Aula 3

F 502 – Eletromagnetismo I

2º semestre de 2020

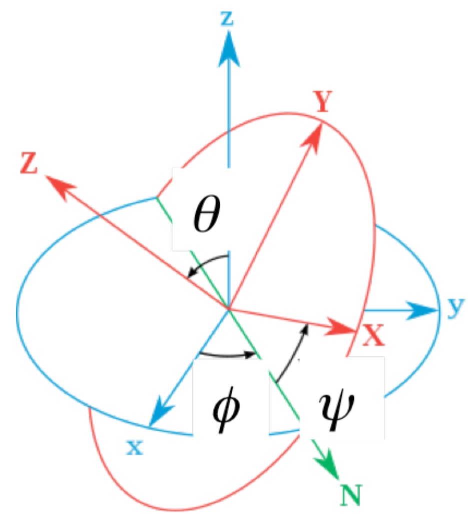
24/09/2020

Aulas passadas

Vetores: propriedades de transformação sob rotações

$$\begin{pmatrix} \overline{A}_x \\ \overline{A}_y \\ \overline{A}_z \end{pmatrix} = \begin{pmatrix} R_{xx} & R_{xy} & R_{xz} \\ R_{yx} & R_{yy} & R_{yz} \\ R_{zx} & R_{zy} & R_{zz} \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

onde: $R^{-1} = R^T$ (matriz ortogonal)



$$\begin{pmatrix} \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi & \cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi & \sin\psi\sin\theta \\ -\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi & -\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi & \cos\psi\sin\theta \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{pmatrix}$$

Aulas passadas

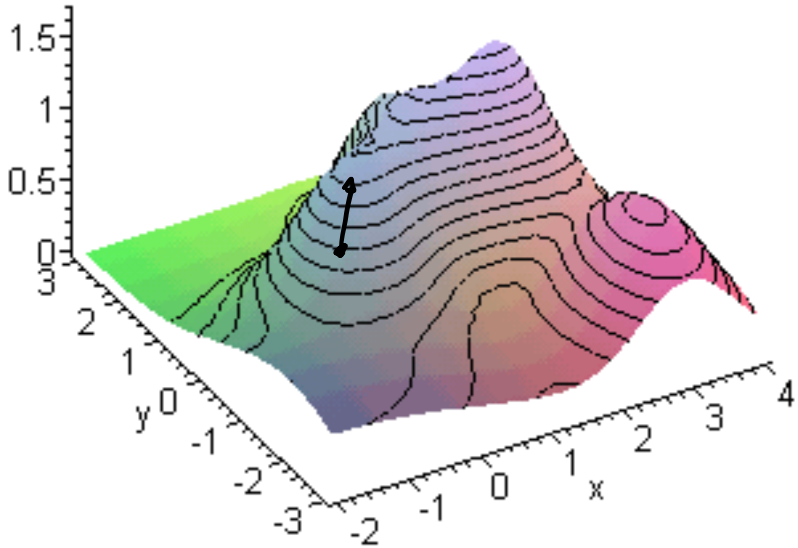
Campo escalar: $f\left(x, y, z, t\right) \quad \equiv \quad f\left(\mathbf{r}, t\right)$

Campo vetorial:
$$\left.\begin{array}{rcl} A_x\left(x, y, z, t\right) & \equiv & A_x\left(\mathbf{r}, t\right) \\ A_y\left(x, y, z, t\right) & \equiv & A_y\left(\mathbf{r}, t\right) \\ A_z\left(x, y, z, t\right) & \equiv & A_z\left(\mathbf{r}, t\right) \end{array}\right\} \Rightarrow \mathbf{A}\left(\mathbf{r}, t\right)$$

Aulas passadas

Gradiente de $f(\mathbf{r},t)$: $\nabla f = \frac{\partial f}{\partial x}\hat{\mathbf{x}} + \frac{\partial f}{\partial y}\hat{\mathbf{y}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}}$

- Direção e sentido de maior crescimento de f
- Módulo: taxa de crescimento naquela direção

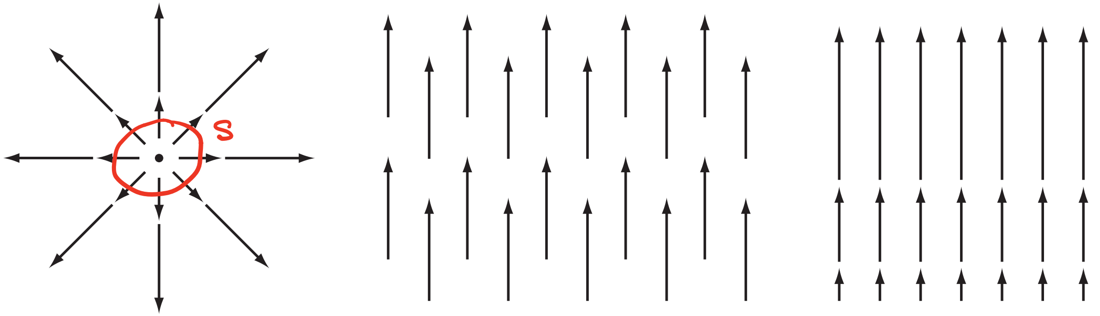


Aulas passadas

Divergente de $\mathbf{v}(\mathbf{r},t)$: $\nabla \cdot \mathbf{v} = \dfrac{\partial v_x}{\partial x} + \dfrac{\partial v_y}{\partial y} + \dfrac{\partial v_z}{\partial z}$

$$\nabla \cdot \mathbf{v} = \lim_{V \to 0} \left(\frac{\oint_{S(V)} \mathbf{v} \cdot d\mathbf{S}}{V} \right)$$

onde V é um volume que contém o ponto em questão e $S(V)$ é a superfície que contém V.



(a) $\nabla \cdot \mathbf{v} \neq 0$

(b) $\nabla \cdot \mathbf{v} = 0$

(c) $\nabla \cdot \mathbf{v} \neq 0$

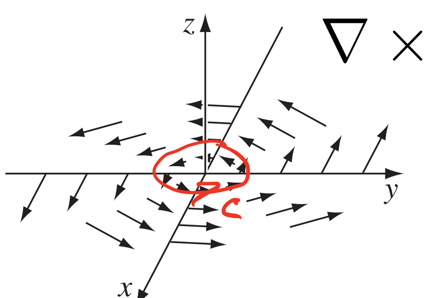
Aulas passadas

Rotacional de $\mathbf{v}(\mathbf{r},t)$:

$$\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$$

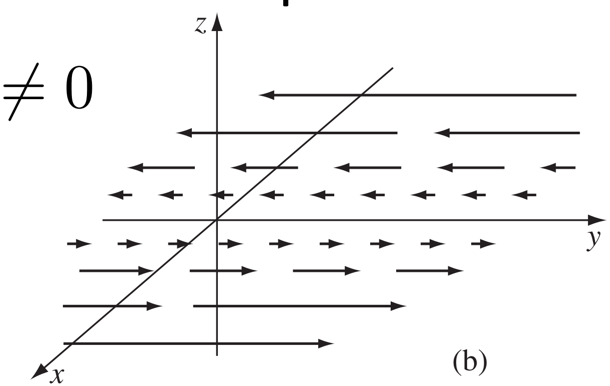
$$\nabla \times \mathbf{v} = \lim_{S\to 0}\left(\frac{\oint_{C(S)} \mathbf{v}\cdot d\mathbf{l}}{S}\right)$$

onde S é uma superfície aberta que contém o ponto em questão, normal à direção procurada do rotacional, e $C(S)$ é a borda da superfície S.



(a)

$$\nabla \times \mathbf{v} \neq 0$$



(b)

Aulas passadas

$$\int_a^b \frac{df}{dx}\,dx = f(b) - f(a)$$

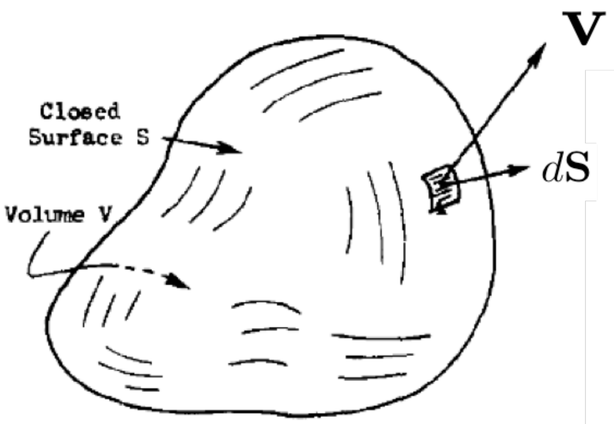
Teoremas fundamentais do cálculo vetorial:

Teorema do gradiente: $$\int_{\mathbf{r}_0}^{\mathbf{r}} (\nabla f) \cdot d\mathbf{l} = f(\mathbf{r}) - f(\mathbf{r}_0)$$

independente do caminho

Teorema do divergente (teorema de Gauss):

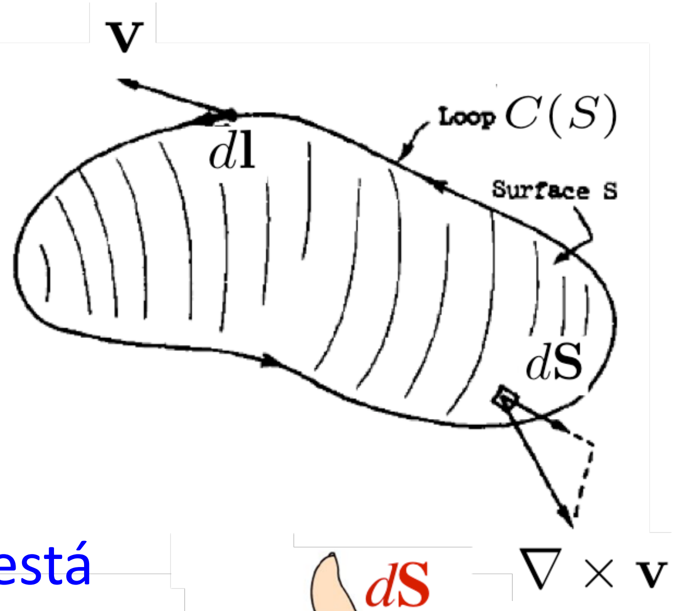
$$\int_V (\nabla \cdot \mathbf{v})\, dV = \oint_{S(V)} \mathbf{v} \cdot d\mathbf{S}$$



A normal $d\mathbf{S}$ sempre aponta pra fora da superfície.

Aulas passadas

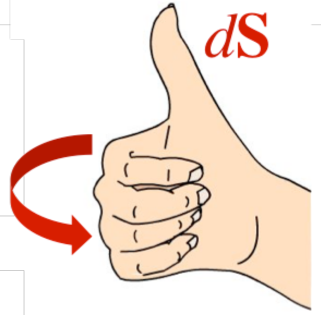
Teoremas fundamentais do cálculo vetorial:



Rotacional (teorema de Stokes):

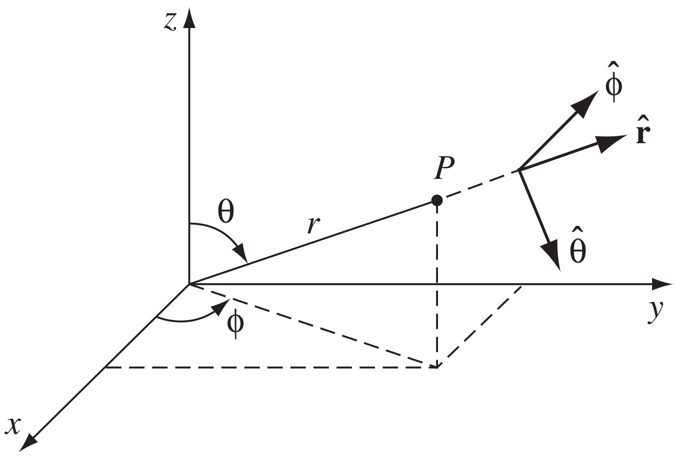
$$\int_S (\nabla \times \mathbf{v}) \cdot d\mathbf{S} = \oint_{C(S)} \mathbf{v} \cdot d\mathbf{l}$$

A direção de circulação de $C(S)$ está "amarrada" à direção da normal $d\mathbf{S}$ pela regra da mão direita.



Aulas passadas

Coordenadas esféricas:



$$
\begin{aligned}
x &= r\sin\theta\cos\phi \\
y &= r\sin\theta\sin\phi \\
z &= r\cos\theta
\end{aligned}
$$

$$
\begin{aligned}
r &= \sqrt{x^2+y^2+z^2} \\
\theta &= \arccos\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) \in [0,\pi] \\
\phi &= \arctan\left(\frac{y}{x}\right) \in [0,2\pi]
\end{aligned}
$$

$$
\begin{aligned}
\hat{\mathbf{r}} &= \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}, \\
\hat{\boldsymbol{\theta}} &= \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}}, \\
\hat{\boldsymbol{\phi}} &= -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}},
\end{aligned}
$$

UNITÁRIOS CURVILÍNEOS DEPENDEM DA POSIÇÃO

Gradient:

$$\nabla T = \frac{\partial T}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{\phi}}. \tag{1.70}$$

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}. \tag{1.71}$$

Curl:

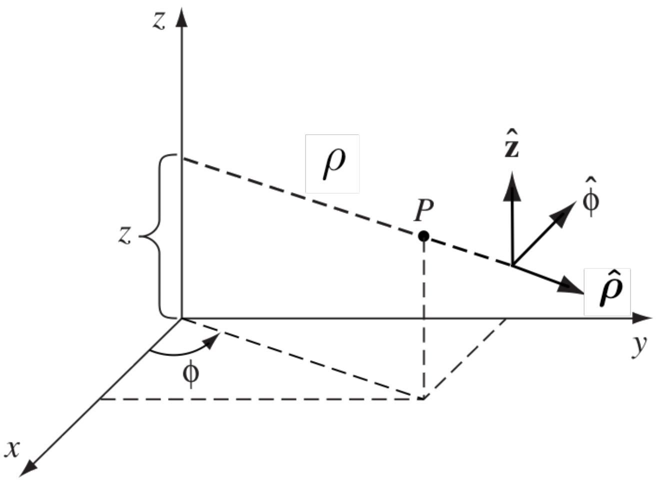
$$\begin{aligned}\nabla \times \mathbf{v} = {} & \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} \\ & + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}. \end{aligned} \tag{1.72}$$

Laplacian:

$$\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial \phi^2}. \tag{1.73}$$

Aulas passadas

Coordenadas cilíndricas:



$$x = \rho\cos\phi$$
$$y = \rho\sin\phi$$
$$z = z$$

$$\rho = \sqrt{x^2+y^2}$$
$$\phi = \arctan\left(\frac{y}{x}\right) \in [0, 2\pi]$$
$$z = z$$

$$\hat{\boldsymbol{\rho}} = \cos\phi\ \hat{\mathbf{x}} + \sin\phi\ \hat{\mathbf{y}}$$
$$\hat{\boldsymbol{\phi}} = -\sin\phi\ \hat{\mathbf{x}} + \cos\phi\ \hat{\mathbf{y}}$$
$$\hat{\mathbf{z}} = \hat{\mathbf{z}}$$

Gradient:

$$\nabla T = \frac{\partial T}{\partial s}\,\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial T}{\partial \phi}\,\hat{\boldsymbol{\phi}} + \frac{\partial T}{\partial z}\,\hat{\mathbf{z}}. \tag{1.79}$$

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}. \tag{1.80}$$

Curl:

$$\nabla \times \mathbf{v} = \left(\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right)\hat{\mathbf{s}} + \left(\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right)\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}}. \tag{1.81}$$

Laplacian:

$$\nabla^2 T = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial T}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2}. \tag{1.82}$$

Teorema de Helmholtz

(Apêndice B do Griffiths) Seja um campo vetorial $\mathbf{F}(\mathbf{r})$ tal que saibamos seu divergente e seu rotacional,

$$\boxed{\begin{aligned} \boldsymbol{\nabla} \cdot \mathbf{F} &= D(\mathbf{r}) \\ \boldsymbol{\nabla} \times \mathbf{F} &= \mathbf{C}(\mathbf{r}) \end{aligned}}$$

e tal que:

- $D(\mathbf{r})$ e $\mathbf{C}(\mathbf{r})$ caem a zero no infinito ($r \longrightarrow \infty$) mais rapidamente que $1/r^2$;
- $\mathbf{F}(\mathbf{r})$ cai a zero no infinito ($r \longrightarrow \infty$).

Segue que $\mathbf{F}(\mathbf{r})$ é <u>único</u> e dado por:

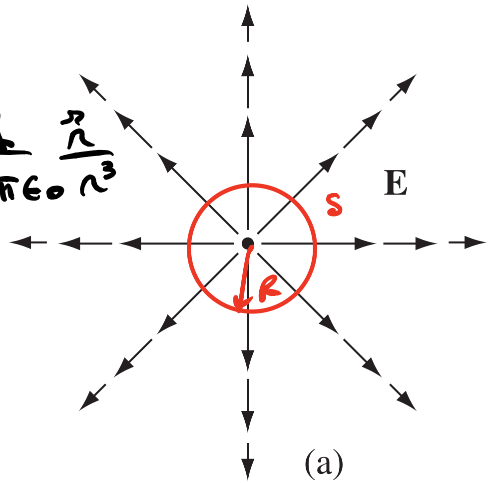
$$\begin{aligned} \mathbf{F} &= -\boldsymbol{\nabla} U + \boldsymbol{\nabla} \times \mathbf{W} \\ U(\mathbf{r}) &= \frac{1}{4\pi} \int \frac{D(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} dV', \\ \mathbf{W}(\mathbf{r}) &= \frac{1}{4\pi} \int \frac{\mathbf{C}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} dV'. \end{aligned}$$

A função delta de Dirac

Motivação: o campo elétrico de uma carga pontual

$$\vec{E}(\vec{r}) = \frac{q}{4\pi\epsilon_0}\frac{\hat{r}}{r^2} = \frac{q}{4\pi\epsilon_0}\frac{\vec{r}}{r^3}$$

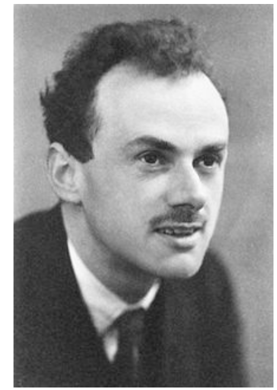
$$\hat{r} = \frac{\vec{r}}{r}$$



(a)

Paul A. M. Dirac

(1902-1984)



Nobel em Física

(1933)

$$\vec{\nabla}\cdot\vec{E} = \frac{q}{4\pi\epsilon_0}\vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(\cancel{r^2}\frac{1}{\cancel{r^2}}\right)\right] = 0$$

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0} \Longleftrightarrow \oint_{S(V)} \vec{E}\cdot d\vec{S} = \frac{1}{\epsilon_0}\int_V \rho\, dV = \frac{Q(V)}{\epsilon_0}$$

$$\frac{q}{\epsilon_0} = \frac{q}{\epsilon_0}$$

$$\int_{S(V)} \vec{E}\cdot d\vec{S} = \int_{S(V)} \left(\frac{q}{4\pi\epsilon_0}\frac{\hat{r}}{r^2}\bigg|_{r=R}\right)\cdot\underbrace{\left(R^2 \sin\theta\, d\theta\, d\phi\right)\hat{r}}_{d\vec{S}}$$

$$= \frac{q}{4\pi\epsilon_0}\underbrace{\int_0^{\pi}\int_0^{2\pi}\frac{1}{\cancel{R^2}}\cancel{R^2}\sin\theta\, d\theta\, d\phi}_{4\pi} = \frac{q}{4\pi\epsilon_0}4\pi = \frac{q}{\epsilon_0}$$

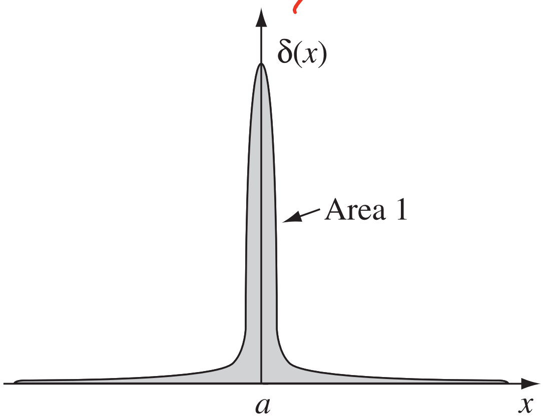
$$\oint_{S(V)} \vec{E}\cdot d\vec{S} = \int_V (\vec{\nabla}\cdot\vec{E})\, dV \overset{?}{=} 0 \qquad !\ !$$

O PROBLEMA VEM DA SINGULARIDADE DE $\vec{E}$ NA ORIGEM ($r=0$). COMO LIDAR COM ELA?

COMO LIDAR COM UMA QUANTIDADE FINITA DE CARGA (q) NUM VOLUME NULO ($V=0$)?

$$\Rightarrow \rho \rightarrow \frac{q}{V} \rightarrow \infty$$

A função delta de Dirac em 1D



$\delta(x)$ DE UMA CARGA PONTUAL EM 1D

VOU TENTAR DEFINIR UMA FUNÇÃO QUE:

a) SEJA MUITO GRANDE $(\to \infty)$ NA ORIGEM

b) SEJA ZERO FORA DA ORIGEM $(\to 0)$

c) SUA INTEGRAL SEJA FINITA (DIGAMOS = 1 PARA $q=1$)

O QUE É PRECISO É DEFINIR $\delta(x)$ COMO UM PROCESSO DE LIMITE. MAIS ESPECIFICAMENTE, VOU DEFINIR UMA FAMÍLIA DE FUNÇÕES A UM PARÂMETRO QUE SATISFAÇA AS CONDIÇÕES ACIMA

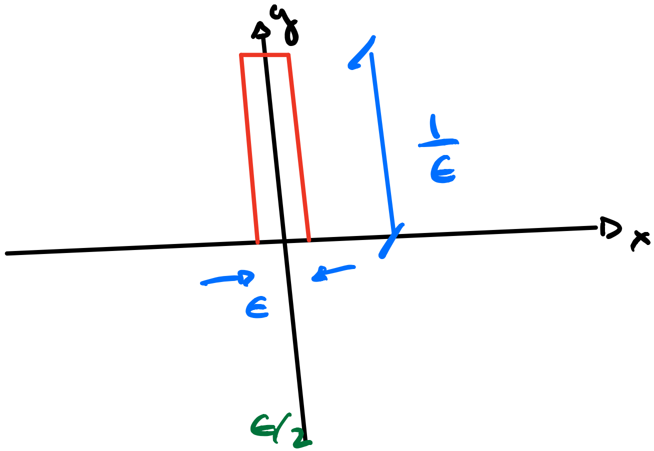
$\Rightarrow \delta_\epsilon(x)$ "DISTRIBUIÇÃO"

a) $\lim\limits_{\epsilon \to 0^+} \delta_\epsilon(0) = \infty$

b) $\lim\limits_{\epsilon \to 0^+} \delta_\epsilon(x) = 0$ SE $x \neq 0$

c) $\lim\limits_{\epsilon \to 0^+} \int_a^b \delta_\epsilon(x)\,dx = 1$ ONDE $x = 0 \in [a, b]$

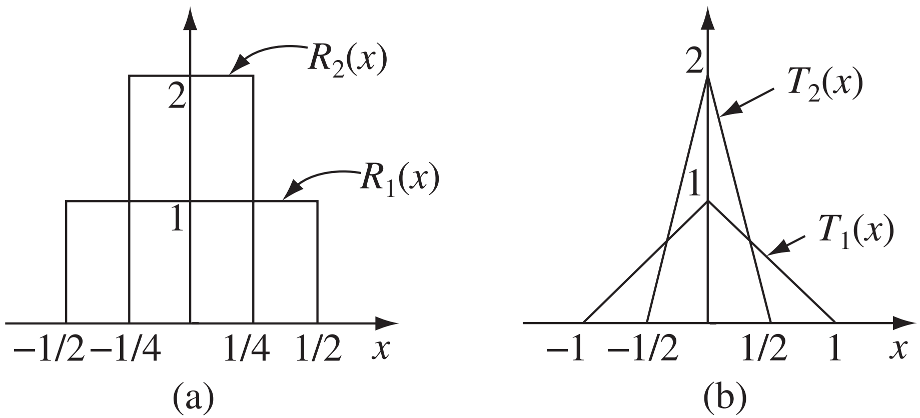
COMO JÁ DITO, ISSO É UM EXEMPLO DE UM NOVO OBJETO MATEMÁTICO CHAMADO "DISTRIBUIÇÃO".



$$\delta_\epsilon(x) = \begin{cases} \frac{1}{\epsilon} & \text{SE } x \in \left[-\frac{\epsilon}{2}, \frac{\epsilon}{2}\right] \\ 0 & \text{DO CONTRÁRIO} \end{cases}$$

$$\int_a^b \delta_\epsilon(x)\,dx = \int_{-\frac{\epsilon}{2}}^{\epsilon/2} \frac{1}{\epsilon}\,dx = 1 \checkmark$$

Sequências de funções que dão a delta



$$\delta_\epsilon(x) = \frac{1}{\pi\epsilon}\,\frac{1}{1+\left(\frac{x}{\epsilon}\right)^2}$$

$$\delta_\epsilon(x) = \frac{\sin\left(\frac{x}{\epsilon}\right)}{\pi x}$$

$$\delta_\epsilon(x) = \frac{1}{\sqrt{\pi}\,\epsilon}\exp\left[-\left(\frac{x}{\epsilon}\right)^2\right]$$

Propriedades importantes da função delta de Dirac

(i) PROPRIEDADE DE FILTRAGEM: (SEMPRE $x=0 \in [a,b]$)

$$\int_a^b f(x)\,\delta(x)\,dx = f(0)$$

ONDE $f(x)$ É UMA FUNÇÃO CONTÍNUA FINITA QUALQUER

COMO $\delta(x) \to 0$ SE $x \neq 0$, EU POSSO ESCREVER QUE

$$\int_a^b f(x)\,\delta(x)\,dx \longrightarrow \int_a^b f(0)\,\delta(x)\,dx = f(0)\int_a^b \delta(x)\,dx = f(0)$$

DE MANEIRA GERAL:

$$\int_{-\infty}^{+\infty} f(x)\,\delta(x-x_0)\,dx = f(x_0)$$

$$\text{(ii)} \int_{-\infty}^{+\infty} f(x)\,\delta(kx)\,dx \qquad (k \neq 0)$$

$y = kx$ TROCA DE VARIÁVEIS

$$\Rightarrow \int_{-\infty}^{+\infty} f\left(\frac{y}{k}\right)\delta(y)\frac{dy}{|k|} = \frac{1}{|k|} f(0)$$

$$\Rightarrow \int_{-\infty}^{+\infty} f(x)\,\delta(kx)\,dx = \frac{f(0)}{|k|}$$

$$\text{(iii)} \int_{-\infty}^{+\infty} f(x)\,\delta'(x)\,dx = -f'(0)$$

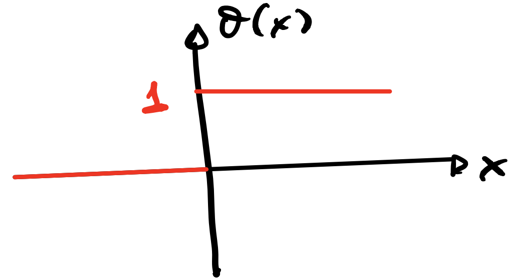
$$\int_{-\infty}^{+\infty} f(x)\,\delta'(x)\,dx = \underbrace{f(x)\,\delta(x)\Big|_{x\to-\infty}^{x\to\infty}}_{0} - \int_{-\infty}^{+\infty} f'(x)\,\delta(x)\,dx = -f'(0)$$

EM GERAL, $\displaystyle\int_{-\infty}^{+\infty} f(x)\,\delta^{(n)}(x)\,dx = (-1)^n f^{(n)}(0)$ ONDE

O SUPER-ÍNDICE (n) INDICA A n-ÉSIMA DERIVADA

(iv) SEJA $\theta(x)$ A FUNÇÃO DEGRAU DE HEAVISIDE

$$\theta(x) = \begin{cases} 1 & \text{SE } x > 0 \\ 0 & \text{SE } x < 0 \end{cases}$$



SEGUE QUE $\dfrac{d\theta}{dx} = \delta(x)$

$$I = \int_{-\infty}^{+\infty} f(x)\,\frac{d\theta}{dx}\,dx = \lim_{A\to\infty}\left[\int_{-A}^{A} f(x)\,\frac{d\theta}{dx}\,dx\right] = f(0)$$

$$\int_{-A}^{A} f(x)\,\frac{d\theta}{dx}\,dx = f(x)\,\theta(x)\Big|_{x=-A}^{x=A} - \int_{-A}^{A}\theta(x)\,\frac{df}{dx}\,dx$$

$$= f(A) - \int_{0}^{A}\frac{df}{dx}\,dx = f(A) - \left[f(A) - f(0)\right] = f(0)$$

PELA PROPRIEDADE DE FILTRAGEM $\dfrac{d\theta}{dx} = \delta(x)$

A função delta de Dirac em 3D

ASSIM, EM 1D, A CARGA PONTUAL q É DESCRITA POR:

$$\rho(x) = q\,\delta(x)$$

EM 3D, ESCREVEMOS: $\rho(\vec{r}) = q\,\underbrace{\delta(x)\delta(y)\delta(z)}_{\delta^{(3)}(\vec{r})}$

$$\rho(\vec{r}) = q\,\delta^{(3)}(\vec{r})$$

$$\int \rho(\vec{r})\,dV = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} dz \; q\,\delta^{(3)}(\vec{r}) = q \quad \text{OU} \quad \int \delta^{(3)}(\vec{r})\,dV = 1$$

FILTRAGEM: $\displaystyle\int_V f(\vec{r})\,\delta^{(3)}(\vec{r})\,dV = \underbrace{f(\vec{r}=0)}_{f(x=0,\,y=0,\,z=0)}$ SE $\vec{r}=0 \in V$

VOLTANDO AO PROBLEMA ORIGINAL: $\vec{E}(\vec{r}) = \dfrac{q}{4\pi\epsilon_0}\dfrac{\hat{r}}{r^2}$

$$\begin{cases} \vec{\nabla}\cdot\vec{E} = 0 & \text{SE } \vec{r} \neq 0 \text{ (FORA DA ORIGEM)} \\ \vec{\nabla}\cdot\vec{E} \neq 0 & \text{SE } \vec{r} = 0 \end{cases}$$

$$\boxed{\vec{\nabla}\cdot\vec{E} = \frac{q}{\epsilon_0}\delta^{(3)}(\vec{r})} \Rightarrow \int_V \vec{\nabla}\cdot\vec{E}\, dV = \int_V \frac{q}{\epsilon_0}\delta^{(3)}(\vec{r})\, dV = \frac{q}{\epsilon_0}$$

$$\Rightarrow \vec{\nabla}\cdot\left[\frac{q}{4\pi\epsilon_0}\frac{\hat{r}}{r^2}\right] = \frac{q}{\epsilon_0}\delta^{(3)}(\vec{r})$$

$$\Rightarrow \frac{\cancel{q}}{4\pi\cancel{\epsilon_0}}\vec{\nabla}\cdot\left[\frac{\hat{r}}{r^2}\right] = \frac{\cancel{q}}{\cancel{\epsilon_0}}\delta^{(3)}(\vec{r}) \Rightarrow \boxed{\vec{\nabla}\cdot\left[\frac{\hat{r}}{r^2}\right] = 4\pi\,\delta^{(3)}(\vec{r})} \quad (1)$$

$$\hat{r} = \frac{\vec{r}}{r} \Rightarrow \boxed{\vec{\nabla}\cdot\left[\frac{\vec{r}}{r^3}\right] = 4\pi\,\delta^{(3)}(\vec{r})} \quad (2)$$

SE A CARGA PONTUAL NÃO ESTIVER NA ORIGEM MAS EM $\vec{r}_0$!

$$\boxed{\vec{\nabla}\cdot\left[\frac{(\vec{r}-\vec{r}_0)}{|\vec{r}-\vec{r}_0|^3}\right] = 4\pi\,\delta^{(3)}(\vec{r}-\vec{r}_0)} \quad (3)$$

QUAL É O GRADIENTE DE $f(\vec{r}) = \frac{1}{r} = \frac{1}{|\vec{r}|}$?

$$\vec{\nabla} f = \vec{\nabla}\left(\frac{1}{r}\right) = \frac{\partial\left(\frac{1}{r}\right)}{\partial r}\,\hat{r} = -\frac{1}{r^2}\,\hat{r} = -\frac{\hat{r}}{r^2}$$

LEVANDO EM (1):

$$\vec{\nabla}\cdot\left[\frac{\hat{r}}{r^2}\right] = -\vec{\nabla}\cdot\left[\vec{\nabla}\left(\frac{1}{r}\right)\right] = -\nabla^2\left(\frac{1}{r}\right) = 4\pi\,\delta^{(3)}(\vec{r})$$

$$\Rightarrow \boxed{\nabla^2\left(\frac{1}{r}\right) = -4\pi\,\delta^{(3)}(\vec{r})} \quad (4)$$

DESLOCANDO NOVAMENTE DA ORIGEM PARA $\vec{r}_0$:

$$r = |\vec{r}| \longrightarrow |\vec{r} - \vec{r}_0|$$

$$\Rightarrow \boxed{\nabla^2 \left[\frac{1}{|\vec{r} - \vec{r}_0|} \right] = -4\pi \, \delta^{(3)}(\vec{r} - \vec{r}_0)} \quad (5)$$