F 415 – Mecânica Geral II

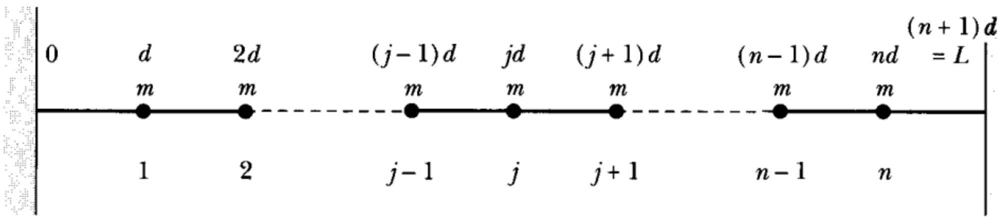
1º semestre de 2024

13/06/2024

Aula 23

Aula passada

A corda presa nas extremidades como limite de um sistema de massas discretas



Limite do contínuo:

$$
\begin{aligned}
n &\rightarrow \infty, d \rightarrow 0, m \rightarrow 0 \\
L &= (n+1)\,d = \text{const.} \\
\rho &= \frac{nm}{(n+1)\,d} = \frac{M}{L} = \frac{m}{d} = \text{const.}
\end{aligned}
$$

$$
\begin{aligned}
jd \in [0, d, 2d, \ldots, (n+1)\,d = L] &\rightarrow x \in [0, L] \\
q_j(t) &\rightarrow q(x,t)
\end{aligned}
$$

$$
m\ddot{q}_j = \frac{\tau}{d}\left(q_{j-1} - 2q_j + q_{j+1}\right) \rightarrow \frac{\partial^2 q(x,t)}{\partial t^2} = \frac{\tau}{\rho}\frac{\partial^2 q(x,t)}{\partial x^2}
$$

Aula passada

EQUAÇÃO DE ONDA

$$\boxed{\frac{1}{c^2}\frac{\partial^2 q(x,t)}{\partial t^2} - \frac{\partial^2 q(x,t)}{\partial x^2} = 0} \qquad c \equiv \sqrt{\frac{\tau}{\rho}}$$

A solução por modos normais da corda presa nas extremidades

$$q_j(t) = \sum_{r=1}^{n} \sin\left(\frac{r\pi j}{n+1}\right)(\mu_r \cos\omega_r t + \nu_r \sin\omega_r t) \to q(x,t) = \sum_{r=1}^{\infty} \sin\left(\frac{r\pi x}{L}\right)\left[\mu_r \cos\left(\frac{r\pi ct}{L}\right) + \nu_r \sin\left(\frac{r\pi ct}{L}\right)\right]$$

$$\mu_r = \frac{2}{n+1}\sum_{j=1}^{n} \sin\left(\frac{r\pi j}{n+1}\right) q_j(0) \qquad \to \mu_r = \frac{2}{L}\int_0^L \sin\left(\frac{r\pi x}{L}\right) q(x,0)\,dx$$

$$\nu_r = \frac{2}{\omega_r(n+1)}\sum_{j=1}^{n} \sin\left(\frac{r\pi j}{n+1}\right) \dot{q}_j(0) \qquad \to \nu_r = \frac{2}{r\pi c}\int_0^L \sin\left(\frac{r\pi x}{L}\right) \left.\frac{\partial q(x,t)}{\partial t}\right|_{t=0} dx$$

$$\boxed{\omega_r = 2\sqrt{\frac{\tau}{md}}\sin\left[\frac{r\pi}{2(n+1)}\right]} \longrightarrow \boxed{\omega_r = \frac{r\pi c}{L}}$$

Aula passada

EQUAÇÃO DE ONDA

$$\boxed{\frac{1}{c^2}\frac{\partial^2 q(x,t)}{\partial t^2} - \frac{\partial^2 q(x,t)}{\partial x^2} = 0} \qquad c \equiv \sqrt{\frac{\tau}{\rho}}$$

Soluções separáveis: $e^{i\frac{\omega}{c}(x-ct)}$ ou $e^{i\frac{\omega}{c}(x+ct)}$ $\quad \omega \in (-\infty, +\infty)$

Superpondo todas as soluções separáveis possíveis, obtém-se a solução geral:

$$\begin{aligned} q^{(\pm)}(x,t) &= \int_{-\infty}^{+\infty} e^{i\frac{\omega}{c}(x\mp ct)} a^{(\pm)}(\omega)\, d\omega = \begin{cases} f(x-ct) \\ g(x+ct) \end{cases} \\ q(x,t) &= q^{(+)}(x,t) + q^{(-)}(x,t) = f(x-ct) + g(x+ct) \end{aligned}$$

Solução geral por pacotes de ondas caminhantes (sem assumir condições de contorno)

$$\boxed{q(x,t) = f(x-ct) + g(x+ct)}$$

Condições iniciais

Dados $\begin{cases} q(x,0)\,, & \forall x \\ \left.\frac{\partial q(x,t)}{\partial t}\right|_{t=0}\,, & \forall x \end{cases}$

$$q(x,t) = f(x - ct) + g(x + ct)$$

$$\frac{\partial q}{\partial t}(x,t) = -c f'(x-ct) + c g'(x+ct)$$

EM $t=0$:

$$q(x,0) = f(x) + g(x) \quad (1)$$

$$\left.\frac{\partial q(x,t)}{\partial t}\right|_{t=0} = c\left[-f'(x) + g'(x)\right] \quad (2)$$

$$(2) \Rightarrow \frac{1}{c}\int_0^x \left.\frac{\partial q(x,t)}{\partial t}\right|_{t=0} dx = -f(x) + g(x) + A \quad (3)$$

$$(1)+(3) \Rightarrow 2g(x) + A = q(x,0) + \left.\frac{\partial q}{\partial t}\right|_{t=0}$$

$$\Rightarrow g(x) = \frac{1}{2}\left[q(x,0) + \int_0^x dx \left.\frac{\partial q(x,t)}{\partial t}\right|_{t=0} - A\right] \quad (4)$$

LEVANDO (4) EM (1):

$$f(x) = \frac{1}{2}\left[g(x,0) - \frac{1}{c}\int_0^x \frac{\partial g(x,t)}{\partial t}\bigg|_{t=0} dx + A\right]$$

A CONSTANTE <u>A</u> NÃO APARECE EM:

$$g(x,t) = f(x-ct) + g(x+ct)$$

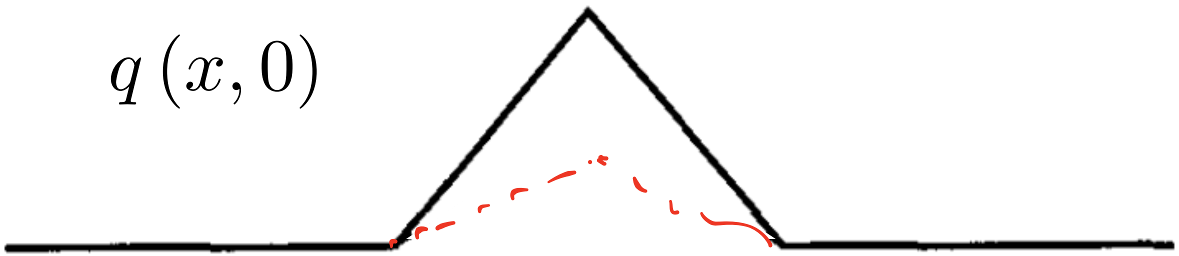
EQUAÇÃO DE ONDA

$$\boxed{\frac{1}{c^2}\frac{\partial^2 q(x,t)}{\partial t^2}-\frac{\partial^2 q(x,t)}{\partial x^2}=0}\quad c\equiv\sqrt{\frac{\tau}{\rho}}$$

Solução por pacotes de ondas caminhantes

$$\boxed{q(x,t)=f(x-ct)+g(x+ct)}$$

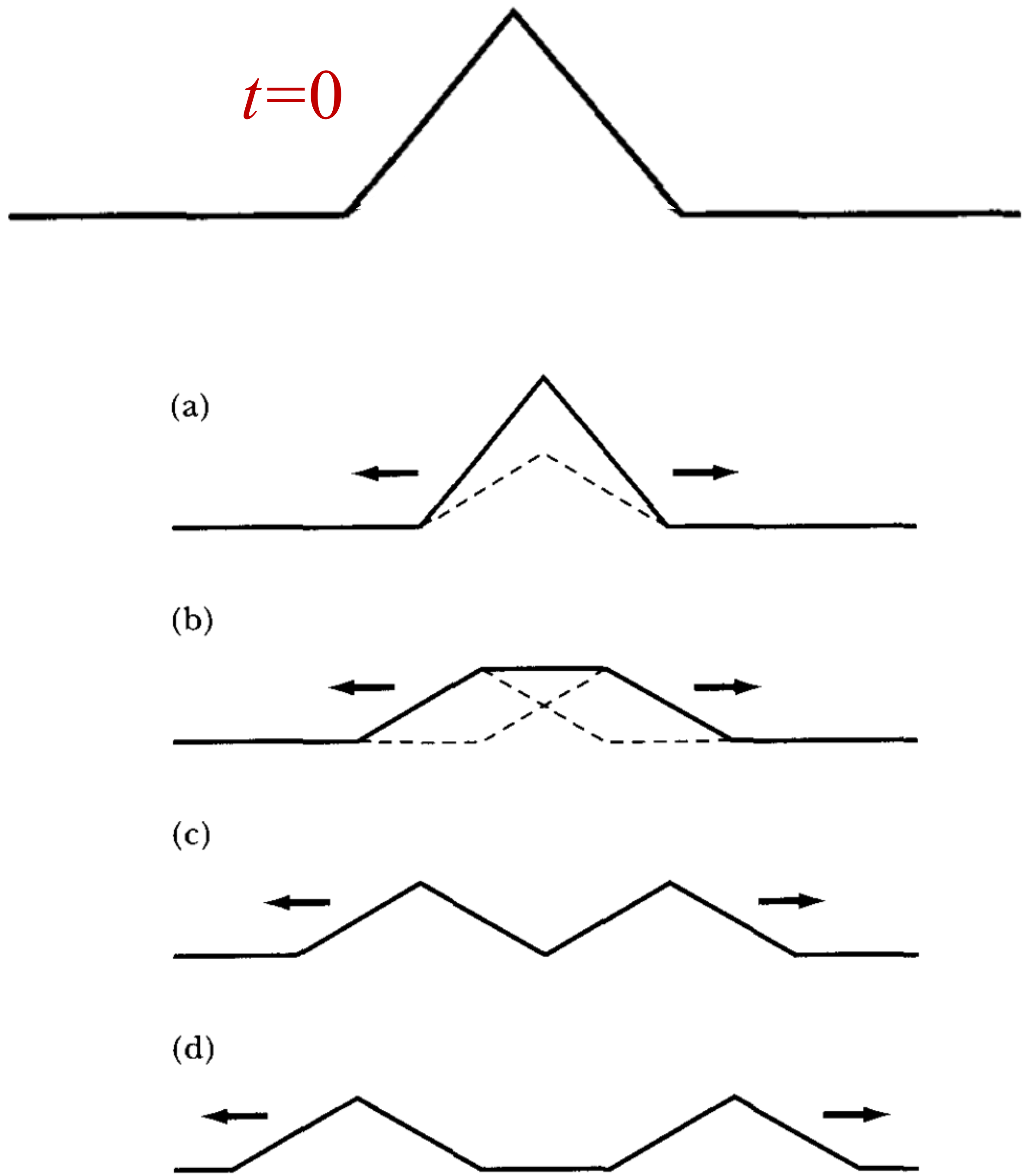
$$\begin{aligned} f(x) &= \frac{1}{2}\left[q(x,0)-\frac{1}{c}\int^{x}\left.\frac{\partial q(x',t)}{\partial t}\right|_{t=0}dx'\right] \\ g(x) &= \frac{1}{2}\left[q(x,0)+\frac{1}{c}\int^{x}\left.\frac{\partial q(x',t)}{\partial t}\right|_{t=0}dx'\right] \end{aligned}$$



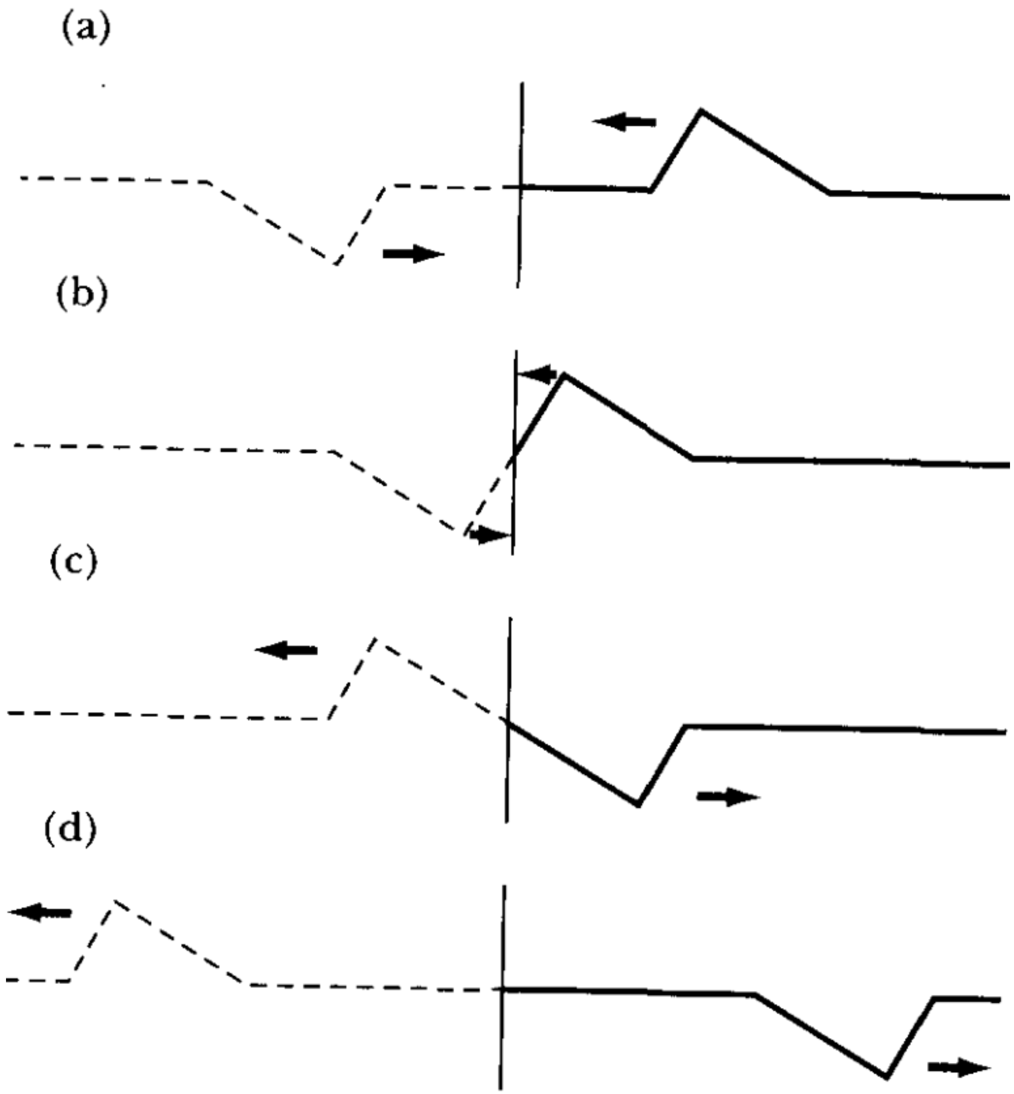
$$\left.\frac{\partial q(x,t)}{\partial t}\right|_{t=0} = 0, \quad \forall x$$

$$\Rightarrow f(x) = g(x) = \frac{1}{2} q(x,0)$$

$$q(x,t) = f(x-ct) + g(x+ct)$$



$t=0$
(a)
(b)
(c)
(d)



Ver solução no livro OU NAS NOTAS

Energia e Lagrangiana da corda

$$T = \frac{m}{2}\sum_{j=1}^{n}\dot{q_j}^2 = \frac{m}{2d}\sum_{j=1}^{n}(\dot{q}_j)^2 d \xrightarrow[\substack{n\to\infty \\ d\to 0}]{} \frac{\rho}{2}\int_0^L \left(\frac{\partial q}{\partial t}\right)^2 dx$$

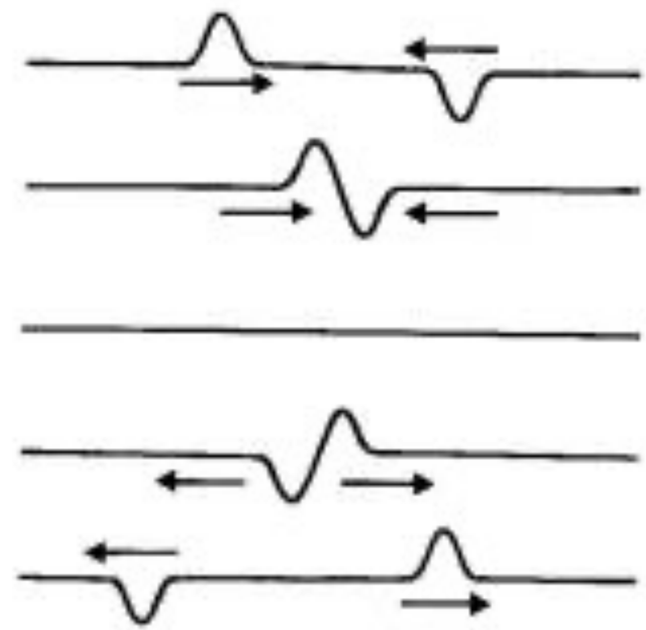
$$U = \frac{\tau}{2d}\sum_{j=1}^{n}(q_j - q_{j-1})^2$$

$$U = \frac{\tau \cancel{d}}{2\cancel{d}}\sum_{j=1}^{n}\left(\frac{q_j - q_{j-1}}{d}\right)^2 d \xrightarrow[\substack{d\to 0 \\ n\to\infty}]{} \frac{\tau}{2}\int_0^L \left(\frac{\partial q}{\partial x}\right)^2 dx$$

ENERGIA TOTAL: $E = T + U = \int_0^L \left[\frac{\rho}{2}\left(\frac{\partial q}{\partial t}\right)^2 + \frac{\tau}{2}\left(\frac{\partial q}{\partial x}\right)^2\right] dx$

$$L = T - U = \int_0^L \left[\frac{\rho}{2}\left(\frac{\partial q}{\partial t}\right)^2 - \frac{\tau}{2}\left(\frac{\partial q}{\partial x}\right)^2\right] dx$$

Se a onda carrega energia, para onde foi a energia no terceiro instante ao lado?



Energia no caso da corda presa nas extremidades

$$q(x,t) = \sum_{r=1}^{\infty} \sin\left(\frac{r\pi x}{L}\right)\eta_r(t) \Rightarrow \frac{\partial q}{\partial t} = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\dot{\eta}_n(t)$$

$$\eta_r(t) = \mu_r \cos\left(\frac{r\pi ct}{L}\right) + \nu_r \sin\left(\frac{r\pi ct}{L}\right) \Rightarrow \frac{\partial q}{\partial x} = \sum_{n=1}^{\infty}\left(\frac{n\pi}{L}\right)\cos\left(\frac{n\pi x}{L}\right)\eta_n(t)$$

$$T = \frac{\rho}{2}\int_0^L \left(\frac{\partial q}{\partial t}\right)^2 dx = \frac{\rho}{2}\int_0^L \sum_{n=1}^{\infty}\sum_{s=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{s\pi x}{L}\right)\dot{\eta}_n(t)\dot{\eta}_s(t)\,dx$$

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{s\pi x}{L}\right)dx = \frac{L}{2}\delta_{n,s} \qquad \omega_n = \frac{n\pi c}{L}$$

$$T = \frac{\rho}{2}\sum_{n,s}\dot{\eta}_n(t)\dot{\eta}_s(t)\frac{L}{2}\delta_{n,s} = \frac{\rho L}{4}\sum_{n=1}^{\infty}\left[\dot{\eta}_n(t)\right]^2$$

$$U = \frac{\tau}{2}\int_0^L \left(\frac{\partial q}{\partial x}\right)^2 dx = \frac{\tau}{2}\int_0^L \sum_{n,s}\left(\frac{n\pi}{L}\right)\left(\frac{s\pi}{L}\right)\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{s\pi x}{L}\right)\eta_n(t)\eta_s(t)\,dx$$

$$\int_0^L \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{s\pi x}{L}\right)dx = \frac{L}{2}\delta_{n,s}; \quad U = \frac{\tau L}{4}\sum_{n=1}^{\infty}\underbrace{\left(\frac{n\pi}{L}\right)^2}_{\frac{\omega_n}{c}}\left[\eta_n(t)\right]^2$$

$$U = \frac{\tau L}{4c^2}\sum_{n=1}^{\infty}\left[\omega_n \eta_n(t)\right]^2 \qquad c^2 = \frac{\tau}{\rho}$$

$$U = \frac{\rho L}{4}\sum_{n=1}^{\infty}\left[\omega_n \eta_n(t)\right]^2$$

$$\Rightarrow E = T + U = \frac{\rho L}{4}\sum_{n=1}^{\infty}\left[\dot{\eta}_n^2(t) + \omega_n^2\eta_n^2(t)\right]$$

$$\eta_r(t) = \mu_r \cos\left(\frac{r\pi ct}{L}\right) + \nu_r \sin\left(\frac{r\pi ct}{L}\right)$$

$$\dot{\eta}_n(t) = \left(\frac{n\pi c}{L}\right)\left[-\mu_n S + \nu_n C\right] = \omega_n\left[-\mu_n S + \nu_n C\right]$$

$$(\dot{\eta}_n)^2 = \omega_n^2\left[-\mu_n S + \nu_n C\right]^2$$

$$\omega_n^2 \eta_n^2 = \omega_n^2\left[\mu_n C + \nu_n S\right]^2$$

$$(\dot{\eta}_n)^2 + \omega_n^2\eta_n^2 = \omega_n^2\left[\mu_n^2 + \nu_n^2\right]$$

$$E = \frac{\rho L}{4}\sum_{n=1}^{\infty}\omega_n^2\left(\mu_n^2 + \nu_n^2\right)$$

QUE NÃO DEPENDE DO TEMPO.

Energia da onda

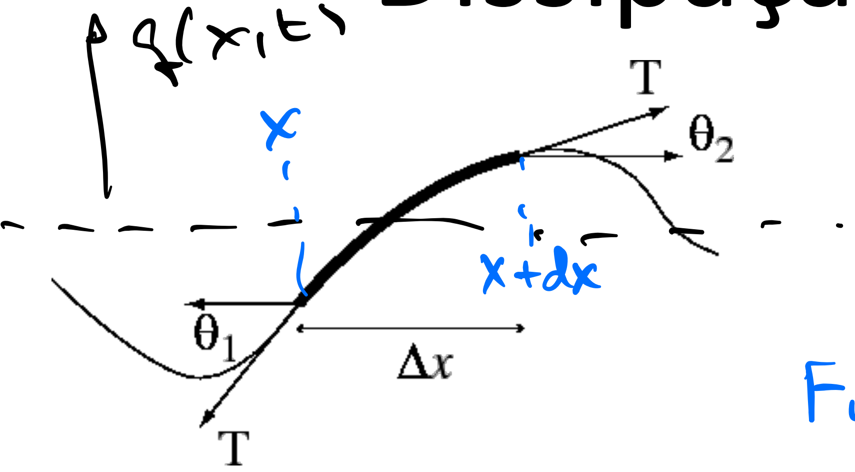
$$T = \int_0^L \frac{\rho}{2}\left(\frac{\partial q}{\partial t}\right)^2 dx$$

$$U = \int_0^L \frac{\tau}{2}\left(\frac{\partial q}{\partial x}\right)^2 dx$$

$$q\left(x,t\right) = \sum_{r=1}^{\infty} \sin\left(\frac{r\pi x}{L}\right)\eta_r\left(t\right) \Rightarrow \quad T = \frac{\rho L}{4}\sum_{r=1}^{\infty} \dot{\eta}_r^2\left(t\right), \quad U = \frac{\rho L}{4}\sum_{r=1}^{\infty} \omega_r^2 \eta_r^2\left(t\right)$$

$$\eta_r\left(t\right) = \mu_r \cos\omega_r t + \nu_r \sin\omega_r t \Rightarrow \; E = T + U = \frac{\rho L}{4}\sum_{r=1}^{\infty} \omega_r^2\left(\mu_r^2 + \nu_r^2\right)$$

Dissipação e força externa



O ELEMENTO DE CORDA Δx SOFRE DUAS FORÇAS DE MÓDULO τ

$$F_y(x+dx) = \tau \sin\theta_2$$

$$F_y(x) = -\tau \sin\theta_1$$

$$\tan\theta_2 = \frac{\partial q}{\partial x}(x+dx) \cong \sin\theta_2$$

$$\tan\theta_1 = \frac{\partial q}{\partial x}(x) \cong \sin\theta_1$$

$$dF_y = F_y(x+dx) + F_y(x)$$

$$= \tau\left[\frac{\partial q}{\partial x}(x+dx) - \frac{\partial q}{\partial x}(x)\right]$$

$$= \tau\, dx \frac{\partial^2 q}{\partial x^2}$$

$$dm\, a_y = (\rho\, dx)\frac{\partial^2 q}{\partial t^2}(x)$$

DISSIPAÇÃO: FORÇA VISCOSA $\propto v$

$$dF_y^D \propto -\frac{\partial q}{\partial t}\,dx \Rightarrow dF_y^D = -D\frac{\partial q}{\partial t}\,dx$$

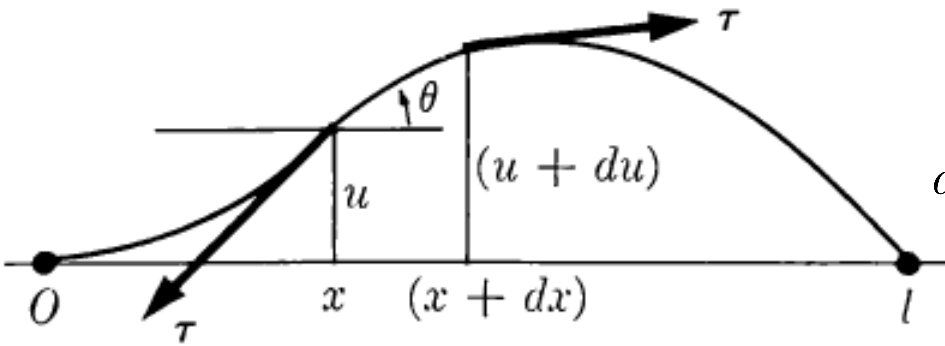
FORÇA EXTERNA: $dF_y^{ext} = F(x,t)\,dx$

$$dF_y^T = \left[\tau\frac{\partial^2 q}{\partial x^2} - D\frac{\partial q}{\partial t} + F(x,t)\right]\cancel{dx} = \rho\,\frac{\partial^2 q}{\partial t^2}\,\cancel{dx}$$

$$\boxed{\rho\frac{\partial^2 q}{\partial t^2} + D\frac{\partial q}{\partial t} - \tau\frac{\partial^2 q}{\partial x^2} = F(x,t)} \quad (5)$$

Dissipação e força externa

Dissipação e força externa:



Força devido à tração

$$F_y^{E\rightarrow R}(x) = -\tau\frac{\partial q}{\partial x} = -F_y^{R\rightarrow E}(x)$$

$$dF = F_y^{E\rightarrow R}(x) + F_y^{R\rightarrow E}(x+dx) = \tau\frac{\partial^2 q}{\partial x^2}dx$$

Força de resistência do ar

$$dF_y^R = -D\frac{\partial q}{\partial t}dx$$

Força externa

$$dF_y^{ext} = F(x,t)\,dx$$

massa X aceleração

$$dma_y = \rho dx\frac{\partial^2 q}{\partial t^2}$$

$$\rho dx\frac{\partial^2 q(x,t)}{\partial t^2} = \tau\frac{\partial^2 q(x,t)}{\partial x^2}dx - D\frac{\partial q(x,t)}{\partial t}dx + F(x,t)\,dx$$

$$\boxed{\rho\frac{\partial^2 q(x,t)}{\partial t^2} - \tau\frac{\partial^2 q(x,t)}{\partial x^2} + D\frac{\partial q(x,t)}{\partial t} = F(x,t)}$$

Solução geral
(corda presa nas extremidades)

SOLUÇÃO POR MODOS NORMAIS:

CHUTE: $q(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \eta_n(t)$

PRESA NAS EXTREMIDADES

LEVANDO EM (5):

$$\sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \left[\rho \ddot{\eta}_n + D\dot{\eta}_n + \tau \left(\frac{n\pi}{L}\right)^2 \eta_n(t) \right] = F(x,t)$$

MULTIPLICO OS DOIS LADOS POR $\sin\left(\frac{s\pi x}{L}\right)$ E

INTEGRO DE 0 A L:

$$\int_0^L dx \sum_{n=1}^{\infty} \sin\left(\frac{s\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) \left[\quad \right] = \underbrace{\int_0^L dx \sin\left(\frac{s\pi x}{L}\right) F(x,t)}_{f_s(t)}$$

USANDO A ORTOGONALIDADE ENTRE OS SENOS:

$$\frac{L}{2}\left[\rho\,\ddot{\eta}_s + D\dot{\eta}_s + \tau\underbrace{\left(\frac{s\pi}{L}\right)^2}_{\frac{\omega_s^2}{c^2}}\eta_s(t)\right] = f_s(t)$$

$$\frac{\omega_s^2}{c^2} \Rightarrow \frac{\tau}{c^2}\omega_s^2 = \rho\,\omega_s^2$$

$$\boxed{\rho\,\ddot{\eta}_s + D\,\dot{\eta}_s + \rho\,\omega_s^2\eta_s = \frac{2}{L}f_s(t)}$$

EQUAÇÃO DO OSC. HARM. AMORTECIDO FORÇADO

DE MANEIRA GERAL, PRECISAMOS DA SOL. GERAL DA HOMOGÊNEA ($\sim e^{\lambda t}$) + UMA SOL. PARTICULAR DA NÃO-HOMOGÊNEA (EXPANSÃO DE FOURIER OU FUNÇÃO DE GREEN).

DEVIDO À DISSIPAÇÃO, A SOL. GERAL DA HOMOGÊNEA DECAI A ZERO COM O PASSAR DO TEMPO (TRANSIENTE)

Exemplo: Corda amortecida sob a ação de uma força externa dada por

$$F(x,t) = F_0 \delta\left(x - \frac{L}{2}\right)\cos\omega t$$

$$f_s(t) = \int_0^L \sin\left(\frac{s\pi x}{L}\right) F_0 \delta\left(x - \frac{L}{2}\right)\cos\omega t \, dx$$

$$= F_0 \sin\left(\frac{s\pi}{2}\right)\cos\omega t$$

$$\rho \ddot{\eta}_s + D\dot{\eta}_s + \rho\omega_s^2 \eta_s = \underbrace{\frac{2F_0}{L}\sin\left(\frac{s\pi}{2}\right)}_{A_s}\cos\omega t = A_s \cos\omega t$$

$$\mathrm{Re}\left[\rho \ddot{\eta}_s + D\dot{\eta}_s + \rho\omega_s^2 \eta_s\right] = \mathrm{Re}\left[A_s e^{i\omega t}\right]$$

$$\rho \ddot{\eta}_s + D\dot{\eta}_s + \rho\omega_s^2 \eta_s = A_s e^{i\omega t}$$

CHUTE: $\eta_s(t) = B_s e^{i\omega t}$