F 415 – Mecânica Geral II

1º semestre de 2024

19/03/2024

Aula 5

Aulas passadas

Conservação do momento angular interno: $\ell = \mu r^2 \dot{\theta} = \text{const.}$

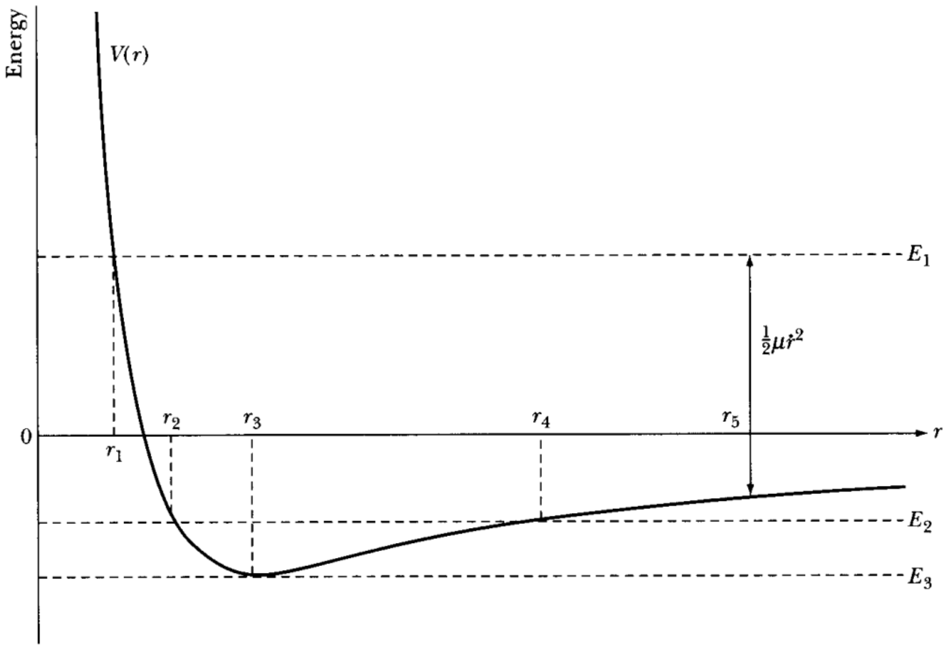
Conservação da energia mecânica total:

$$E = \frac{\mu}{2}\dot{r}^2 + \frac{\mu}{2}r^2\dot{\theta}^2 + U\left(r\right) = \text{const.}$$

$$E = \frac{\mu}{2}\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + U\left(r\right) = \text{const.}$$

Potencial efetivo:

$$V\left(r\right) = \frac{\ell^2}{2\mu r^2} + U\left(r\right) \Rightarrow \text{pontos de retorno}$$



Equação da trajetória: $u\left(\theta\right) \equiv \frac{1}{r\left(\theta\right)} \Rightarrow \frac{d^2u}{d\theta^2} + u = -\frac{\mu}{\ell^2 u^2}F\left(\frac{1}{u}\right)$

Aula passada

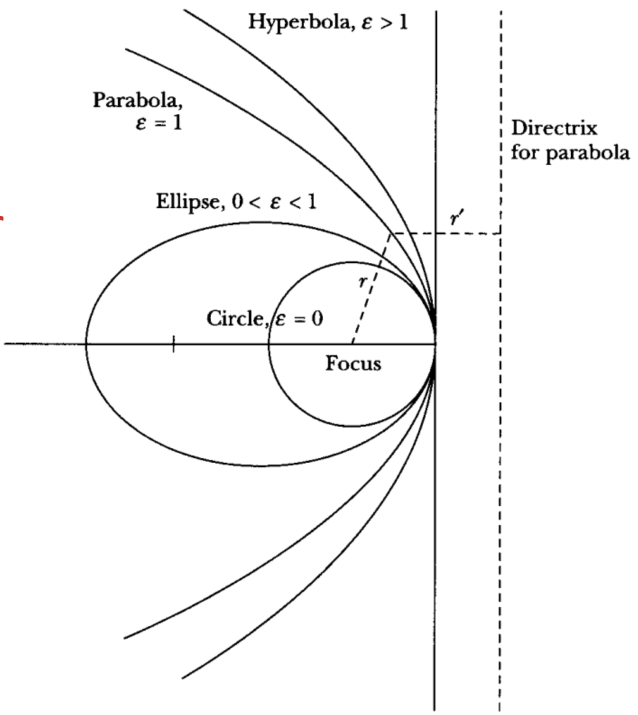
Problema de Kepler:

$$k = GMm$$

$$\begin{aligned} U(r) &= -\frac{k}{r} \Rightarrow r(\theta) = \frac{\alpha}{1+\varepsilon\cos\theta} \\ \alpha &= \frac{\ell^2}{\mu k} \\ \varepsilon &= \sqrt{1+\frac{2\ell^2 E}{\mu k^2}} = \sqrt{1-\frac{E}{V_{min}}} \end{aligned}$$

$$\begin{aligned} E = V_{min} &\Rightarrow \varepsilon = 0 \Rightarrow && \text{círculo} \\ V_{min} < E < 0 &\Rightarrow 0 < \varepsilon < 1 \Rightarrow && \text{elipse} \\ E = 0 &\Rightarrow \varepsilon = 1 \Rightarrow && \text{parábola} \\ E > 0 &\Rightarrow \varepsilon > 1 \Rightarrow && \text{hipérbole} \end{aligned}$$

1ª lei de Kepler



Aula passada

$$
\begin{aligned}
r_{min} &= \frac{\alpha}{1+\varepsilon} \quad (\theta = 0) \\
r_{max} &= \frac{\alpha}{1-\varepsilon} \quad (\theta = \pi) \\
r_{max} + r_{min} &= 2a \\
r_{max} - r_{min} &= 2c = 2\varepsilon a \\
a &= \frac{\alpha}{1-\varepsilon^2} = \frac{k}{2|E|} \Rightarrow E = -\frac{k}{2a} \\
b &= \sqrt{1-\varepsilon^2}a = \frac{\alpha}{\sqrt{1-\varepsilon^2}} = \frac{\ell}{\sqrt{2\mu|E|}}
\end{aligned}
$$

Terceira lei de Kepler: $$\frac{T^2}{a^3} = 4\pi^2\frac{\mu}{k} = \frac{4\pi^2}{G(m_1+m_2)} \xrightarrow[m_2 \gg m_1]{} \frac{4\pi^2}{Gm_2}$$

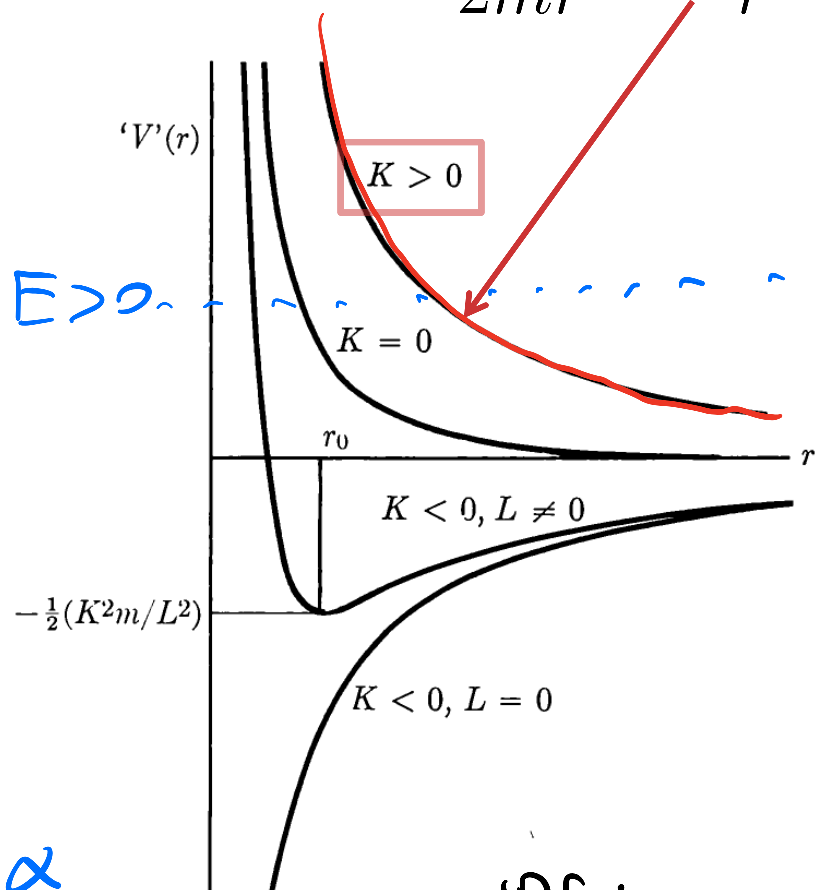
Potencial $1/r$ repulsivo

$$U(r) = \frac{k}{r} \quad (k > 0) \Rightarrow F(r) = -\frac{dU}{dr} = \frac{k}{r^2}$$

$$V(r) = \frac{L^2}{2mr^2} + \frac{K}{r}$$

$$V(r) = \frac{l^2}{2\mu r^2} + \frac{k}{r} > 0$$

$$\frac{d^2u}{d\theta^2} + u = -\frac{\mu k}{l^2}$$



$$\Rightarrow u(\theta) = A\cos(\theta - \theta_0) - \frac{\mu k}{l^2}$$

$\theta_0 = 0$ E $A > 0$

$$u(\theta) = A\cos(\theta) - \frac{\mu k}{l^2} = \frac{1}{r(\theta)}$$

$$r(\theta) = \frac{1}{A\cos\theta - \frac{\mu k}{l^2}} = \frac{\overbrace{l^2/\mu k}^{\alpha}}{\varepsilon\cos\theta - 1}$$

ONDE:

$$\varepsilon = \frac{A l^2}{\mu k}$$

Equação da trajetória

$$r(\theta) = \frac{\alpha}{\varepsilon \cos\theta - 1}$$

PARA QUE HAJA CURVA: $\boxed{\varepsilon > 1}$

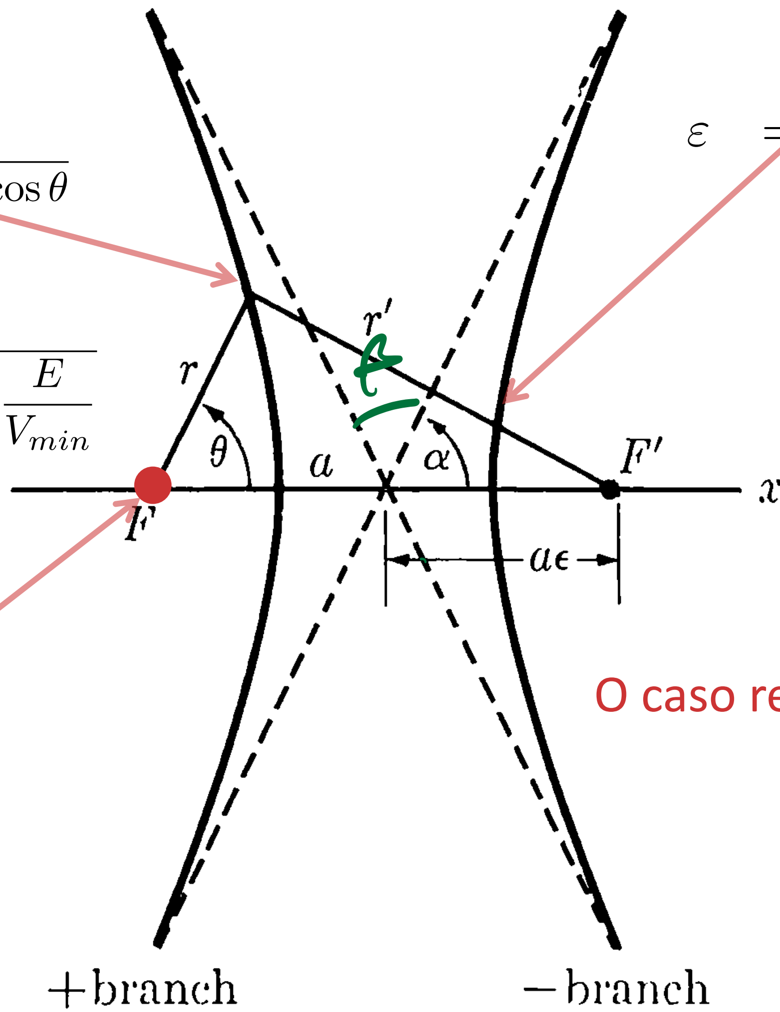
DO CONTRÁRIO: $\varepsilon \cos\theta - 1 < 0 \Rightarrow r < 0$

Potencial $1/r$ repulsivo

$$
\begin{aligned}
U(r) &= \frac{k}{r} \Rightarrow r(\theta) = \frac{\alpha}{-1+\varepsilon\cos\theta} \\
\alpha &= \frac{\ell^2}{\mu k} \\
\varepsilon &= \sqrt{1+\frac{2\ell^2 E}{\mu k^2}} \quad (E>0)
\end{aligned}
$$

O caso atrativo é o ramo (+) da hipérbole

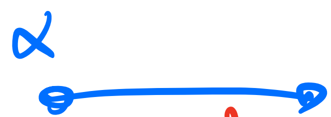
$$
\begin{aligned}
U(r) &= -\frac{k}{r} \Rightarrow r(\theta) = \frac{\alpha}{1+\varepsilon\cos\theta} \\
\alpha &= \frac{\ell^2}{\mu k} \\
\varepsilon &= \sqrt{1+\frac{2\ell^2 E}{\mu k^2}} = \sqrt{1-\frac{E}{V_{min}}}
\end{aligned}
$$



centro de força

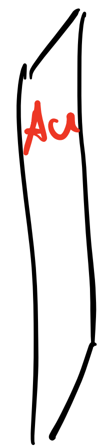
O caso repulsivo é o ramo (-) da hipérbole

Espalhamento de Rutherford



$$F(r) = \overbrace{\frac{q_1 q_2}{4\pi\varepsilon_0}}^{k} \frac{1}{r^2}.$$

$$U(r) = \frac{k}{r}.$$



α = NÚCLEO DO He

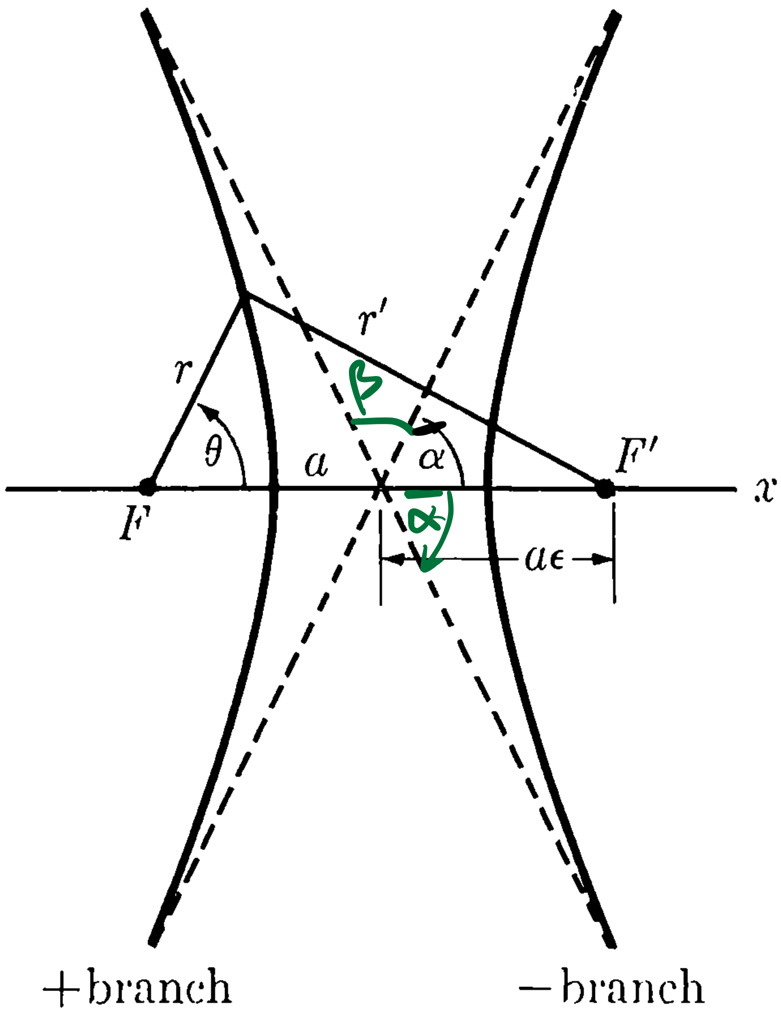
= 2 p's + 2 n's

Au ⇒ Z = 79

M_{Au} = 197 u.m.a.

A ÚNICA QUANTIDADE RELEVANTE PARA NÓS É O DESVIO DA TRAJETÓRIA INICIAL.

Os ângulos assintóticos



$$r(\theta) = \frac{\alpha}{\varepsilon \cos\theta - 1}$$

$\bar{\alpha} = ?$

$\bar{\alpha}$ CORRESPONDE AO ÂNGULO ONDE $r(\theta \to \bar{\alpha}) \to \infty$

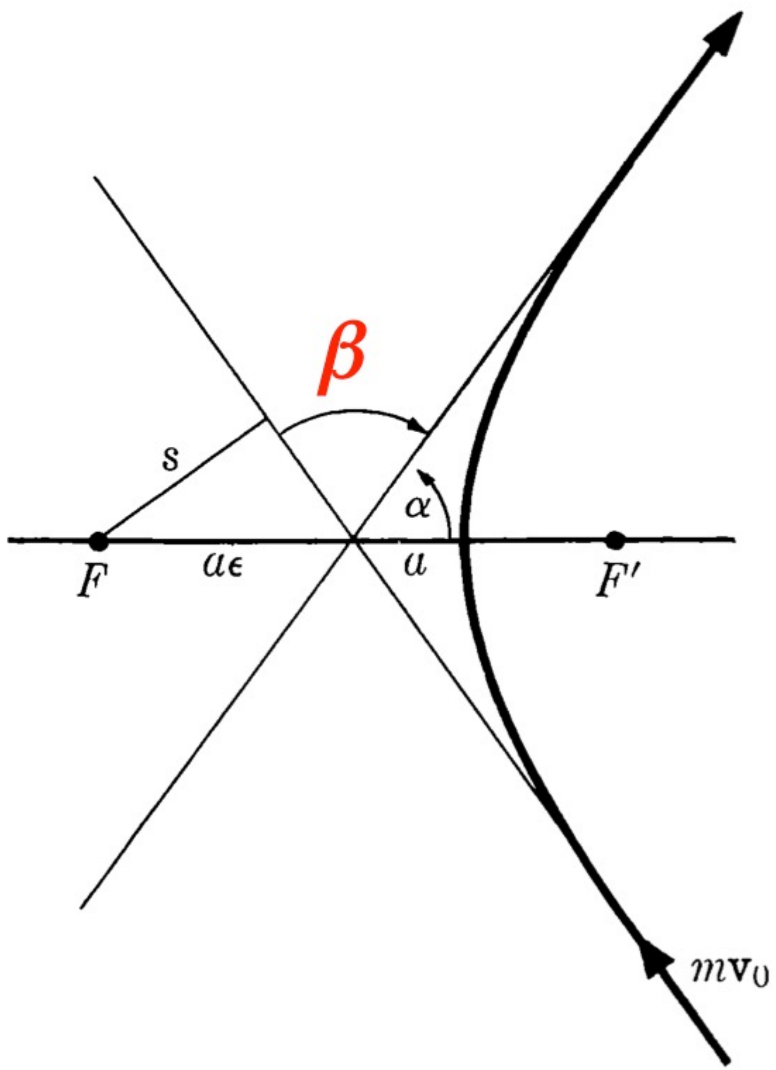
$\Rightarrow \varepsilon \cos\bar{\alpha} - 1 = 0$

$\Rightarrow \boxed{\cos\bar{\alpha} = \frac{1}{\varepsilon}}$

$\Rightarrow \beta = \pi - 2\bar{\alpha}$

$$\tan\left(\frac{\beta}{2}\right) = \tan\left(\frac{\pi}{2} - \bar{\alpha}\right) = \operatorname{cotan}\bar{\alpha}$$

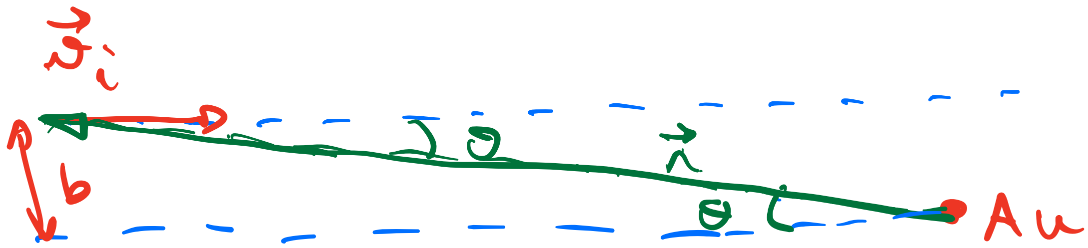
A deflexão total β



$$\tan\frac{\beta}{2} = \cot\bar{\alpha}$$

$$\Rightarrow \tan\frac{\beta}{2} = \frac{1}{\sqrt{\varepsilon^2 - 1}}$$

$$\boxed{\tan\frac{\beta}{2} = \sqrt{\frac{m k^2}{2 \ell^2 E}}}$$



PARÁMETRO DE IMPACTO

$r \sin\theta = b$

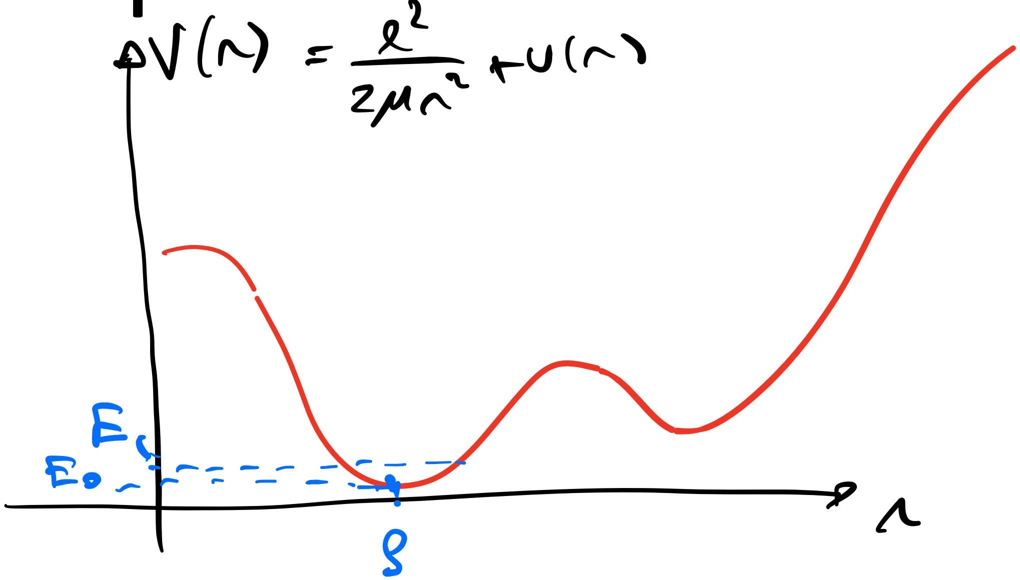
$$E = T + U(r) = \frac{\mu}{2} v^2 + \frac{k}{r} \xrightarrow[r \to \infty]{} E = \frac{\mu}{2} v^2 = \frac{\mu}{2} v_i^2$$

$\ell = ?$

$$\ell = |\mu \vec{r} \times \vec{v}| = \mu r v \sin\theta = \mu v b$$

$$\Rightarrow \boxed{\tan\left(\frac{\beta}{2}\right) = \frac{k}{\mu b v_i^2}}$$

Pequenos desvios de órbitas circulares



SE $E = E_0$: $r = \rho = \text{const}$

$\dot{\theta} = \frac{\ell}{\mu r^2} = \frac{\ell}{\mu \rho^2} = \text{CONST.}$

$\Rightarrow$ MCU

SE $E = E_1 \Rightarrow r \in [\rho - \epsilon, \rho + \epsilon]$, PODEMOS SUBSTITUIR O POTENCIAL EXATO POR UMA PARÁBOLA:

$$\Rightarrow V(r) \cong V(\rho) + \cancel{\left.\frac{dV}{dr}\right|_{r=\rho}}^{\,0} (r-\rho) + \frac{1}{2} \left.\frac{d^2V}{dr^2}\right|_{r=\rho} (r-\rho)^2$$

$$V(r) \cong \text{CONST.} + \frac{1}{2} k (r-\rho)^2$$

$$k = \left.\frac{d^2V}{dr^2}\right|_{r=\rho} > 0$$

MÍNIMO

$$-\frac{dV}{dr} = -k(r-\rho)$$

$$\mu \ddot{r} = -\frac{dV}{dr} = -k(r-\rho)$$

DEFINO: $x = r - \rho \Rightarrow \ddot{x} = \ddot{r} \Rightarrow \boxed{\mu \ddot{x} + kx = 0}$

$$x(t) = A\cos(\omega t + \delta) \qquad \omega = \sqrt{\frac{k}{\mu}} = \sqrt{\frac{V''(\rho)}{\mu}}$$

$$r(t) = \rho + A\cos(\omega t + \delta)$$

TRAJETÓRIAS SERÃO FECHADAS SE

$$\frac{\omega}{\dot{\theta}} = \frac{p}{q} \qquad p, q \in \mathbb{N}^+$$

$$\dot{\theta} = \frac{\ell}{\mu r^2} \simeq \frac{\ell}{\mu \rho^2} \qquad \frac{\sqrt{\frac{V''(\rho)}{\mu}}}{\ell / \mu \rho^2} \stackrel{?}{=} \frac{p}{q}$$

EXEMPLO (VER NOTAS)

$$V(r) = -\frac{k}{r^n} \qquad V(r) = \frac{\ell^2}{2\mu r^2} - \frac{k}{r^n}$$

$$\left.\frac{dV}{dr}\right|_{r=\rho} = 0 \qquad \rho^{(n-2)} = \frac{n k \mu}{\ell^2}$$

$$\left.\frac{d^2V}{dr^2}\right|_{r=\rho} > 0 \Rightarrow \boxed{n < 2}$$

$$\omega = \frac{\ell}{\mu \rho^2}\sqrt{2-n} = \sqrt{\frac{V''(\rho)}{\mu}}$$

$$\frac{\omega}{\dot\theta} = \sqrt{2-n} = \begin{cases} n=1: \quad 1 \Rightarrow \text{FECHADA (KEPLER)} \\ n=-2: (k<0) \quad 2 \Rightarrow \text{FECHADA (OH3D)} \\ \cancel{n=-7} \end{cases}$$

(INDO ALÉM DE $(r-\rho)^2$)

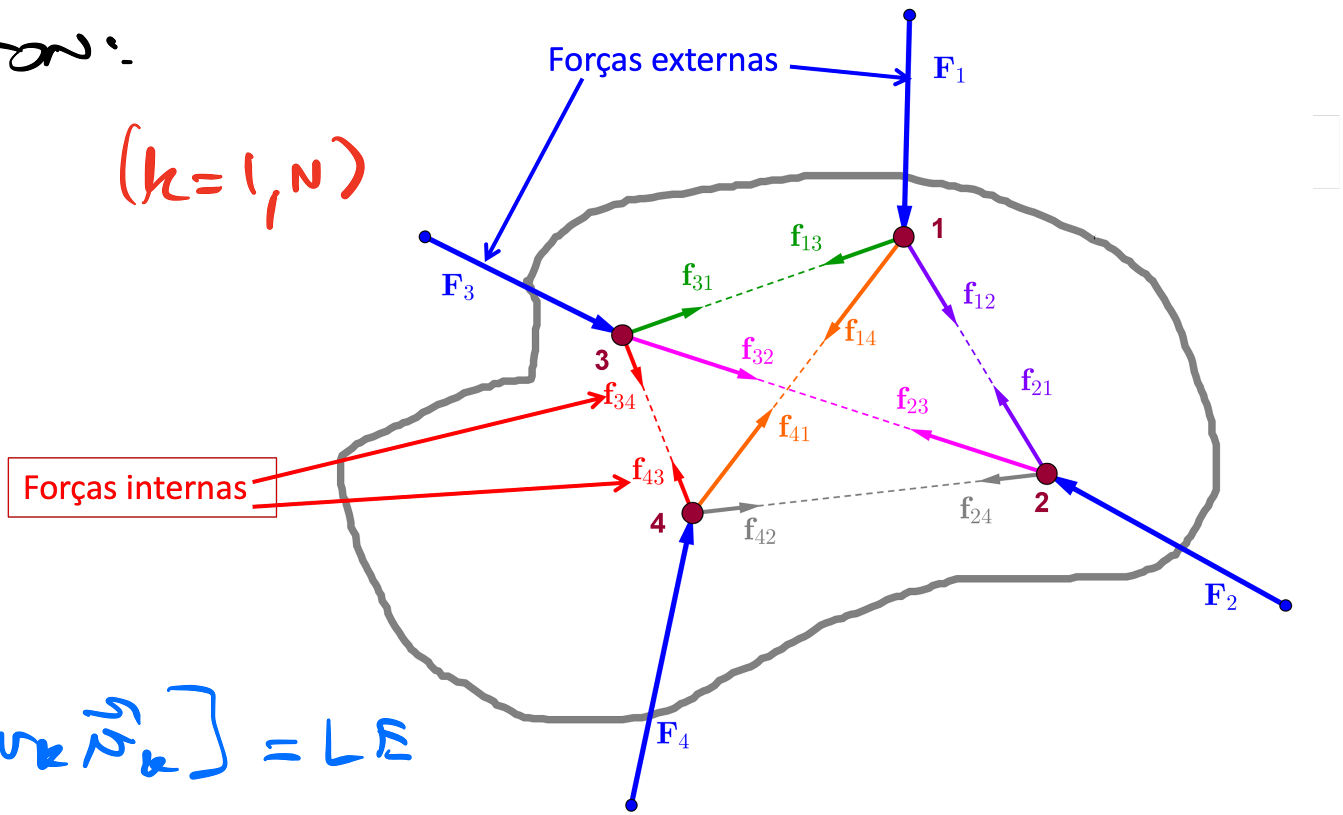
Cap. 9
Dinâmica de um sistema de muitas partículas

Leis de conservação: momento linear

2ª LEI DE NEWTON:

$$m_k \frac{d^2\vec{r}_k}{dt^2} = \vec{F}_k + \vec{f}_k \qquad (k=1,N)$$

$$\vec{f}_k = \sum_{\substack{j=1 \\ j\neq k}}^{N} \vec{f}_{kj}$$



$$\frac{d}{dt}\left[m_k \frac{d\vec{r}_k}{dt}\right] = \frac{d}{dt}\left[m_k \vec{v}_k\right] = LE$$

SOMO SOBRE $\underline{k}$:

$$LE = \sum_{k=1}^{N} \frac{d}{dt}\left[m_k \vec{v}_k\right] = \frac{d}{dt}\left[\sum_k \overbrace{m_k \vec{v}_k}^{\vec{p}_k}\right] = \frac{d}{dt}\left[\sum_k \vec{p}_k\right]$$

$$= \frac{d\vec{P}}{dt}$$

ONDE $\vec{P} = \sum_k \vec{p}_k$ = MOM. LINEAR TOTAL

$$\frac{d\vec{p}}{dt} = \underbrace{\sum_k \vec{F}_k}_{\vec{F}} + \underbrace{\cancel{\sum_k \vec{f}_k}}_{?} \to 0 \Rightarrow \boxed{\frac{d\vec{p}}{dt} = \vec{F}}$$

$\vec{F}$ = FORÇA EXTERNA TOTAL

$$\sum_k \vec{f}_k = \sum_k \sum_{\substack{j \\ j \neq k}} \vec{f}_{kj} = 0$$

$N = 4$:

$k=1$: $\cancel{\vec{f}_{12}} + \cancel{\vec{f}_{13}} + \cancel{\vec{f}_{14}}$

$k=2$: $\cancel{\vec{f}_{21}} + \cancel{\vec{f}_{23}} + \cancel{\vec{f}_{24}}$

$k=3$: $\cancel{\vec{f}_{31}} + \cancel{\vec{f}_{32}} + \cancel{\vec{f}_{34}}$

$k=4$: $\cancel{\vec{f}_{41}} + \cancel{\vec{f}_{42}} + \cancel{\vec{f}_{43}}$

3ª LEI DE NEWTON

$$\frac{d\vec{p}}{dt} = \vec{F}$$

SE $\vec{F} = 0 \Rightarrow \vec{P} = \text{CONST.}$

LEI DE CONSERVAÇÃO DO MOMENTO LINEAR TOTAL DE UM SISTEMA DE $\underline{N}$ PARTÍCULAS.