Aula 12

F 502 – Eletromagnetismo I

2º semestre de 2020

27/10/2020

Aulas passadas

Problema: resolver a Equação de Laplace numa certa região R do espaço, dadas condições de contorno na fronteira da região. $\nabla^2 V = 0$ EQ. DE LAPLACE

A escolha do sistema de coordenadas é ditada pela geometria do problema: p. ex., esférica.

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}=0.$$

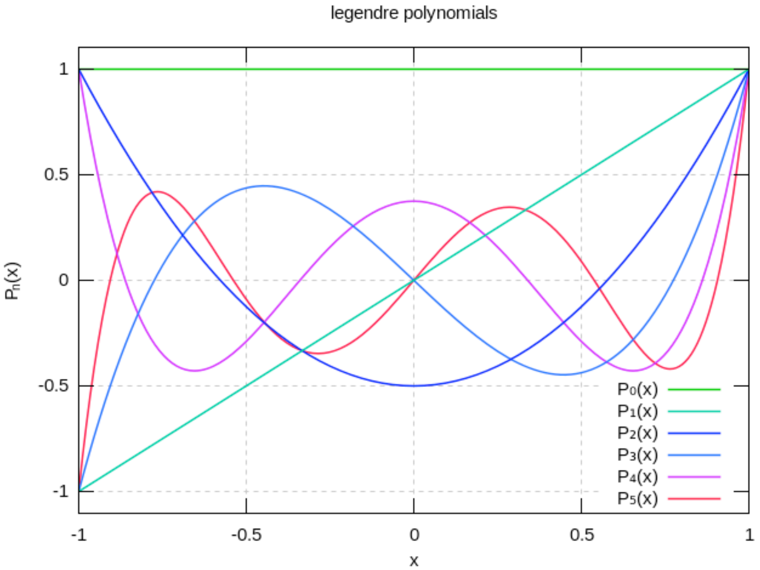
Aulas passadas

Solução geral com simetria azimutal:

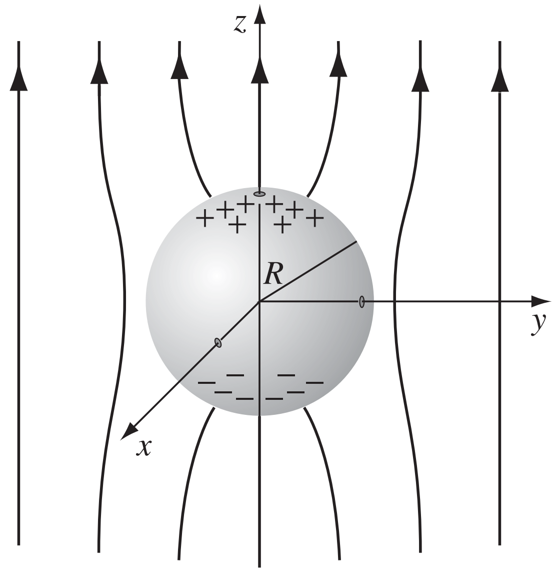
$$V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{(l+1)}}\right) P_l(\cos\theta)$$

$P_l(x)$ são polinômios de Legendre

$$\begin{aligned}
P_0(x) &= 1 \\
P_1(x) &= x \\
P_2(x) &= (3x^2-1)/2 \\
P_3(x) &= (5x^3-3x)/2 \\
P_4(x) &= (35x^4-30x^2+3)/8 \\
P_5(x) &= (63x^5-70x^3+15x)/8
\end{aligned}$$



Exemplo 3.8: Uma esfera condutora neutra num campo uniforme



Campo externo: $\mathbf{E} = E_0\,\mathbf{z}$

Condições de contorno: REGIÃO $r \geq R$

$$V\left(r \to \infty, \theta\right) \to -E_0 z + \text{const.} = -E_0 r\cos\theta + \text{const.}$$

$$V\left(R, \theta\right) = \text{const.} \equiv V_0,\ \ \forall\theta$$

$$V\left(R,\theta\right) = \sum_{l=0}^{\infty} \underbrace{\left(A_l R^l + \frac{B_l}{R^{(l+1)}}\right)}_{a_l} P_l\left(\cos\theta\right) = V_0,\ \ \forall\theta$$

$$V\left(R,\theta\right) = \sum_{l=0}^{\infty} a_l P_l\left(\cos\theta\right) = V_0 P_0\left(\cos\theta\right)$$

$$\Rightarrow \quad a_0 = V_0$$

$$a_l = 0,\ \ l = 1, 2, 3, \ldots$$

$$\left\{\begin{array}{l} A_0 + \dfrac{B_0}{R} = V_0 \Rightarrow \boxed{B_0 = R\left(V_0 - A_0\right)} \\ A_l R^l + \dfrac{B_l}{R^{(l+1)}} = 0 \qquad (l = 1, 2, 3, \ldots) \end{array}\right.$$

$$\Rightarrow \boxed{B_\ell = -A_\ell R^{(2\ell+1)} \quad (\ell = 1, 2, 3, \ldots)}$$

REESCREVENDO O POTENCIAL:

$$V(r,\theta) = A_0 - \frac{R(A_0 - V_0)}{r} + \sum_{\ell=1}^{\infty} A_\ell \left(r^\ell - \frac{R^{(2\ell+1)}}{r^{(\ell+1)}} \right) P_\ell(\cos\theta)$$

A OUTRA CONDIÇÃO DE CONTORNO: $r \to \infty$

$$V(r \to \infty, \theta) = A_0 + \sum_{\ell=1}^{\infty} A_\ell r^\ell P_\ell(\cos\theta) = -E_0 r \cos\theta + \text{CONST.}$$

$$= A_0 + A_1 r \underbrace{P_1(\cos\theta)}_{\cos\theta} + \sum_{\ell=2}^{\infty} A_\ell r^\ell P_\ell(\cos\theta) = -E_0 r \cos\theta + \text{CONST.}$$

$$\Rightarrow \boxed{A_1 = -E_0} \qquad \boxed{A_\ell = 0 \quad \ell = 2, 3, 4 \ldots}$$

$$\Rightarrow V(r,\theta) = A_0 + \frac{(V_0 - A_0)R}{r} - E_0 \left(r - \frac{R^3}{r^2} \right) \cos\theta$$

O POTENCIAL DA CARGA INDUZIDA É $\left[(V_0 - A_0)\frac{R}{r} + E_0 \frac{R^3}{r^2} \cos\theta \right]$

QUAL É A DISTRIBUIÇÃO $\sigma(R,\theta)$ DA CARGA INDUZIDA NA ESFERA?

$$\sigma(R,\theta) = \varepsilon_0 \, \hat{n}\cdot\vec{E}(R,\theta) = -\epsilon_0 \, \hat{n}\cdot\vec{\nabla}V\Big|_{r=R^+} = -\epsilon_0 \, \hat{r}\cdot\vec{\nabla}V\Big|_{r=R^+}$$

$$\sigma(R,\theta) = -\epsilon_0 \frac{\partial V}{\partial r}\Big|_{r=R^+}$$

$$\frac{\partial V}{\partial r}\Big|_{r=R^+} = -\frac{(V_0-A_0)}{R} - E_0\left[1 + 2\frac{R^3}{R^3}\right]\cos\theta = \frac{A_0-V_0}{R} - 3E_0\cos\theta$$

$$\Rightarrow \sigma(R,\theta) = \epsilon_0\left[\frac{V_0-A_0}{R} + 3E_0\cos\theta\right]$$

$$q_{ind} = \int \sigma(R,\theta)\, dS = \int \sigma(R,\theta)\, R^2 \sin\theta \, d\theta \, d\phi = 2\pi R^2 \int_0^\pi \sigma(R,\theta)\sin\theta \, d\theta$$

$$= 2\pi\epsilon_0 R^2\left[\left(\frac{V_0-A_0}{R}\right)\underbrace{\int_0^\pi \sin\theta \, d\theta}_{2} + 3E_0 \underbrace{\int_0^\pi \cos\theta \sin\theta \, d\theta}_{0}\right] = 4\pi\epsilon_0 R(V_0-A_0)$$

COMO A CARGA TOTAL DA ESFERA É $q_{ind} = 0$

$$\Rightarrow V_0 = A_0$$

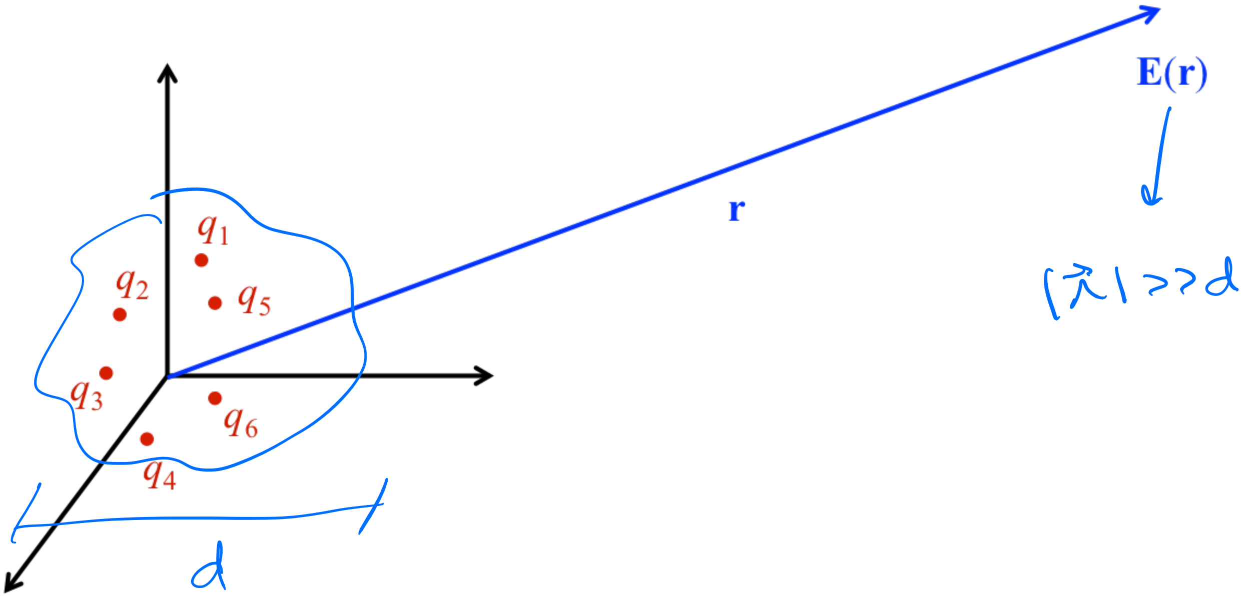
$$\boxed{V(r,\theta) = V_0 - E_0\left(r - \frac{R^3}{r^2}\right)\cos\theta}$$

$$\boxed{\sigma(R,\theta) = 3\epsilon_0 E_0 \cos\theta}$$

$$\boxed{V_{IND}(r,\theta) = E_0 \frac{R^3}{r^2}\cos\theta}$$

Expansão multipolar

Objetivo: obter o campo/potencial de uma configuração localizada de cargas, num ponto longe da distribuição.



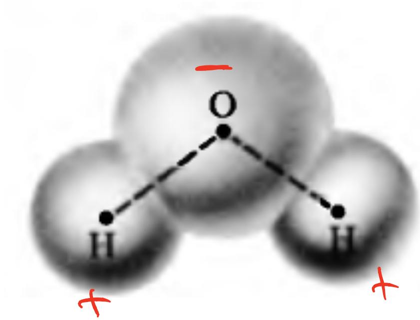
Expansão multipolar

Primeira resposta: longe da distribuição, o campo é

$$V(\mathbf{r}) = \frac{Q}{4\pi\epsilon_0}\frac{1}{r}, \quad \mathbf{E}(\mathbf{r}) = \frac{Q}{4\pi\epsilon_0}\frac{\hat{\mathbf{r}}}{r^2}$$

onde Q é a carga total. $Q = \sum_i q_i$

Mas e se a carga total for zero?



Expansão multipolar

SOLUÇÃO GERAL:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}\,')\,dV'}{|\vec{r}-\vec{r}\,'|}$$

$\vec{r}\,'$ VARIA NA REGIÃO ONDE HÁ CARGA

$|\vec{r}| >> |\vec{r}\,'|$ (MUITO LONGE DAS CARGAS)

$$|\vec{r}-\vec{r}\,'| = [r^2 + r'^2 - 2\vec{r}\cdot\vec{r}\,']^{1/2} = [r^2 + r'^2 - 2rr'\cos\gamma']^{1/2}$$

ONDE γ' É O ÂNGULO ENTRE $\vec{r}$ E $\vec{r}\,'$

$$\Rightarrow |\vec{r}-\vec{r}\,'| = r\left[1 + \frac{r'^2}{r^2} - 2\frac{r'}{r}\cos\gamma'\right]^{1/2} \Rightarrow \frac{1}{|\vec{r}-\vec{r}\,'|} = \frac{1}{r}\,\frac{1}{\Big[\underbrace{1+\frac{r'^2}{r^2} - 2\frac{r'}{r}\cos\gamma'}_{x}\Big]^{1/2}}$$

$$\frac{1}{\sqrt{1+x}} \cong 1 - \frac{x}{2} + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \dots \qquad |x| << 1$$

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \frac{1}{r}\left[1 + \left(\frac{r'}{r}\right)\cos\gamma' + \left(\frac{r'}{r}\right)^2\frac{3\cos^2\gamma' - 1}{2} + O\left(\frac{r'}{r}\right)^3\right]$$

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r}\left[\underbrace{\int \rho(\vec{r}\,')\,dv'}_{Q\,=\,\text{CARGA TOTAL}} + \int \frac{r'}{r}\cos\gamma'\,\rho(\vec{r}\,')\,dv' + O\left(\frac{r'}{r}\right)^2\right]$$

O 1º TERMO É O QUE EU TINHA ANTECIPADO. E O 2º?

$$V_1(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\int \underbrace{r\,r'\cos\gamma'}_{\vec{r}\cdot\vec{r}\,'}\,\rho(\vec{r}\,')\,dv' = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\int(\vec{r}\cdot\vec{r}\,')\,\rho(\vec{r}\,')\,dv'$$

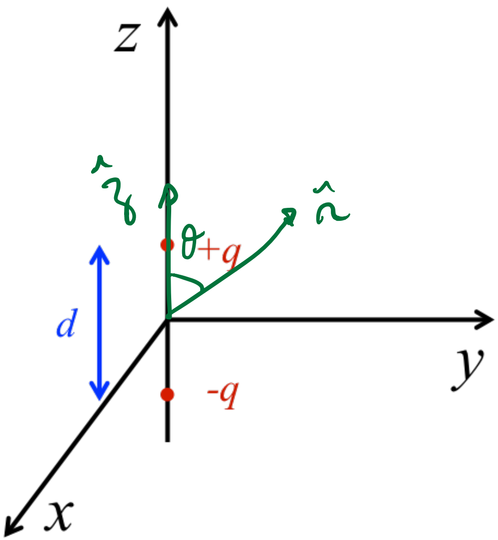
$$= \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\,\vec{r}\cdot\Big[\underbrace{\int \vec{r}\,'\rho(\vec{r}\,')\,dv'}_{\vec{p}}\Big]$$

$$p_x = \int x'\rho(\vec{r}\,')\,dv' \qquad p_y = \int y'\rho(\vec{r}\,')\,dv' \qquad p_z = \int z'\rho(\vec{r}\,')\,dv'$$

$\vec{p}$ É O MOMENTO DE DIPOLO ELÉTRICO DA DISTRIBUIÇÃO $\rho(\vec{r})$

$$\Rightarrow \boxed{V_1(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\vec{r}}{r^3} = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2}}$$

O dipolo elétrico



NO CASO DE DUAS CARGAS +q E −q DISTANTES DE $\underline{d}$:

$$\vec{p} = \int \vec{r}\,\rho(\vec{r})\,dV = \sum_i \vec{r}_i q_i$$

$$= q\left(\frac{d}{2}\right)\hat{z} - q\left(-\frac{d}{2}\right)\hat{z} = \boxed{q d\,\hat{z} = \vec{p}}$$

$$|\vec{p}| = q d$$

$\vec{p}$ APONTA DA CARGA NEGATIVA PARA A CARGA POSITIVA

$$V_1(\vec{r}) = \frac{d\,q}{4\pi\epsilon_0}\,\frac{\hat{z}\cdot\hat{r}}{r^2} = \left(\frac{q\,d}{4\pi\epsilon_0}\right)\frac{\cos\theta}{r^2}$$

Dipolos elétricos somam-se vetorialmente

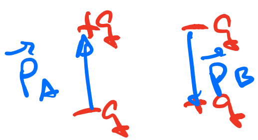
SEJA UMA DISTRIBUIÇÃO COM DUAS CONTRIBUIÇÕES!

$$\rho(\vec{r}) = \rho_A(\vec{r}) + \rho_B(\vec{r})$$

$$\Rightarrow \vec{p} = \int \vec{r}\,\rho(\vec{r})\,dV = \int \vec{r}\left[\rho_A(\vec{r}) + \rho_B(\vec{r})\right]dV$$

$$= \underbrace{\int \vec{r}\,\rho_A(\vec{r})\,dV}_{\vec{p}_A} + \underbrace{\int \vec{r}\,\rho_B(\vec{r})\,dV}_{\vec{p}_B}$$

$$\vec{p} = \vec{p}_A + \vec{p}_B$$



$$\Rightarrow \vec{p}_A + \vec{p}_B = 0$$

O campo elétrico de um dipolo

$V_1(\vec{r}) = \frac{p}{4\pi\epsilon_0} \frac{\cos\theta}{r^2}$ PARA UM DIPOLO NA DIREÇÃO $\hat{z}$: $\vec{p} = p\hat{z}$

$\vec{E}_1 = -\vec{\nabla} V_1 \Rightarrow E_{1r} = -\frac{\partial V}{\partial r} \qquad E_{1\theta} = -\frac{1}{r}\frac{\partial V}{\partial \theta}$

$$\vec{E}_1 = E_{1r}\,\hat{r} + E_{1\theta}\,\hat{\theta}$$

TOMANDO AS DERIVADAS (VER NOTAS DE AULA):

$$\vec{E}_1 = \frac{p}{4\pi\epsilon_0 r^3}\left(3\cos\theta\,\hat{r} - \hat{z}\right)$$

MAS $p\cos\theta = \vec{p}\cdot\hat{r}$ E $p\hat{z} = \vec{p}$

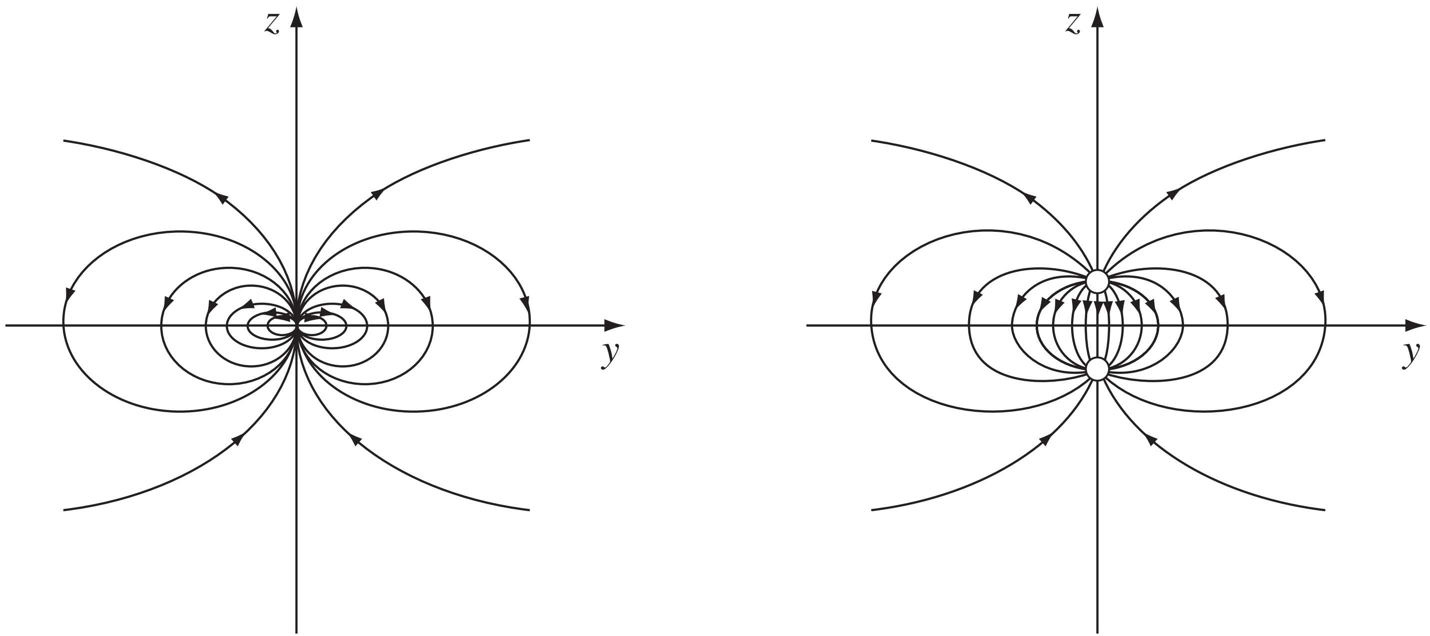
$$\boxed{\vec{E}_1 = \frac{1}{4\pi\epsilon_0 r^3}\left[3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}\right]}$$

ESSA FORMA É GENÉRICA MESMO QUE $\vec{p}$ NÃO SEJA NA DIREÇÃO $\hat{z}$

DIPOLO "PURO" TEM DIMENSÃO ZERO: PARA CARGAS $\pm q$

$d \to 0$

$q \to \infty$ TAL QUE $p = qd = \text{CONST.}$



(a) Field of a "pure" dipole

(b) Field of a "physical" dipole

O resto da expansão

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{r}\frac{1}{\sqrt{1-2\left(r'/r\right)\cos\gamma' + \left(r'/r\right)^2}}$$

$s \equiv \frac{r'}{r}$

$x \equiv \cos\gamma'$

Mas, das propriedades dos polinômios de Legendre:

$$\frac{1}{\sqrt{1-2sx+s^2}} = \sum_{l=0}^{\infty} s^l P_l\left(x\right) \quad \left(|s|<1\right)$$

"Função geratriz"

$$\boxed{\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{r}\sum_{l=0}^{\infty}\left(\frac{r'}{r}\right)^l P_l\left(\cos\gamma'\right)}$$

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{1}{r} \sum_{\ell=0}^{\infty} \int \left(\frac{r'}{r}\right)^{\ell} P_\ell(\cos\gamma') \rho(\vec{r}') dV'$$

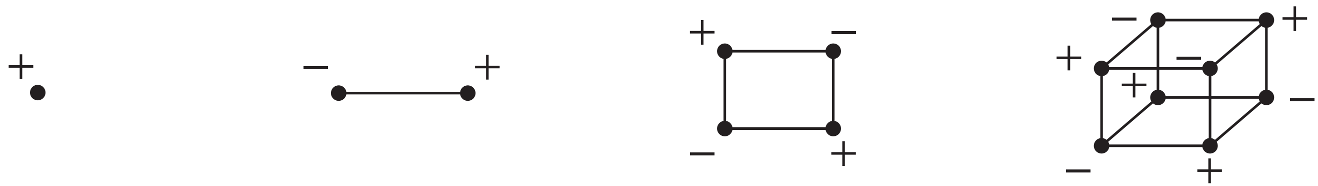
$$= \frac{1}{4\pi\epsilon_0} \left[\frac{Q}{r} + \frac{\vec{p}\cdot\hat{r}}{r^2} + \underbrace{\frac{1}{r^2}\int (r')^2 \left(\frac{3\cos^2\gamma' - 1}{2}\right) \rho(\vec{r}') dV'}_{3^{o}} + \right.$$

$$\left. + \sum_{\ell \geq 3}^{\infty} \frac{1}{r^{\ell}} \int (r')^{\ell} P_\ell(\cos\gamma') \rho(\vec{r}') dV' \right]$$

3º TERMO É O TERMO DE QUADRUPOLO ELÉTRICO

CADA TERMO É CHAMADO DE MULTIPOLO ELÉTRICO

MULTIPOLOS ELEMENTARES:



Monopole ($V \sim 1/r$) | Dipole ($V \sim 1/r^2$) | Quadrupole ($V \sim 1/r^3$) | Octopole ($V \sim 1/r^4$)

Problema 3.26 (3ª. ed.)/3.27 (4ª. ed.)

Problem 3.27 A sphere of radius R, centered at the origin, carries charge density

$$\rho(r,\theta) = k\frac{R}{r^2}(R-2r)\sin\theta, \qquad r \leq R$$

where k is a constant, and r, θ are the usual spherical coordinates. Find the approximate potential for points on the z axis, far from the sphere.

MONOPOLO: $Q = \int \rho(\vec{r})\,dV = kR \int \frac{1}{\cancel{r^2}}(R-2r)\sin\theta\,\cancel{r^2}\sin\theta\,dr\,d\theta\,d\phi$

INTEGRAL RADIAL: $\int_0^R (R-2r)\,dr = R^2 - r^2\Big|_0^R = R^2 - R^2 = 0$

DIPOLO: $\int r' \underbrace{P_1(\cos\gamma')}_{\cos\theta'} \frac{kR}{r'^2}(R-2r')\sin\theta'\, r'^2 \sin\theta'\, dr'\, d\theta'\, d\phi'$

$P_1(\cos\gamma') = \cos\gamma'$

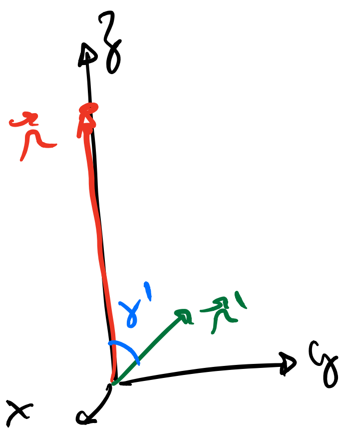
COMO SÓ QUERO O POTENCIAL NO EIXO $\hat{z}$ $(\vec{r} = z\hat{z}) \Rightarrow \cos\gamma' = \cos\theta'$

$$\int_0^\pi \sin^2\theta' \cos\theta' \, d\theta' = \frac{1}{3}\sin^3\theta \Big|_{\theta=0}^{\theta=\pi} = 0$$

QUADRUPOLO: $$\int (r')^2 \underbrace{P_2(\cos\theta')}_{\frac{3\cos^2\theta'-1}{2}} \frac{kR}{r^2}(R-2r)\sin\theta' \, r'^2 dr' \sin\theta' d\theta' d\phi$$

FAZENDO AS INTEGRAIS:

$$\boxed{V(\vec{r}) \cong \frac{\pi}{192\,\epsilon_0}\frac{kR^5}{|z|^3}}$$



$\gamma' \Rightarrow$ ÂNGULO ENTRE $\vec{r}'$ E $\hat{z}$

AS COORDENADAS ESFÉRICAS DE $\vec{r}'$ SÃO: r', $\theta' = \gamma'$, ϕ'