F 415 – Mecânica Geral II

1º semestre de 2024

12/03/2024

Aula 3

Aula passada

$$\boldsymbol{F} = F\left(r\right)\hat{\boldsymbol{r}} = -\frac{dU}{dr}\hat{\boldsymbol{r}}$$

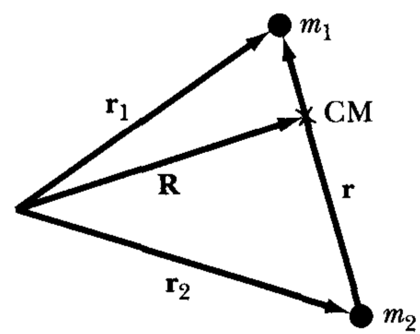
$$L = \frac{m_1}{2}\left|\boldsymbol{v_1}\right|^2 + \frac{m_2}{2}\left|\boldsymbol{v_2}\right|^2 - U\left(r\right)$$

$$\boldsymbol{R} = \frac{m_1\boldsymbol{r_1} + m_2\boldsymbol{r_2}}{m_1 + m_2}$$

$$\boldsymbol{r} = \boldsymbol{r_1} - \boldsymbol{r_2}$$

$$L = \frac{M}{2}\left|\dot{\boldsymbol{R}}\right|^2 + \frac{\mu}{2}\left|\dot{\boldsymbol{r}}\right|^2 - U\left(r\right)$$

$$M = m_1 + m_2\,;\ \mu = \frac{m_1 m_2}{m_1 + m_2}$$



$$M\dot{\boldsymbol{R}} = \text{const.} \Rightarrow m_1\dot{\boldsymbol{r}}_{\boldsymbol{1}} + m_2\dot{\boldsymbol{r}}_{\boldsymbol{2}} = \cancel{0}\ \text{CONST.}$$

- Centro de massa se move com velocidade constante.
- Conservação do momento linear total: invariância da Lagrangiana sob translações rígidas.

Aula passada

$$\boldsymbol{F} = F(r)\,\hat{\boldsymbol{r}} = -\frac{dU}{dr}\hat{\boldsymbol{r}}$$

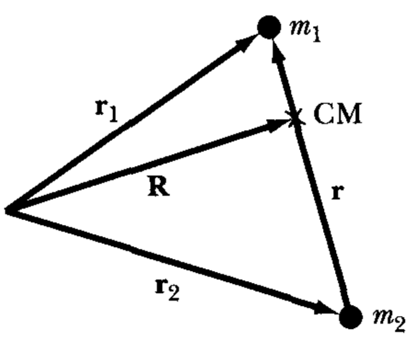
$$L = \frac{m_1}{2}\left|\boldsymbol{v_1}\right|^2 + \frac{m_2}{2}\left|\boldsymbol{v_2}\right|^2 - U(r)$$

$$\boldsymbol{R} = \frac{m_1\boldsymbol{r_1} + m_2\boldsymbol{r_2}}{m_1 + m_2}$$

$$\boldsymbol{r} = \boldsymbol{r_1} - \boldsymbol{r_2}$$

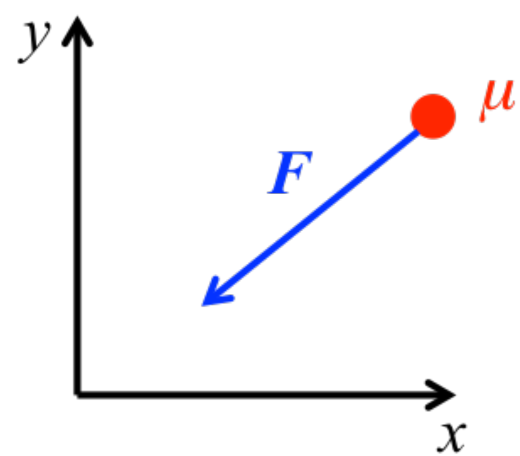
$$L = \frac{M}{2}\left|\dot{\boldsymbol{R}}\right|^2 + \frac{\mu}{2}\left|\dot{\boldsymbol{r}}\right|^2 - U(r)$$

$$M = m_1 + m_2\,;\ \mu = \frac{m_1 m_2}{m_1 + m_2}$$



A dinâmica de $\boldsymbol{r}$ é equivalente ao problema de uma partícula de massa μ sujeito a uma força que aponta pra origem.

$$L_{CM} = \frac{\mu}{2}\left|\dot{\boldsymbol{r}}\right|^2 - U(r) \longrightarrow \boxed{\mu\ddot{\boldsymbol{r}} = \boldsymbol{F} = -\frac{dU}{dr}\hat{\boldsymbol{r}}}$$



Aula passada

Conservação do momento angular total:

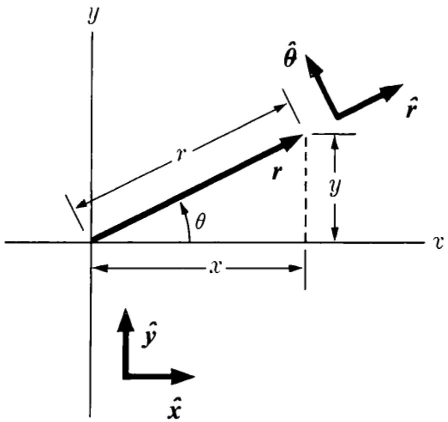
$$\boldsymbol{L} = m_1 \boldsymbol{r}_1 \times \dot{\boldsymbol{r}}_1 + m_2 \boldsymbol{r}_2 \times \dot{\boldsymbol{r}}_2 = M\boldsymbol{R} \times \dot{\boldsymbol{R}} + \mu \boldsymbol{r} \times \dot{\boldsymbol{r}} = \text{const.}$$

Movimento global

Movimento interno

Conservação do momento angular interno: $\boldsymbol{\ell} = \mu \boldsymbol{r} \times \dot{\boldsymbol{r}} = \text{const.}$

O movimento relativo é sempre num plano perpendicular ao vetor fixo $\boldsymbol{\ell}$: 3D → 2D.



Coordenadas polares no plano:

$$L_{CM} = \frac{\mu}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - U\left(r\right)$$

Aula passada

$$L_{CM} = \frac{\mu}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - U\left(r\right)$$

Equações de Euler-Lagrange:

Para θ, que é uma coordenada ignorável:

$$\frac{\partial L}{\partial \dot{\theta}} = \boxed{\mu r^2\dot{\theta} = \text{const.} = \ell}$$

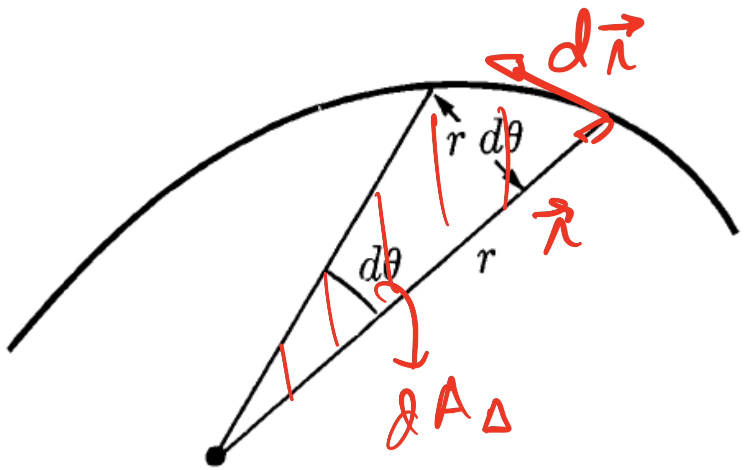
Conservação do momento angular interno

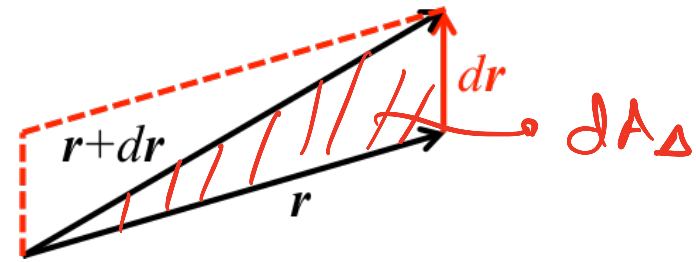
$$\Rightarrow r^2\dot{\theta} = \frac{\ell}{\mu} = \text{CONST.}$$

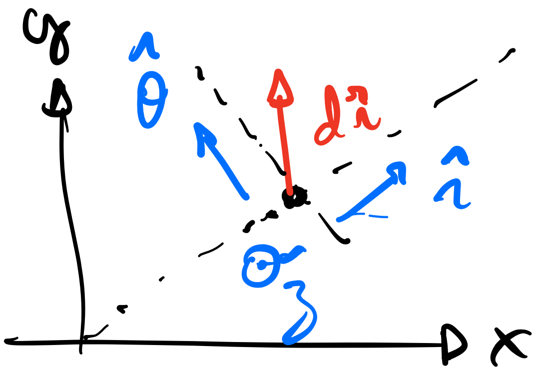
Para *r*:

$$\mu\ddot{r} - \mu r\dot{\theta}^2 + \frac{dU}{dr} = 0$$

Conservação do momento angular: 2a. lei de Kepler (lei das áreas)







$$A_\Delta = \frac{A_\square}{2} = \frac{1}{2}\left[\,|\vec{r} \times d\vec{r}|\,\right]$$

$$= \frac{1}{2}\left|r\hat{r} \times (dr\,\hat{r} + r\,d\theta\,\hat{\theta})\right|$$

$$= \frac{1}{2}\left|r^2 d\theta\,\underbrace{(\hat{r}\times\hat{\theta})}_{\hat{z}}\right| = \frac{1}{2} r^2 d\theta$$

$$d\vec{r} = dr_r\,\hat{r} + dr_\theta\,\hat{\theta} = dr\,\hat{r} + r\,d\theta\,\hat{\theta}$$

$$dr_r = dr \;;\; dr_\theta = r\,d\theta$$

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{1}{2} r^2 \dot{\theta} = \frac{\ell}{2\mu} = \text{const.}$$

✓

As leis de Kepler do movimento dos planetas

1. A órbita de cada planeta é uma elipse com o Sol num dos focos.

2. O segmento de reta que vai do planeta ao Sol varre áreas iguais em períodos de tempo iguais. ✓

3. Os quadrados dos períodos de revolução dos planetas são proporcionais aos cubos dos semi-eixos maiores das suas elipses.

Mas note que a 2a. lei é válida pra qualquer força central $U(r)$

Energia mecânica total

$L_{CM} = \frac{\mu}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - U(r)$ NÃO DEPENDE EXPLICITAMENTE DO TEMPO.

$\Rightarrow H = \sum_i p_i \dot{q}_i - L = p_\theta \dot{\theta} + p_r \dot{r} - L_{CM}$

$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \mu r^2 \dot{\theta}$; $p_r = \frac{\partial L}{\partial \dot{r}} = \mu \dot{r}$

$\Rightarrow H = \mu r^2 \dot{\theta}^2 + \mu \dot{r}^2 - \left[\frac{\mu}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - U(r)\right]$

$= \boxed{\underbrace{\frac{\mu}{2}(\dot{r}^2 + r^2\dot{\theta}^2)}_{T} + \underbrace{U(r)}_{U}} = T + U =$ EN. MEC. TOTAL

$= E$

$$\boxed{E = T + U = \text{CONST.}}$$

1. O centro de massa do sistema de dois corpos move-se com velocidade constante. ✓

2. A dinâmica da coordenada relativa é a mesma de um corpo de massa μ, sujeito a uma força que aponta para a origem. ✓

3. O momento angular total é conservado:
   a. O momento angular "do centro de massa" é conservado. ✓
   b. O momento angular "interno" é conservado. ✓

4. A energia mecânica total é conservada. ✓

2 graus de liberdade, 2 leis de conservação: problema solúvel!

$$\boxed{\ell = \mu r^2 \dot{\theta} = \text{const.}} \quad (1) \qquad \boxed{E = \frac{\mu}{2}\dot{r}^2 + \frac{\mu}{2} r^2 \dot{\theta}^2 + U(r) = \text{const.}} \quad (2)$$

COM ISSO, POSSO REDUZIR O PROBLEMA A QUADRATURAS.

DE (1): $\dot{\theta} = \frac{\ell}{\mu r^2}$ LEVO EM (2):

$$\Rightarrow E = \frac{\mu}{2}\dot{r}^2 + \frac{\mu}{2} r^2 \left(\frac{\ell}{\mu r^2}\right)^2 + U(r) = \frac{\mu}{2}\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + U(r) \quad (3)$$

$$\dot{r} = \frac{dr}{dt} = \pm\left[\frac{2}{\mu}[E - U(r)] - \frac{\ell^2}{\mu^2 r^2}\right]^{1/2} \Leftrightarrow \frac{dr}{dt} = f(r)$$

$$\int dt = \int \frac{dr}{f(r)} \Rightarrow t - t_0 = \int_{r_0}^{r} \frac{dr}{f(r)} = \int_{r_0}^{r} \frac{\pm\, dr}{\left[\frac{2}{\mu}(E - U(r)) - \frac{\ell^2}{\mu^2 r^2}\right]^{1/2}} \Rightarrow \boxed{r = r(t)}$$

DE (1): $\dot{\theta} = \frac{d\theta}{dt} = \frac{l}{\mu r^2} = \frac{l}{\mu r^2(t)}$

$$\Rightarrow \theta(t) = \int_{t_0}^{t} \frac{d\theta}{dt}\, dt = \int_{t_0}^{t} \frac{l\, dt}{\mu\, r^2(t}$$

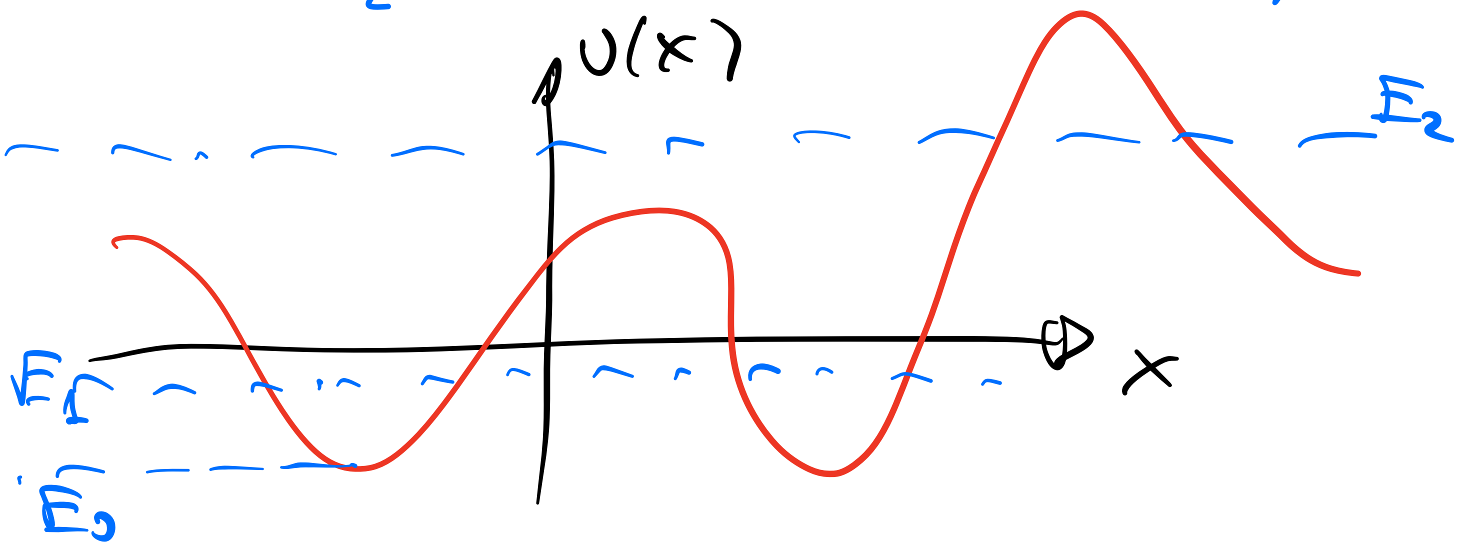
O potencial efetivo

$$E = \frac{\mu}{2}\dot{r}^2 + \frac{1}{2}\frac{\ell^2}{\mu r^2} + U(r) = \text{CONST.}$$

UM PROBLEMA 1D CONSERVATIVO:

$$E = \frac{m}{2}\dot{x}^2 + U(x) \Rightarrow \dot{x}^2 = \frac{2}{m}\left[E - U(x)\right] \geq 0$$

$$\Rightarrow U(x) \leq E$$

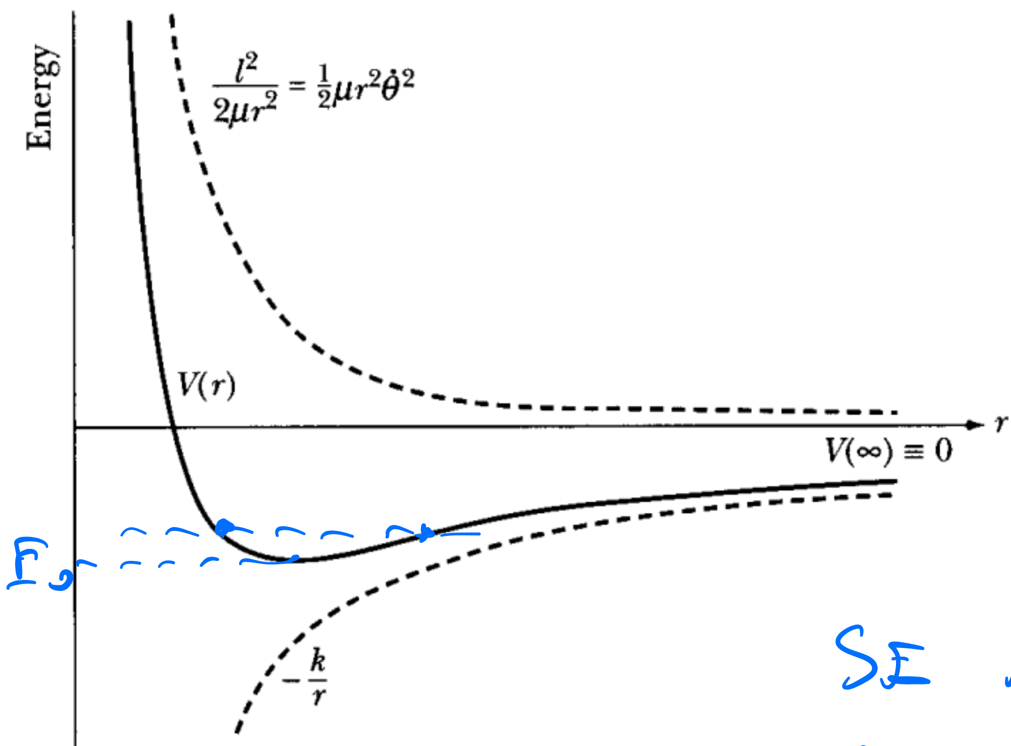


$$V_{eff}(r) = \frac{\ell^2}{2\mu r^2} + U(r)$$

Potencial efetivo

$$\dot{\theta} = \frac{\ell}{\mu r^2}$$

$$\boxed{V_{eff}(r) = \frac{\ell^2}{2\mu r^2} + U(r)}$$

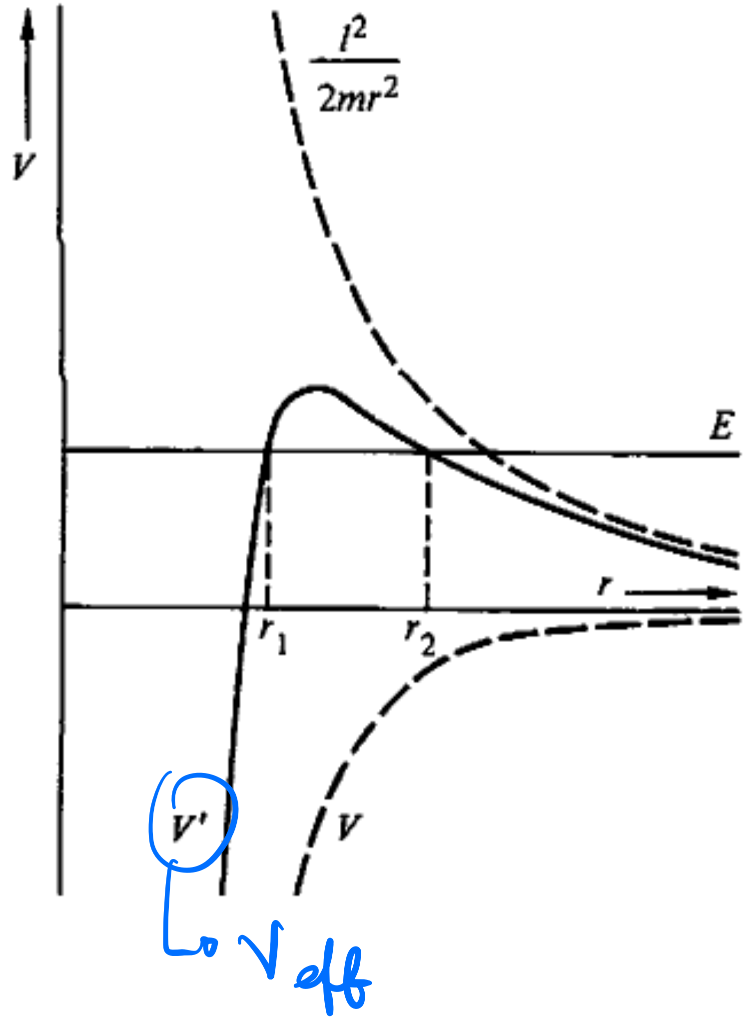


$$U(r) = -\frac{k}{r}$$

SE r = CONST.

$$\dot{\theta} = \frac{\ell}{\mu r^2} = \text{CONST} \Rightarrow \theta(t) = \frac{\ell}{\mu r^2} t + \theta_0$$

Potencial efetivo

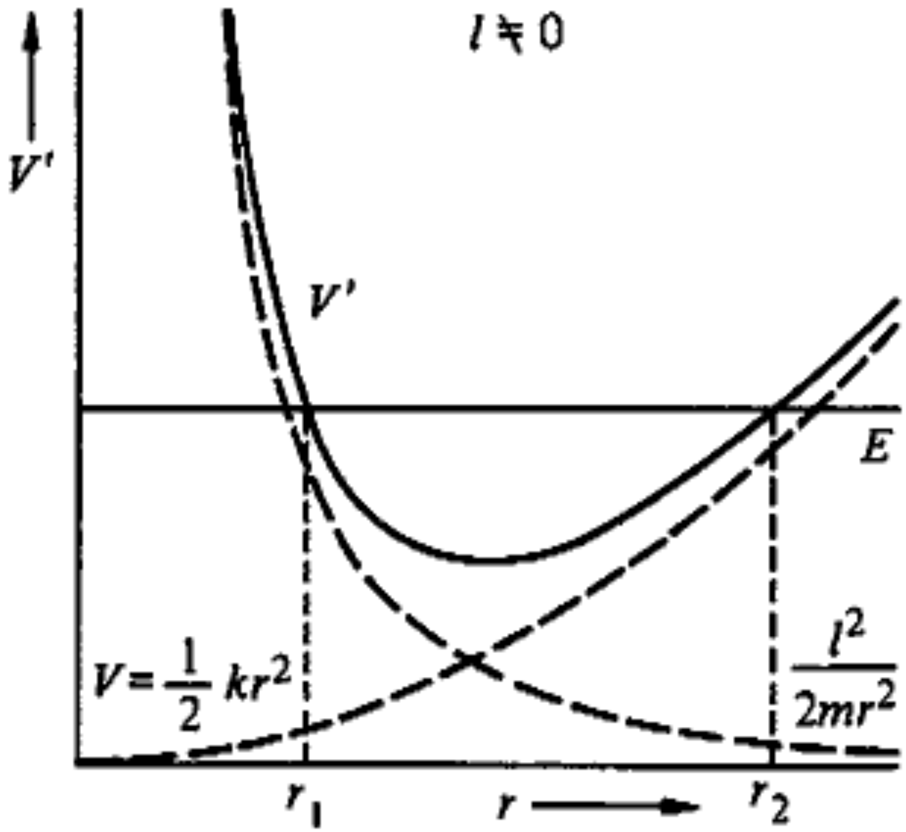


$$\boxed{V_{eff}(r) = \frac{\ell^2}{2\mu r^2} + U(r)}$$

$$U(r) = -\frac{k}{r^3}$$

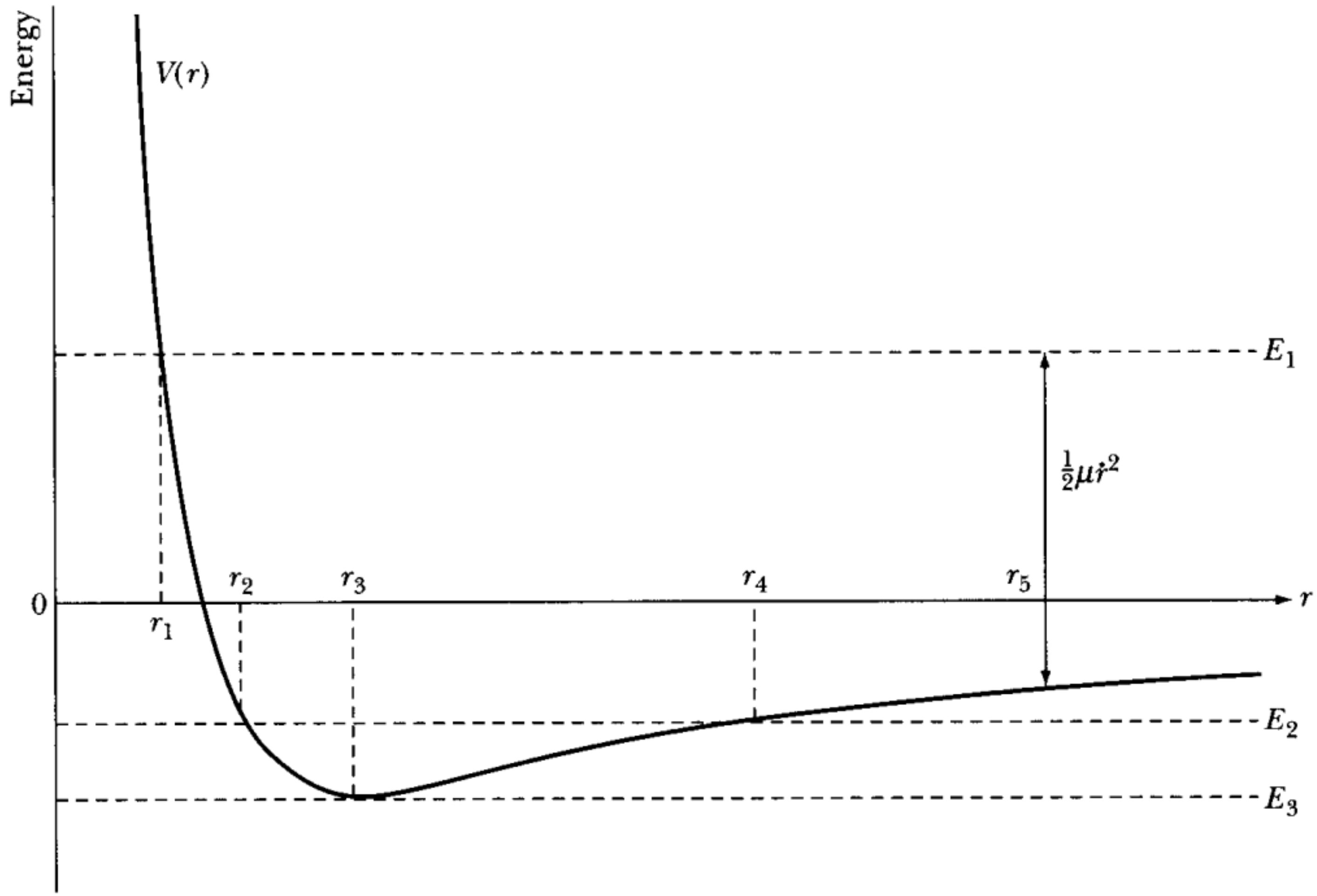
Potencial efetivo

$$\boxed{V(r) = \frac{\ell^2}{2\mu r^2} + U(r)}$$



$$U(r) = \frac{k}{2}r^2$$

$$\boxed{V(r) = \frac{\ell^2}{2\mu r^2} + U(r)} \qquad U(r) = -\frac{k}{r} \qquad \dot{\theta} = \frac{\ell}{\mu r^2}$$

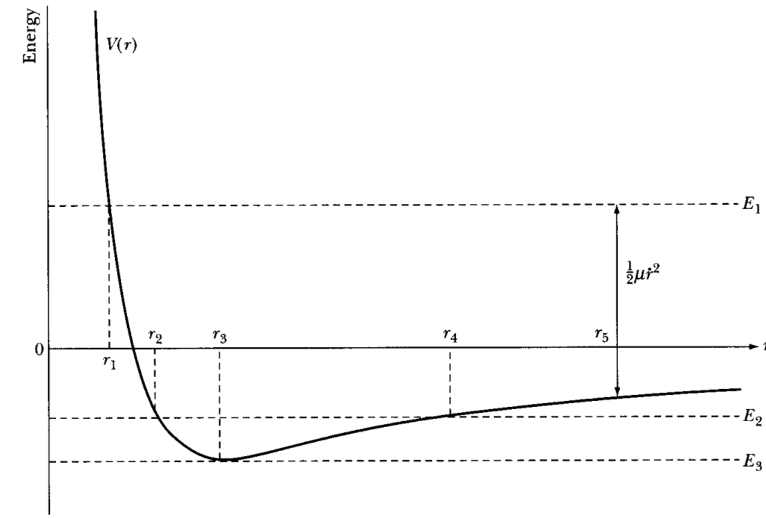


Trajetórias fechadas ou abertas

$$\dot{\theta} = \frac{\ell}{\mu r^2} \qquad \dot{r} = \pm\sqrt{\frac{2}{\mu}\left[E - V_{eff}(r)\right]}$$

$$+ \quad : \quad \dot{r} > 0 \Rightarrow r \text{ crescente}$$

$$- \quad : \quad \dot{r} < 0 \Rightarrow r \text{ decrescente}$$



$$\frac{d\theta}{dr} = \frac{d\theta}{dt}\frac{dt}{dr} = \frac{\dot{\theta}}{\dot{r}} = \frac{\ell}{\mu r^2 \dot{r}} = \frac{\ell/\mu r^2}{\pm\left[\frac{2}{\mu}(E - V_{eff}(r))\right]^{1/2}}$$

$$\Delta\theta = \int_{r_{min}}^{r_{max}} \frac{\ell/\mu r^2}{[\quad]^{1/2}}\,dr + \int_{r_{max}}^{r_{min}} \frac{\ell/\mu r^2}{-[\quad]^{1/2}}\,dr$$

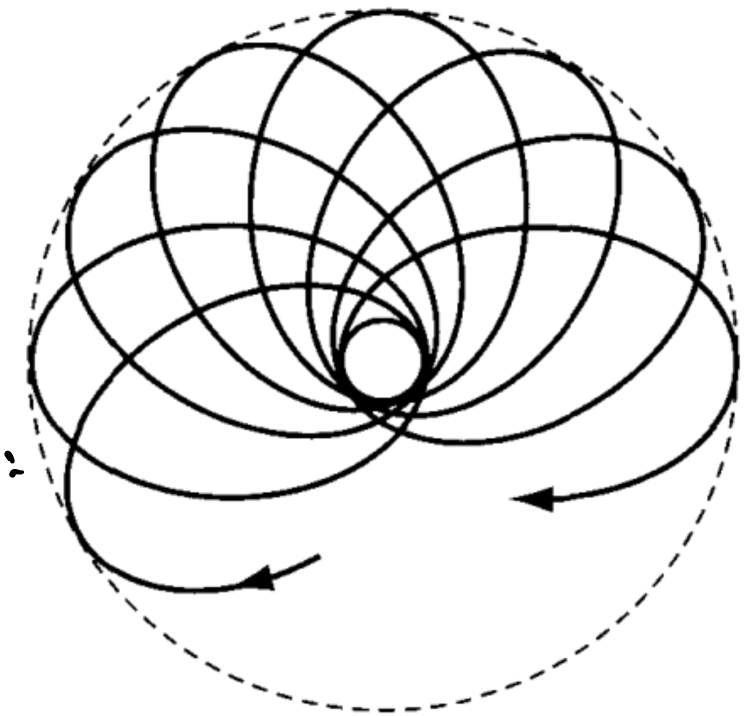
$$= 2\int_{r_{min}}^{r_{max}} \frac{\ell/\mu r^2}{[\quad]^{1/2}}\,dr$$

SE :

$$\Delta\theta = 2\pi \frac{p}{q}$$ p, q SÃO INTEIROS

DO CONTRÁRIO A TRAJETÓRIA NUNCA SE FECHA

Uma trajetória aberta



TEOREMA DE BERTRAND:

SE $U(r) = k\,r^{n}$

TRAJETÓRIAS SEMPRE FECHADAS SE:

$n = -1$ OU $n = 2$

Equação da trajetória

$$r(t)\,,\theta(t) \Rightarrow \boxed{r(\theta)}$$

Análogo a $x(t)\,,y(t) \Rightarrow y(x)$

EQ. RADIAL:

a r:

$$\mu\ddot{r} - \mu r\dot{\theta}^2 + \frac{dU}{dr} = 0$$

$$\dot{\theta} = \frac{\ell}{\mu r^2}$$

$$\Rightarrow \mu\ddot{r} - \frac{\ell^2}{\mu r^3} + \frac{dU}{dr} = 0 \Rightarrow \mu\ddot{r} = -\frac{dU}{dr} + \frac{\ell^2}{\mu r^3} = -\frac{dV_{ef}(r)}{dr} \quad (4)$$

DEFINO: $u = \frac{1}{r} \Rightarrow u(\theta) = \frac{1}{r(\theta)}$

$$\frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta} = -\frac{1}{r^2}\frac{dr}{dt}\frac{dt}{d\theta} = -\frac{1}{r^2}\frac{\dot{r}}{\dot{\theta}} = -\frac{\mu}{\ell}\dot{r}$$

$$\frac{d^2u}{d\theta^2} = -\frac{\mu}{\ell}\frac{d\dot{r}}{d\theta} = -\frac{\mu}{\ell}\frac{d\dot{r}}{dt}\frac{dt}{d\theta} = -\frac{\mu}{\ell}\frac{\ddot{r}}{\dot{\theta}} = -\frac{\mu^2}{\ell^2}r^2\ddot{r} \quad (5)$$

LEVANDO (5) NA (4):

F(r)

$$\Rightarrow \frac{d^2u}{d\theta^2} + u = -\frac{\mu}{\ell^2 u^2} F\left(\frac{1}{u}\right)$$

EM PARTICULAR: $F(r) = -\dfrac{k}{r^2}$

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu k}{\ell^2}$$