}\pi}{\cancel{2}L/n} = \frac{n\pi}{L}$$

Exemplo



QUERO: $q(x,t)$

$$q(x,0) = \begin{cases} \dfrac{h}{L/2}\,x\,, & x \in \left[0, \frac{L}{2}\right] \\ -\dfrac{2h}{L}\,x + 2h\,, & x \in \left[\frac{L}{2}, L\right] \end{cases}$$

$$\left.\frac{\partial q(x,t)}{\partial t}\right|_{t=0} = 0 \Rightarrow \boxed{\nu_r = 0}$$

$$\rightarrow \mu_r = \frac{2}{L} \int_0^L \sin\left(\frac{r\pi x}{L}\right) q\,(x,0)\,dx$$

$$\rightarrow \nu_r = \frac{2}{r\pi c} \int_0^L \sin\left(\frac{r\pi x}{L}\right) \left.\frac{\partial q\,(x,t)}{\partial t}\right|_{t=0} dx$$

$$\mu_r = \frac{2}{L}\left[\int_0^{L/2} \frac{2h}{L}\,x \sin\left(\frac{r\pi x}{L}\right) dx + \int_{L/2}^{L} \left[\frac{2h}{L}(L-x)\right] \sin\left(\frac{r\pi x}{L}\right) dx\right]$$

$$\mu_r = \frac{8h}{r^2\pi^2} \sin\left(\frac{r\pi}{2}\right) = \begin{cases} 0\,, & \text{SE } r \text{ É PAR} \\ \dfrac{8h}{r^2\pi^2}(-1)^{(r-1)/2}\,, & \text{SE } r \text{ É ÍMPAR} \end{cases}$$

Solução da equação de onda

(sem impôr condições de contorno: corda “infinita”)

$$\frac{1}{c^2}\frac{\partial^2 q}{\partial t^2} - \frac{\partial^2 q}{\partial x^2} = 0$$

SEPARAÇÃO DE VARIÁVEIS:

$$q(x,t) = \psi(x)\,\chi(t)$$

$$\Rightarrow \frac{1}{c^2}\psi(x)\frac{d^2\chi}{dt^2} - \chi(t)\frac{d^2\psi}{dx^2} = 0 \quad ,$$

DIVIDO POR $\psi(x)\chi(t)$

$$\Rightarrow \underbrace{\frac{1}{c^2}\frac{1}{\chi(t)}\frac{d^2\chi(t)}{dt}}_{-\frac{\omega^2}{c^2}} = \underbrace{\frac{1}{\psi(x)}\frac{d^2\psi(x)}{dx^2}}_{-\frac{\omega^2}{c^2}} \Rightarrow \begin{cases} \ddot{\chi} + \omega^2\chi = 0 \\ \psi'' + \frac{\omega^2}{c^2}\psi = 0 \end{cases}$$

$$\chi(t) = e^{\pm i\omega t} \quad ; \quad \psi(x) = e^{\pm i\frac{\omega}{c}x}$$

$$q(x,t) = e^{i\frac{\omega}{c}(x-ct)} \quad \text{OU} \quad e^{i\frac{\omega}{c}(x+ct)}$$

$$e^{-i\frac{\omega}{c}(x-ct)} \quad \text{OU} \quad e^{-i\frac{\omega}{c}(x+ct)}$$

SUPERPONDO PARA TODOS OS VALORES DE $\underline{\omega}$

$$q(x,t) = q_+(x,t) + q_-(x,t) \quad \text{ONDE}$$

$$q_\pm(x,t) = \int_{-\infty}^{+\infty} d\omega\, a^{(\pm)}(\omega)\, e^{i\frac{\omega}{c}(x\mp ct)}$$

$$\Rightarrow q_+(x,t) = f(x-ct)$$

$$q_-(x,t) = g(x+ct)$$

$$q(x,t) = f(x-ct) + g(x+ct)$$

$f(x-ct) \rightarrow$ ONDA CAMINHANTE DA ESQ. PARA A DIREITA COM VELOCIDA[DE] DE $\underline{c}$

$g(x+ct) \rightarrow$ DA DIREITA P/ A ESQUERDA