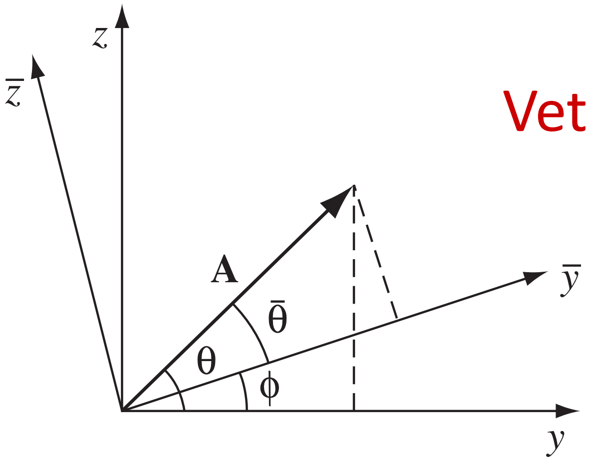
Aula 2

F 502 – Eletromagnetismo I

2º semestre de 2020

22/09/2020

Aula passada



Vetores: propriedades de transformação

Em 2D: $$\begin{pmatrix} \overline{A}_y \\ \overline{A}_z \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} A_y \\ A_z \end{pmatrix}$$

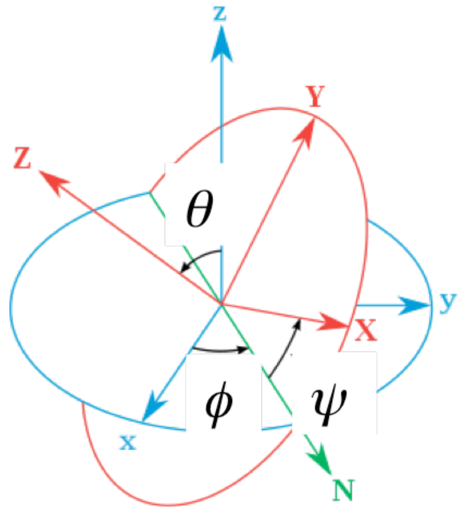
Em 3D: $$\begin{pmatrix} \overline{A}_x \\ \overline{A}_y \\ \overline{A}_z \end{pmatrix} = \begin{pmatrix} R_{xx} & R_{xy} & R_{xz} \\ R_{yx} & R_{yy} & R_{yz} \\ R_{zx} & R_{zy} & R_{zz} \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

onde: $R^{-1} = R^T$ (matriz ortogonal)

Aula passada

Exemplo:

$$\begin{pmatrix} \overline{A}_x \\ \overline{A}_y \\ \overline{A}_z \end{pmatrix} = \begin{pmatrix} R_{xx} & R_{xy} & R_{xz} \\ R_{yx} & R_{yy} & R_{yz} \\ R_{zx} & R_{zy} & R_{zz} \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$



$$\begin{pmatrix} \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi & \cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi & \sin\psi\sin\theta \\ -\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi & -\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi & \cos\psi\sin\theta \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{pmatrix}$$

Aula passada

Campo escalar: $f\left(x,y,z,t\right) \quad \equiv \quad f\left(\mathbf{r},t\right)$

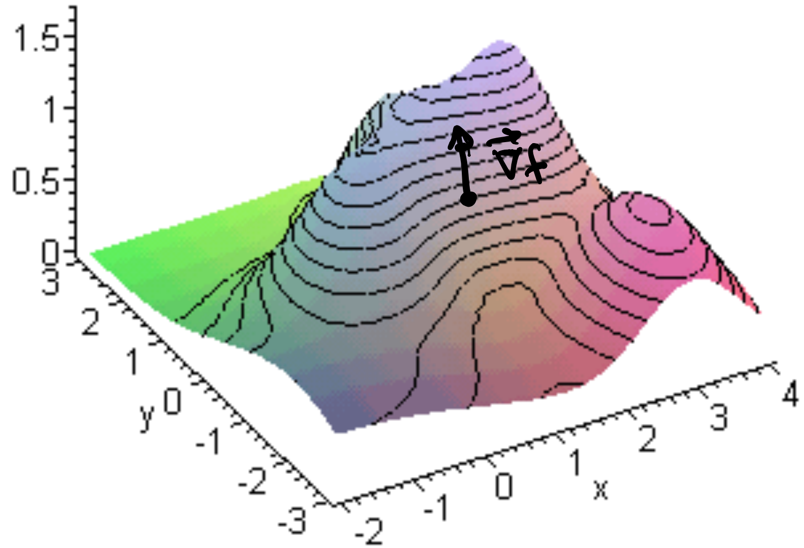
Campo vetorial:
$$\left.\begin{array}{rcl} A_x\left(x,y,z,t\right) & \equiv & A_x\left(\mathbf{r},t\right) \\ A_y\left(x,y,z,t\right) & \equiv & A_y\left(\mathbf{r},t\right) \\ A_z\left(x,y,z,t\right) & \equiv & A_z\left(\mathbf{r},t\right) \end{array}\right\} \Rightarrow \mathbf{A}\left(\mathbf{r},t\right)$$

$\vec{A}(\vec{r},t)$

Aula passada

Gradiente de $f(\mathbf{r},t)$: $\boldsymbol{\nabla} f = \frac{\partial f}{\partial x}\hat{\mathbf{x}} + \frac{\partial f}{\partial y}\hat{\mathbf{y}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}}$

- Direção e sentido de maior crescimento de f
- Módulo: taxa de crescimento naquela direção

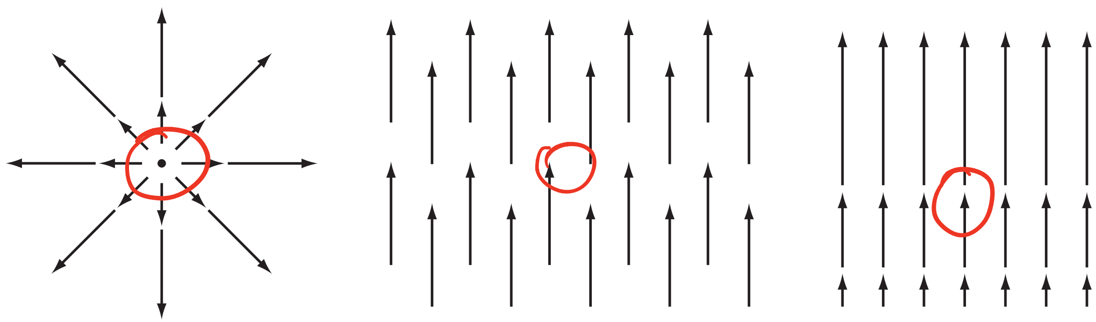


Aula passada

Divergente de $\mathbf{v}(\mathbf{r},t)$: $$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

$$\nabla \cdot \mathbf{v} = \lim_{V \to 0} \left(\frac{\oint_{S(V)} \mathbf{v} \cdot d\mathbf{S}}{V} \right)$$

onde *V* é um volume que contém o ponto em questão e *S(V)* é a superfície que contém *V*.



(a) $\nabla \cdot \mathbf{v} \neq 0$

(b) $\nabla \cdot \mathbf{v} = 0$

(c) $\nabla \cdot \mathbf{v} \neq 0$

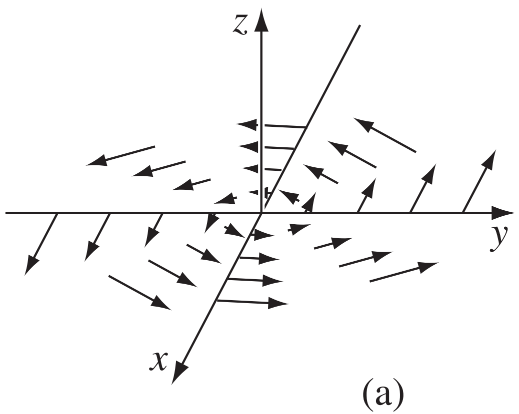
O rotacional

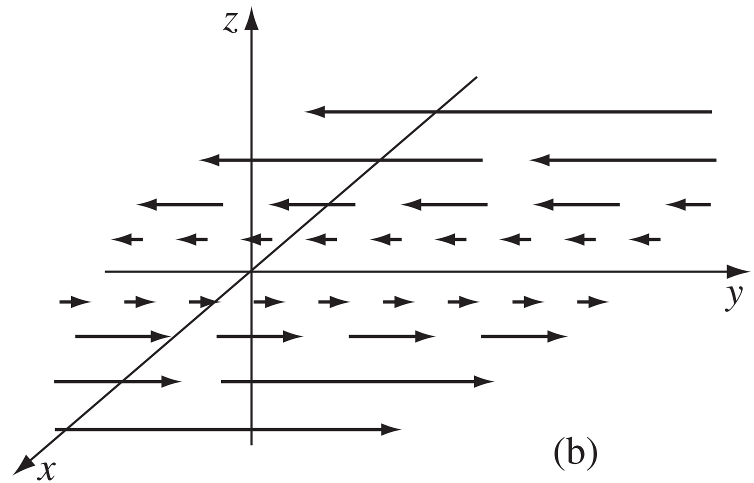
LEVA UM CAMPO VETORIAL $\vec{V}$ A UM OUTRO CAMPO VETORIAL : $\vec{\nabla} \times \vec{V}$:

$$\vec{\nabla} \times \vec{V} = \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)\hat{z} + \left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right)\hat{y} + \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right)\hat{x}$$

$$\vec{\nabla} \times \vec{V} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$

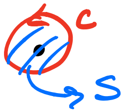
$$\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$$





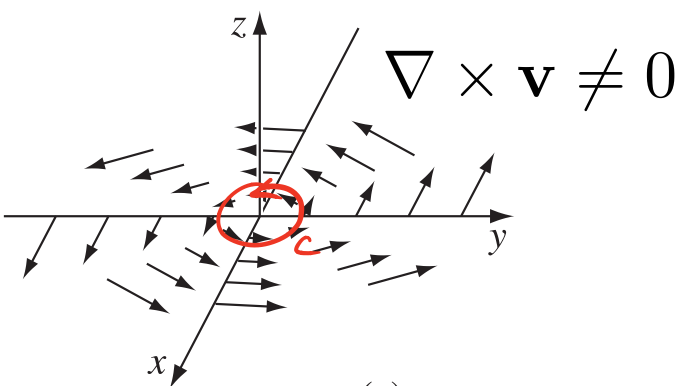
$$\nabla \times \mathbf{v} \neq 0$$

Interpretação física do rotacional

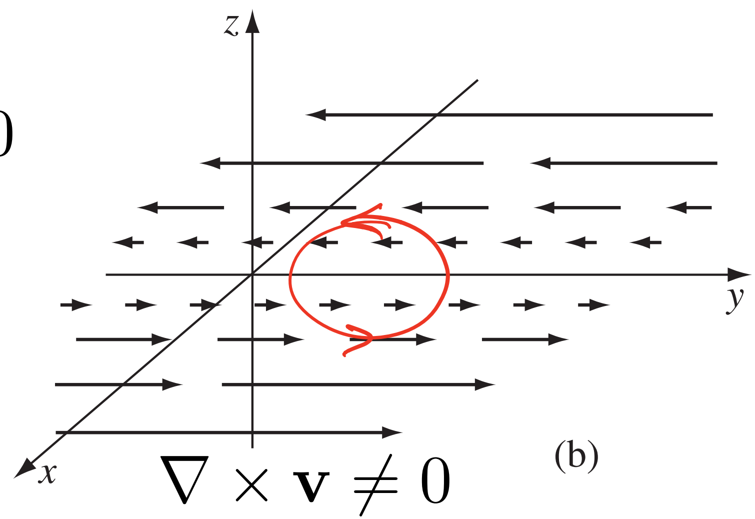


$$\nabla \times \mathbf{v} = \lim_{S \to 0} \left(\frac{\oint_{C(S)} \mathbf{v} \cdot d\mathbf{l}}{S} \right)$$

onde *S* é uma superfície aberta que contém o ponto em questão, normal à direção procurada do rotacional, e *C(S)* é a borda da superfície *S*.



(a)



(b)

Identidades importantes

$$
\begin{aligned}
\nabla (f+g) &= \nabla f + \nabla g && (1)\\
\nabla \cdot (\mathbf{A}+\mathbf{B}) &= \nabla \cdot \mathbf{A} + \nabla \cdot \mathbf{B} && (2)\\
\nabla \times (\mathbf{A}+\mathbf{B}) &= \nabla \times \mathbf{A} + \nabla \times \mathbf{B} && (3)\\
\nabla (fg) &= g\nabla f + f\nabla g, && (4)\\
\nabla \cdot (f\mathbf{A}) &= f(\nabla \cdot \mathbf{A}) + \mathbf{A}\cdot(\nabla f) && (5)\\
\nabla \times (f\mathbf{A}) &= f(\nabla \times \mathbf{A}) + (\nabla f)\times \mathbf{A} \quad (**) && (6)\\
\nabla \cdot (\mathbf{A}\times\mathbf{B}) &= (\nabla\times\mathbf{A})\cdot\mathbf{B} - \mathbf{A}\cdot(\nabla\times\mathbf{B}) \quad (**) && (7)\\
\nabla \times (\mathbf{A}\times\mathbf{B}) &= \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A}) + (\mathbf{B}\cdot\nabla)\overset{*}{\mathbf{A}} - (\mathbf{A}\cdot\nabla)\mathbf{B} \quad (**) && (8)\\
\nabla (\mathbf{A}\cdot\mathbf{B}) &= \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A} \quad (**) && (9)
\end{aligned}
$$

$$A_x \frac{\partial f}{\partial x} + A_y \frac{\partial f}{\partial y} + A_z \frac{\partial f}{\partial z}$$

$(**) \Rightarrow$ A ordem é importante!

$$(*)\ (\vec{B}\cdot\vec{\nabla})\vec{A} = \left(B_x \frac{\partial}{\partial x} + B_y \frac{\partial}{\partial y} + B_z \frac{\partial}{\partial z}\right)\left(A_x \hat{x} + A_y \hat{y} + A_z \hat{z}\right) =$$

$$= B_x \frac{\partial A_x}{\partial x}\hat{x} + B_x \frac{\partial A_y}{\partial x}\hat{y} + B_x \frac{\partial A_z}{\partial x}\hat{z} + B_y \frac{\partial A_x}{\partial y}\hat{x} + \cdots$$

Identidades importantes

$\vec{\nabla}\cdot\vec{\nabla} = \nabla^2$

$$\nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \equiv \nabla^2 f \quad (10)$$

∇^2 = LAPLACIANO

$$\nabla \times (\nabla f) = 0 \quad (11)$$

$\bar{A}\times(\bar{B}\times\bar{C}) = (\vec{A}\cdot\vec{C})\vec{B} - (\vec{A}\cdot\vec{B})\vec{C}$

(*)
$$\nabla \cdot (\nabla \times \mathbf{A}) = 0 \quad (12)$$

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \quad (13)$$

$$\nabla^2 \mathbf{A} = \left(\frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_x}{\partial y^2} + \frac{\partial^2 A_x}{\partial z^2}\right)\hat{\mathbf{x}} + \left(\frac{\partial^2 A_y}{\partial x^2} + \frac{\partial^2 A_y}{\partial y^2} + \frac{\partial^2 A_y}{\partial z^2}\right)\hat{\mathbf{y}} + \left(\frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2} + \frac{\partial^2 A_z}{\partial z^2}\right)\hat{\mathbf{z}}$$

DEFINIÇÃO DE $\nabla^2\vec{A}$

$\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B})$

(*) SIMBOLICAMENTE: $\bar{\nabla}\cdot(\bar{\nabla}\times\vec{A}) = \vec{A}\cdot(\bar{\nabla}\times\cancel{\bar{\nabla}}) = 0$

NÃO É RIGOROSO
É SÓ UMA MANEIRA
DE LEMBRAR

Teorema fundamental do cálculo

$$\int_a^b f(x)\,dx = F(b) - F(a) \quad \text{ONDE} \quad f(x) = \frac{dF}{dx}$$

EQUIVALENTEMENTE:

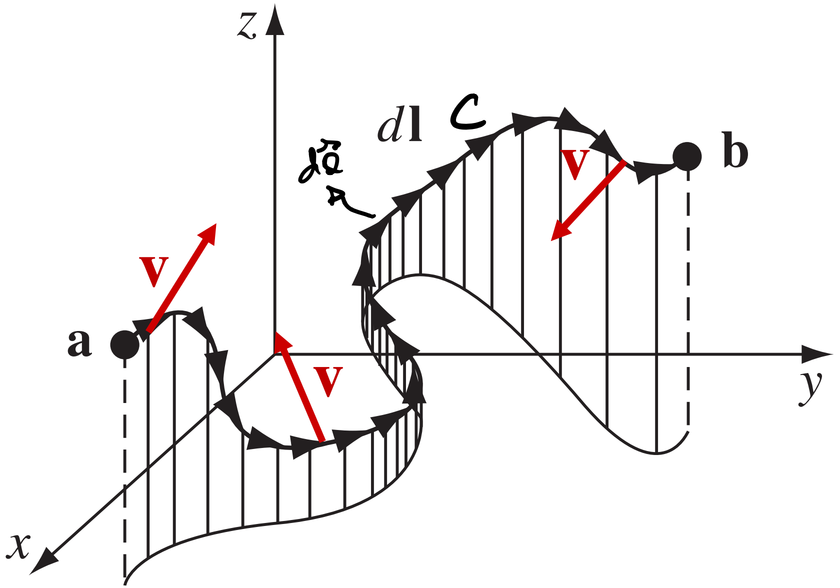
$$\int_a^b \frac{dF}{dx}\,dx = F(b) - F(a)$$

RELACIONA A INTEGRAL DA DERIVADA DE UMA FUNÇÃO A SEUS VALORES NAS BORDAS DO INTERVALO

Integral de linha

$$\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{v} \cdot d\mathbf{l}$$

$\mathcal{C}$



Teoremas fundamentais do cálculo vetorial

Teorema do gradiente: $$\int_{\mathbf{r}_0}^{\mathbf{r}} (\nabla f) \cdot d\mathbf{l} = f(\mathbf{r}) - f(\mathbf{r}_0)$$

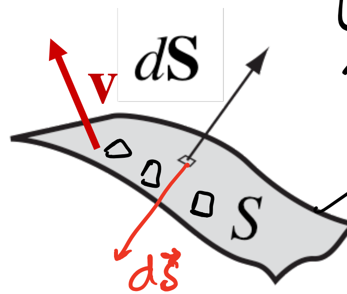
O RESULTADO, NESSE CASO, INDEPENDE DO CAMINHO QUE VAI DE $\vec{r}_0$ A $\vec{r}$

Integrais de superfície

Superfície aberta: $\int_S \mathbf{v} \cdot d\mathbf{S}$

TEM BORDAS

O SENTIDO DOS $d\vec{S}$ TEM UMA AMBIGUIDADE

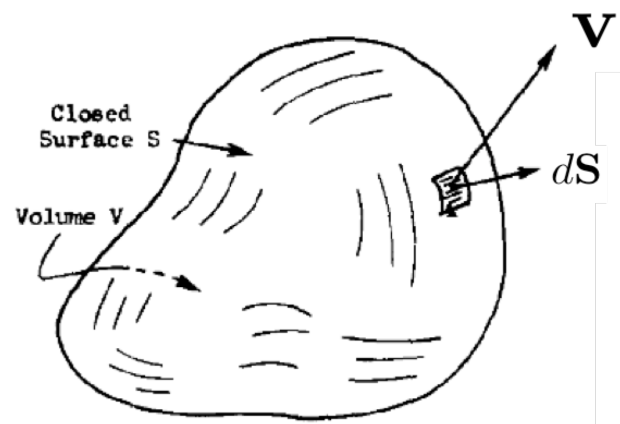


$d\vec{S} = \begin{cases} \text{NORMAL} \\ \text{A S} \\ \text{MÓDULO É} \end{cases}$ A ÁREA DO ELEMENTO

BORDA

Superfície fechada: $\oint_{S(V)} \mathbf{v} \cdot d\mathbf{S}$

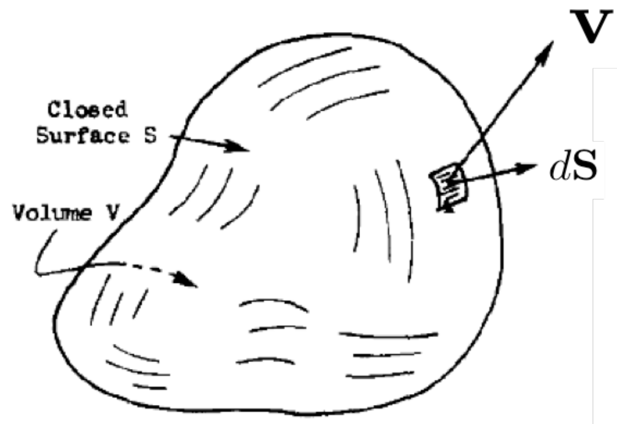
CONVENCIONA-SE $d\vec{S}$ APONTANDO PARA FORA DO VOLUME V



Teoremas fundamentais do cálculo vetorial

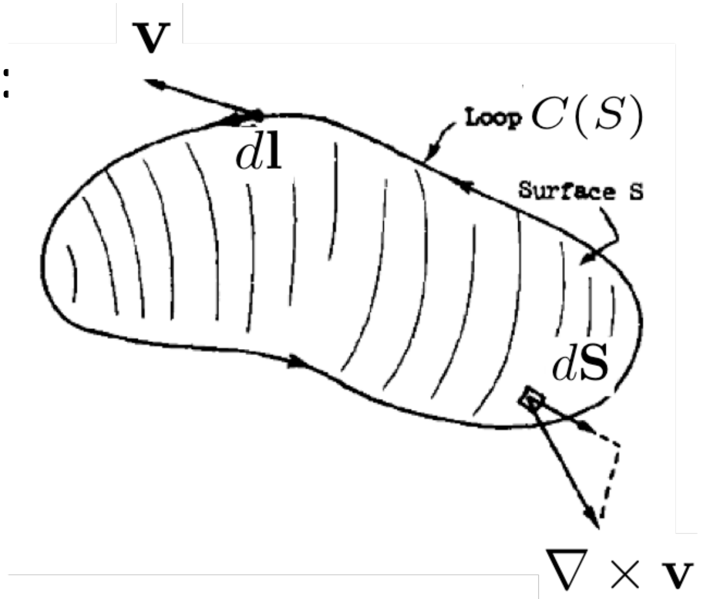
Teorema do divergente (teorema de Gauss):

$$\int_V (\nabla \cdot \mathbf{v})\, dV = \oint_{S(V)} \mathbf{v} \cdot d\mathbf{S}$$



Teoremas fundamentais do cálculo vetorial

Rotacional (teorema de Stokes):



$$\int_S (\nabla \times \mathbf{v}) \cdot d\mathbf{S} = \oint_{C(S)} \mathbf{v} \cdot d\mathbf{l}$$

SENTIDO DE $d\vec{S}$:

- ORIENTO A BORDA $C(S)$
- O SENTIDO DE $d\vec{S}$ É DADO PELA REGRA DA MÃO DIREITA.

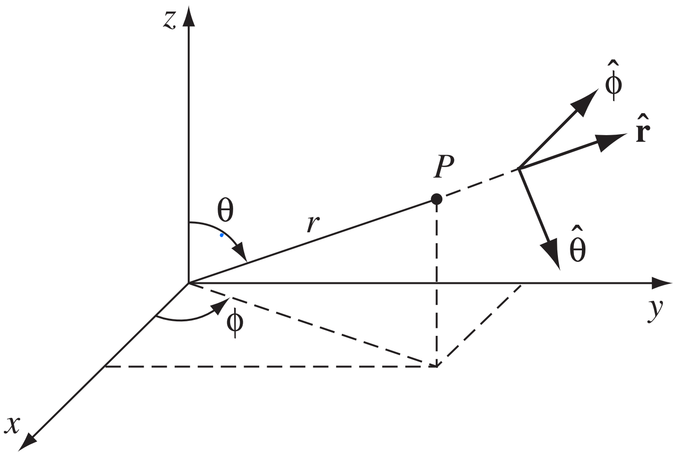
O SENTIDO DE $d\vec{S}$ ESTÁ "TRANCADO" PELA ORIENTAÇÃO ESCOLHIDA PARA $C(S)$

Coordenadas curvilíneas

Coordenadas esféricas:

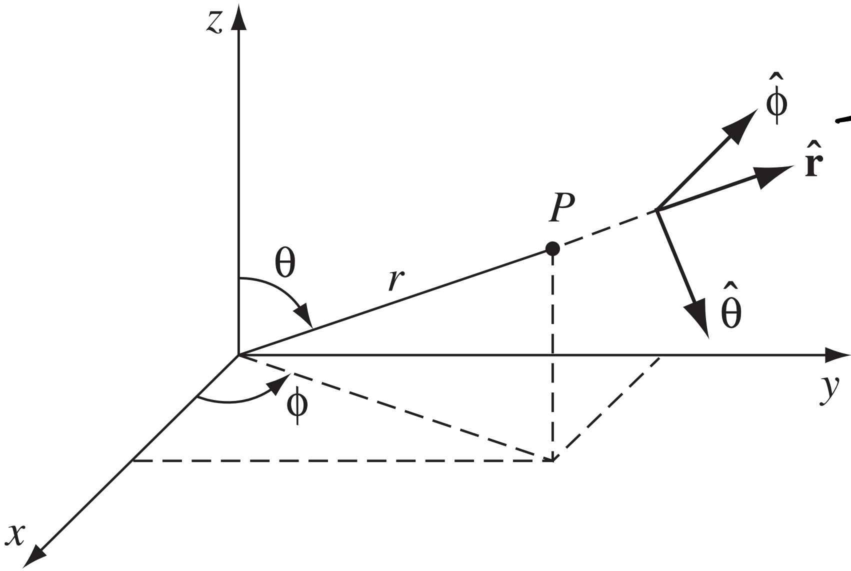
(r, θ, ϕ)

$$
\left.
\begin{aligned}
x &= \underline{r\sin\theta}\cos\phi \\
y &= \underline{r\sin\theta}\sin\phi \\
z &= r\cos\theta
\end{aligned}
\right\}
$$



$$
\begin{aligned}
r &= \sqrt{x^2+y^2+z^2} \\
\theta &= \arccos\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) \in [0,\pi] \\
\phi &= \arctan\left(\frac{y}{x}\right) \in [0,2\pi)
\end{aligned}
$$

Unitários para coords. esféricas



→ TRÍADE ORTONORMAL

$\hat{r} \times \hat{\theta} = \hat{\phi}$

$\vdots$

$$\begin{cases} \hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}, \\ \hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}}, \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}, \end{cases}$$

$\hat{r} = \dfrac{\vec{r}}{|\vec{r}|} = \dfrac{\vec{r}}{r}$

$\hat{r}, \hat{\theta}, \hat{\phi}$ DEPENDEM DA POSIÇÃO NO ESPAÇO (xxx)

Elementos de superfície e volume

ELEMENTO DE VOLUME:

$$dV = dx\,dy\,dz$$

$$dV = r^2 \sin\theta \, dr\, d\theta\, d\phi$$

ELEMENTO DE SUPERFÍCIE DEPENDE DA SUPERFÍCIE

NO CASO PARTICULAR DE UMA SUPERFÍCIE ESFÉRICA

DE RAIO R CENTRADA NA ORIGEM:

$$d\vec{S} = R^2 \sin\theta \, d\theta \, d\phi \, \hat{r}$$

Campos em coordenadas esféricas

EM COORDENADAS CARTESIANAS:

ESCALAR: $f(x,y,z)$

VETORIAL: $\vec{A}(x,y,z) = A_x(x,y,z)\hat{x} + A_y(x,y,z)\hat{y} + A_z(x,y,z)\hat{z}$

COORDENADAS ESFÉRICAS:

ESCALAR: $f(r,\theta,\phi)$

VETORIAL: $\vec{A}(r,\theta,\phi) = A_r(r,\theta,\phi)\hat{r} + A_\theta(r,\theta,\phi)\hat{\theta} + A_\phi(r,\theta,\phi)\hat{\phi}$

Gradient:

$$\nabla T = \frac{\partial T}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{\phi}}. \tag{1.70}$$

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}. \tag{1.71}$$

Curl:

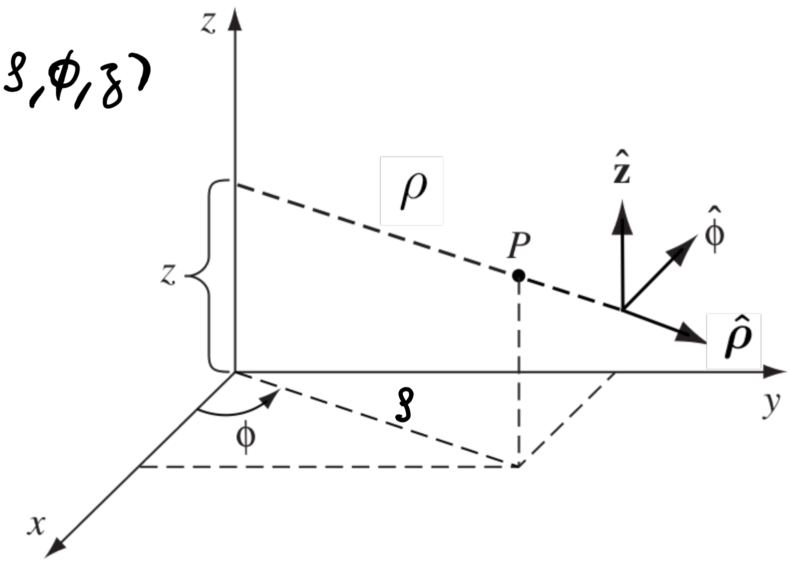
$$\begin{aligned}\nabla \times \mathbf{v} = {} & \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} \\ & + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}. \end{aligned} \tag{1.72}$$

Laplacian:

$$\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial \phi^2}. \tag{1.73}$$

Coordenadas curvilíneas

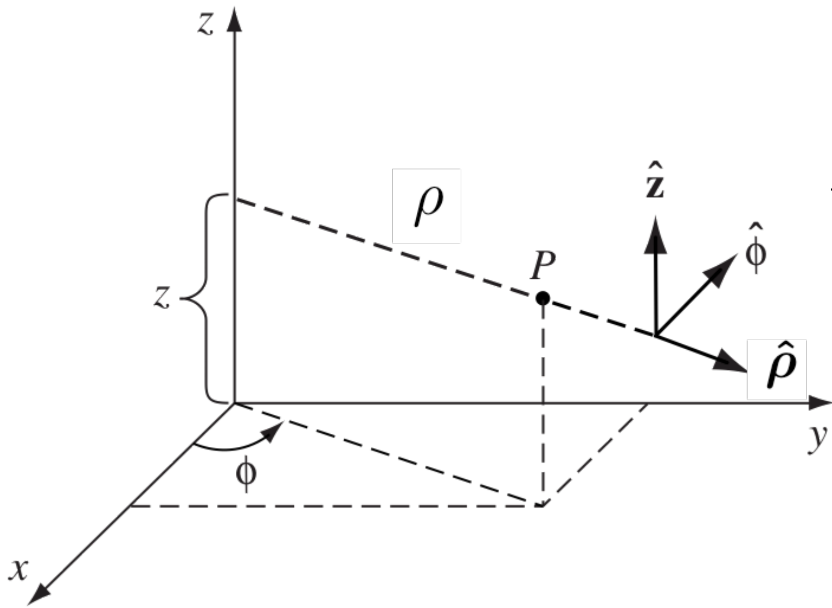
Coordenadas cilíndricas: (ρ, ϕ, z)



$$
\begin{aligned}
x &= \rho \cos \phi \\
y &= \rho \sin \phi \\
z &= z
\end{aligned}
$$

$$
\begin{aligned}
\rho &= \sqrt{x^2 + y^2} \geq 0 \\
\phi &= \arctan\left(\frac{y}{x}\right) \in [0, 2\pi) \\
z &= z
\end{aligned}
$$

Unitários para coords. cilíndricas



TRÍADE ORTONORMAL

$$\hat{\rho} \times \hat{\phi} = \hat{z}$$

$\vdots$

$$\begin{aligned} \hat{\boldsymbol{\rho}} &= \cos\phi\ \hat{\mathbf{x}} + \sin\phi\ \hat{\mathbf{y}} \\ \hat{\boldsymbol{\phi}} &= -\sin\phi\ \hat{\mathbf{x}} + \cos\phi\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} &= \hat{\mathbf{z}} \end{aligned}$$

TANTO $\hat{\rho}$ QUANTO $\hat{\phi}$ DEPENDEM DA POSIÇÃO (***)

Elementos de superfície e volume

ELEMENTO DE VOLUME:

$$dV = \rho \, d\rho \, d\phi \, dz$$

ELEMENTO DE SUPERFÍCIE DEPENDE DA SUPERFÍCIE, MAS NO CASO DE UM CILINDRO DE RAIO R CUJO EIXO É O EIXO z

$$d\vec{S} = R \, d\phi \, dz \, \hat{\rho}$$

Campos em coordenadas cilíndricas

ESCALAR: $f(\rho, \phi, z)$

VETORIAL: $\vec{A}(\rho, \phi, z) = A_\rho(\rho, \phi, z)\,\hat{\rho} + A_\phi(\rho, \phi, z)\,\hat{\phi} + A_z(\rho, \phi, z)\,\hat{z}$

Gradient:

$$\nabla T = \frac{\partial T}{\partial s}\,\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial T}{\partial \phi}\,\hat{\boldsymbol{\phi}} + \frac{\partial T}{\partial z}\,\hat{\mathbf{z}}. \tag{1.79}$$

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}. \tag{1.80}$$

Curl:

$$\nabla \times \mathbf{v} = \left(\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right)\hat{\mathbf{s}} + \left(\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right)\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}}. \tag{1.81}$$

Laplacian:

$$\nabla^2 T = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial T}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2}. \tag{1.82}$$

Teorema de Helmholtz

(Apêndice B do Griffiths) Seja um campo vetorial $\mathbf{F}(\mathbf{r})$ tal que saibamos seu divergente e seu rotacional,

$$\boxed{\begin{aligned} \nabla \cdot \mathbf{F} &= D(\mathbf{r}) \\ \nabla \times \mathbf{F} &= \mathbf{C}(\mathbf{r}) \end{aligned}}$$

e tal que:

- $D(\mathbf{r})$ e $\mathbf{C}(\mathbf{r})$ caem a zero no infinito ($r \to \infty$) mais rapidamente que $1/r^2$;
- $\mathbf{F}(\mathbf{r})$ cai a zero no infinito ($r \to \infty$).

Segue que $\mathbf{F}(\mathbf{r})$ é único e dado por:

$$\begin{aligned} \mathbf{F} &= -\nabla U + \nabla \times \mathbf{W} \\ U(\mathbf{r}) &= \frac{1}{4\pi} \int \frac{D(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dV', \\ \mathbf{W}(\mathbf{r}) &= \frac{1}{4\pi} \int \frac{\mathbf{C}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dV'. \end{aligned}$$