# F 689 – Mecânica Quântica I

2º Semestre de 2022
21/09/2022
Aula 10

# Aula passada

Se dois operadores, $A$ e $B$, comutam então, na base $\left\{\left|u_n^i\right\rangle\right\}$ que diagonaliza $A$:

$$A=\begin{pmatrix} a_1 & 0 & 0 & 0 & 0 & \cdots & \cdots \\ 0 & a_1 & 0 & 0 & 0 & \cdots & \cdots \\ 0 & 0 & a_1 & 0 & 0 & \cdots & \cdots \\ 0 & 0 & 0 & a_2 & 0 & \cdots & \cdots \\ 0 & 0 & 0 & 0 & a_2 & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \qquad B=\begin{pmatrix} b_{11}^1 & b_{12}^1 & b_{13}^1 & 0 & 0 & \cdots & \cdots \\ b_{21}^1 & b_{22}^1 & b_{23}^1 & 0 & 0 & \cdots & \cdots \\ b_{31}^1 & b_{32}^1 & b_{33}^1 & 0 & 0 & \cdots & \cdots \\ 0 & 0 & 0 & b_{11}^2 & b_{12}^2 & \cdots & \cdots \\ 0 & 0 & 0 & b_{21}^2 & b_{22}^2 & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Mudando para a base que diagonaliza cada bloco de $B$ separadamente e, notando que os blocos correspondentes de $A$ não mudam:

$$A=\begin{pmatrix} a_1 & 0 & 0 & 0 & 0 & \cdots & \cdots \\ 0 & a_1 & 0 & 0 & 0 & \cdots & \cdots \\ 0 & 0 & a_1 & 0 & 0 & \cdots & \cdots \\ 0 & 0 & 0 & a_2 & 0 & \cdots & \cdots \\ 0 & 0 & 0 & 0 & a_2 & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \qquad B=\begin{pmatrix} b_1 & 0 & 0 & 0 & 0 & \cdots & \cdots \\ 0 & b_2 & 0 & 0 & 0 & \cdots & \cdots \\ 0 & 0 & b_3 & 0 & 0 & \cdots & \cdots \\ 0 & 0 & 0 & b_3 & 0 & \cdots & \cdots \\ 0 & 0 & 0 & 0 & b_4 & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

# Aula passada

Há duas possibilidades:

A) os auto-valores simultâneos de $A$ e $B$ identificam univocamente cada auto-estado

$$A = \begin{pmatrix} a_1 & 0 & 0 & 0 & 0 & \cdots & \cdots \\ 0 & a_1 & 0 & 0 & 0 & \cdots & \cdots \\ 0 & 0 & a_1 & 0 & 0 & \cdots & \cdots \\ 0 & 0 & 0 & a_2 & 0 & \cdots & \cdots \\ 0 & 0 & 0 & 0 & a_2 & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \qquad B = \begin{pmatrix} b_1 & 0 & 0 & 0 & 0 & \cdots & \cdots \\ 0 & b_2 & 0 & 0 & 0 & \cdots & \cdots \\ 0 & 0 & b_3 & 0 & 0 & \cdots & \cdots \\ 0 & 0 & 0 & b_3 & 0 & \cdots & \cdots \\ 0 & 0 & 0 & 0 & b_4 & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Nesse caso, $A$ e $B$ formam um CCOC: Conjunto Completo de Operadores que Comutam. Existe uma única base de auto-vetores simultâneos de $A$ e $B$.

# Aula passada

B) os auto-valores simultâneos de $A$ e $B$ não identificam univocamente cada auto-estado

$$A = \begin{pmatrix} a_1 & 0 & 0 & 0 & 0 & \cdots & \cdots \\ 0 & a_1 & 0 & 0 & 0 & \cdots & \cdots \\ 0 & 0 & a_1 & 0 & 0 & \cdots & \cdots \\ 0 & 0 & 0 & a_2 & 0 & \cdots & \cdots \\ 0 & 0 & 0 & 0 & a_2 & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \qquad B = \begin{pmatrix} b_1 & 0 & 0 & 0 & 0 & \cdots & \cdots \\ 0 & b_2 & 0 & 0 & 0 & \cdots & \cdots \\ 0 & 0 & b_2 & 0 & 0 & \cdots & \cdots \\ 0 & 0 & 0 & b_2 & 0 & \cdots & \cdots \\ 0 & 0 & 0 & 0 & b_3 & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Se quisermos encontrar um CCOC, precisamos adicionar mais operadores ($C$, $D$,...), todos comutando entre si, até que os auto-valores simultâneos de todos eles identifiquem univocamente cada auto-vetor.

# As representações |r> e |p>

BASES CONTÍNUAS EM $\mathcal{F}$: $|w_\alpha\rangle \quad \alpha \in [a, b]$

DOIS CASOS PARTICULARES:

$$\begin{cases} v_{\vec{p}_0}(\vec{r}) = \dfrac{e^{i\vec{p}_0\cdot\vec{r}/\hbar}}{(2\pi\hbar)^{3/2}} & \text{(ONDAS PLANAS)} \quad \vec{p}_0 = (p_{0x}, p_{0y}, p_{0z}) \\ \xi_{\vec{r}_0}(\vec{r}) = \delta(\vec{r}-\vec{r}_0) & \text{(DELTAS DE DIRAC)} \quad \vec{r}_0 (x_0, y_0, z_0) \end{cases}$$

$\rightarrow \notin \mathcal{F}$

$v_{\vec{p}_0}(\vec{r}) \notin \mathcal{F} \longrightarrow |\vec{p}_0\rangle \notin \mathcal{E}$

$\xi_{\vec{r}_0}(\vec{r}) \notin \mathcal{F} \longrightarrow |\vec{r}_0\rangle \notin \mathcal{E}$

NÃO ESQUECER QUE $|\vec{p}_0\rangle, |\vec{r}_0\rangle$ SÃO NÃO NORMALIZÁVEIS (KETS GENERALIZADOS).

# Ortonormalidade

CASO GERAL : $\langle w_{\alpha'} | w_\alpha \rangle = \delta(\alpha - \alpha')$

$$\langle \vec{r}_0' | \vec{r}_0 \rangle = \int d^3r \, \xi^*_{\vec{r}_0'}(\vec{r}) \, \xi_{\vec{r}_0}(\vec{r}) = \int d^3r \, \delta(\vec{r} - \vec{r}_0') \, \delta(\vec{r} - \vec{r}_0)$$

$$= \delta(\vec{r}_0 - \vec{r}_0') = \delta(\vec{r}_0' - \vec{r}_0)$$

$$\langle \vec{p}_0' | \vec{p}_0 \rangle = \int d^3r \, v^*_{\vec{p}_0'}(\vec{r}) \, v_{\vec{p}_0}(\vec{r}) = \int d^3r \, \frac{e^{-i\vec{p}_0'\cdot\vec{r}/\hbar} \, e^{i\vec{p}_0\cdot\vec{r}/\hbar}}{(2\pi\hbar)^3}$$

$$= \frac{1}{(2\pi\hbar)^3} \int d^3r \, e^{i\vec{r}\cdot(\vec{p}_0 - \vec{p}_0')/\hbar} = \delta(\vec{p}_0 - \vec{p}_0')$$

USANDO $\int dx \, e^{ixp} = 2\pi \delta(p)$

# Fechamento

DE MANEIRA GERAL: $\int d\alpha \, |w_\alpha\rangle\langle w_\alpha| = \mathbb{1}$

SEGUE QUE:

$$\int d^3r_0 \, |\vec{r}_0\rangle\langle\vec{r}_0| = \mathbb{1}$$

$$\int d^3p_0 \, |\vec{p}_0\rangle\langle\vec{p}_0| = \mathbb{1}$$

# Expansão de kets nas duas bases

DADO $|\psi\rangle$ : $|\psi\rangle = \int d\alpha \, c(\alpha) \, |w_\alpha\rangle$ ; $c(\alpha) = \langle w_\alpha|\psi\rangle$

JÁ QUE: $|\psi\rangle = \underbrace{\int d\alpha \, |w_\alpha\rangle\langle w_\alpha|}_{\mathbb{1}} \psi\rangle = \int d\alpha \, c(\alpha) \, |w_\alpha\rangle$

NA REPRESENTAÇÃO $|\vec{r}_0\rangle$:

$$|\psi\rangle = \int d^3r_0 \, |\vec{r}_0\rangle\langle\vec{r}_0|\psi\rangle = \int d^3r \, [\langle\vec{r}_0|\psi\rangle] \, |\vec{r}_0\rangle$$

$$\langle\vec{r}_0|\psi\rangle = \int d^3r \, \xi^*_{\vec{r}_0}(\vec{r}) \, \psi(\vec{r}) = \int d^3r \, \delta(\vec{r}-\vec{r}_0) \, \psi(\vec{r}) = \psi(\vec{r}_0)$$

$$\Rightarrow |\psi\rangle = \int d^3r \, \psi(\vec{r}_0) \, |\vec{r}_0\rangle$$

$\psi(\vec{r}_0)$ (FUNÇÃO DE ONDA) PODE SER VISTA COMO OS COEFICIENTES DA EXPANSÃO DE $|\psi\rangle$ NA BASE $|\vec{r}_0\rangle$.

PARA ONDAS PLANAS:

$$|\psi\rangle = \int d^3p_0 \, |\vec{p}_0\rangle\langle\vec{p}_0|\psi\rangle$$

$$\langle\vec{p}_0|\psi\rangle = \int d^3r \; \phi^*_{\vec{p}_0}(\vec{r}) \, \psi(\vec{r})$$

$$= \int d^3r \; \frac{e^{-i\vec{p}_0\cdot\vec{r}/\hbar}}{(2\pi\hbar)^{3/2}} \, \psi(\vec{r}) \equiv \Psi(\vec{p}_0)$$

$$|\psi\rangle = \int d^3p_0 \; \Psi(\vec{p}_0) \, |\vec{p}_0\rangle$$

A função de onda e sua transformada de Fourier podem ser vistas como coeficientes de expansão nas bases |**r**> e |**p**>, ou ainda como produtos escalares com os vetores dessas bases:

$$\boxed{\begin{aligned} \psi(\mathbf{r}) &= \langle \mathbf{r}|\psi\rangle \\ \overline{\psi}(\mathbf{p}) &= \langle \mathbf{p}|\psi\rangle \end{aligned}}$$

$$\langle \psi|\vec{r}\rangle = (\langle \vec{r}|\psi\rangle)^* = \psi^*(\vec{r})$$

$$\langle \psi|\vec{p}\rangle = (\langle \vec{p}|\psi\rangle)^* = \overline{\psi}^*(\vec{p})$$

# A relação de fechamento em bases discretas

Para bases discretas: $\langle u_i | u_j \rangle = \delta_{ij}$

$$\sum_i |u_i\rangle \langle u_i| = \mathbb{1}$$

APLICANDO O BRA $\langle \vec{r}|$ PELA ESQUERDA E O KET $|\vec{r}'\rangle$ PELA DIREITA NO FECHAMENTO:

$$\sum_i \underbrace{\langle \vec{r} | u_i \rangle}_{u_i(\vec{r})} \underbrace{\langle u_i | \vec{r}' \rangle}_{u_i^*(\vec{r}')} = \langle \vec{r} | \vec{r}' \rangle = \delta(\vec{r} - \vec{r}')$$

$$\Rightarrow \sum_i u_i^*(\vec{r}') \, u_i(\vec{r}) = \delta(\vec{r} - \vec{r}')$$

QUE É O FECHAMENTO EM $\mathcal{F}$

# Recuperando a expressão usual do produto escalar em $\mathcal{F}$

DOIS ESTADOS $|\psi\rangle$ E $|\varphi\rangle$ :

$$\langle\varphi|\psi\rangle = \int d^3r \, \underbrace{\langle\varphi|\vec{r}\rangle}_{\varphi^*(\vec{r})}\underbrace{\langle\vec{r}|\psi\rangle}_{\psi(\vec{r})} = \int d^3r \, \varphi^*(\vec{r})\,\psi(\vec{r})$$

$$\langle\varphi|\psi\rangle = \int d^3p \, \underbrace{\langle\varphi|\vec{p}\rangle}_{\bar{\varphi}^*(\vec{p})}\underbrace{\langle\vec{p}|\psi\rangle}_{\bar{\psi}(\vec{p})} = \int d^3p \, \bar{\varphi}^*(\vec{p})\,\bar{\psi}(\vec{p})$$

QUE PODE SER PROVADA TAMBÉM
(APÊNDICE I, 2c DO COHEN)

# Mudança de base/representação entre |r> e |p>

Mudança de bases no caso discreto:

$$c_i = \sum_j S_{ij} d_j$$

$$\boxed{\begin{matrix} S_{ij} = \langle u_i | t_j \rangle \\ S^\dagger = S^{-1} \end{matrix}}$$

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} & S_{13} & \cdots \\ S_{21} & S_{22} & S_{23} & \cdots \\ S_{31} & S_{32} & S_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \end{pmatrix} \qquad \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \end{pmatrix} = \begin{pmatrix} S^\dagger_{11} & S^\dagger_{12} & S^\dagger_{13} & \cdots \\ S^\dagger_{21} & S^\dagger_{22} & S^\dagger_{23} & \cdots \\ S^\dagger_{31} & S^\dagger_{32} & S^\dagger_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \end{pmatrix}$$

ANALOGAMENTE:

$$S(\vec{r},\vec{p}) = \langle \vec{r} | \vec{p} \rangle = \frac{e^{i\vec{p}\cdot\vec{r}/\hbar}}{(2\pi\hbar)^{3/2}}$$

$$\langle \vec{r} | \psi \rangle = \psi(\vec{r}) = \int d^3p \, S(\vec{r},\vec{p}) \langle \vec{p} | \psi \rangle = \int d^3p \, \frac{e^{i\vec{p}\cdot\vec{r}/\hbar}}{(2\pi\hbar)^{3/2}} \Phi(\vec{p})$$

$$\langle \vec{p} | \psi \rangle = \Phi(\vec{p}) = \int d^3r \, \underbrace{S^\dagger(\vec{p},\vec{r})}_{S^*(\vec{r},\vec{p}) = \frac{e^{-i\vec{p}\cdot\vec{r}/\hbar}}{(2\pi\hbar)^{3/2}}} \langle \vec{r} | \psi \rangle$$

$$\Rightarrow \Phi(\vec{p}) = \int d^3r \, \frac{e^{-i\vec{p}\cdot\vec{r}/\hbar}}{(2\pi\hbar)^{3/2}} \psi(\vec{r})$$

# Auto-vetores e auto-valores de $X$ e $P_x$

JÁ TÍNHAMOS VISTO: $X\,\xi_{\vec{r}_0}(\vec{r}) = x\,\xi_{\vec{r}_0}(\vec{r}) = x\,\delta(\vec{r}-\vec{r}_0)$

$= x\,\delta(x-x_0)\,\delta(y-y_0)\,\delta(z-z_0)$

MAS $f(x)\,\delta(x-x_0) = f(x_0)\,\delta(x-x_0)$

JÁ QUE $\int f(x)\,\delta(x-x_0)\,dx = f(x_0) = \int f(x_0)\,\delta(x-x_0)\,dx$

$X\,\xi_{\vec{r}_0}(\vec{r}) = x_0\,\delta(\vec{r}-\vec{r}_0) = x_0\,\xi_{\vec{r}_0}(\vec{r})$

$\xi_{\vec{r}_0}(\vec{r})$ É AUTO-FUNÇÃO DE $X$ COM AUTO-VALOR $x_0$.

$\Rightarrow \boxed{X\,|\vec{r}_0\rangle = x_0\,|\vec{r}_0\rangle}$

$Y\,|\vec{r}_0\rangle = y_0\,|\vec{r}_0\rangle$

$Z\,|\vec{r}_0\rangle = z_0\,|\vec{r}_0\rangle$

$$P_x \varphi_{\vec{p}_0}(\vec{r}) = \frac{\hbar}{i}\frac{\partial}{\partial x}\left[\frac{e^{i\vec{p}_0\cdot\vec{r}/\hbar}}{(2\pi\hbar)^{3/2}}\right]$$

$\vec{p}_0\cdot\vec{r} = p_{0x}x + p_{0y}y + p_{0z}z$

$$= \frac{\cancel{\hbar}}{\cancel{i}}\left(\frac{\cancel{i}p_{0x}}{\cancel{\hbar}}\right)\frac{e^{i\vec{p}_0\cdot\vec{r}/\hbar}}{(2\pi\hbar)^{3/2}}$$

$$P_x \varphi_{\vec{p}_0}(\vec{r}) = p_{0x}\varphi_{\vec{p}_0}(\vec{r})$$

$\varphi_{\vec{p}_0}(\vec{r})$ É AUTO-FUNÇÃO DE $P_x$ COM AUTO-VALOR $p_{0x}$:

$$P_x|\vec{p}_0\rangle = p_{0x}|\vec{p}_0\rangle$$

$$P_y|\vec{p}_0\rangle = p_{0y}|\vec{p}_0\rangle$$

$$P_z|\vec{p}_0\rangle = p_{0z}|\vec{p}_0\rangle$$

COMPACTAMENTE:

$$\vec{R}|\vec{r}_0\rangle = \vec{r}_0|\vec{r}_0\rangle$$

$$\vec{P}|\vec{p}_0\rangle = \vec{p}_0|\vec{p}_0\rangle$$

Como $|\mathbf{r}>$ e $|\mathbf{p}>$, são, cada um, bases do espaço $\mathcal{E}$, os operadores $(X,Y,Z)$ e $(P_x,P_y,P_z)$ são observáveis.

$(X,Y,Z)$ formam um conjunto completo de operadores que comutam. O mesmo vale para $(P_x,P_y,P_z)$.

# Como $X$ e $P_x$ atuam em $\overline{\psi}(\mathbf{p})$

SABEMOS QUE: $X\psi(\vec{r}) = x\psi(\vec{r})$ $\quad P_x\psi(\vec{r}) = \frac{\hbar}{i}\frac{\partial}{\partial x}\psi(\vec{r})$

COMO $X$ E $P_x$ ATUAM NA TRANSFORMADA DE FOURIER $\overline{\psi}(\vec{p})$?

$X|\vec{r}\rangle = x|\vec{r}\rangle$

$\langle\vec{r}|X = x\langle\vec{r}|$

→ 1º PARTE: EM NOTAÇÃO DE DIRAC:

$$X|\psi\rangle = |\psi'\rangle$$

TOMAND $\langle\vec{r}|$ NA EQUAÇÃO ACIMA:

$$\langle\vec{r}|X|\psi\rangle = \langle\vec{r}|\psi'\rangle = \psi'(\vec{r})$$

$$(\langle\vec{r}|X|\psi\rangle)^* = \langle\psi|X^\dagger|\vec{r}\rangle = \langle\psi|\left[X|\vec{r}\rangle\right] = x\langle\psi|\vec{r}\rangle$$

$$\Rightarrow \langle\vec{r}|X|\psi\rangle = x\langle\vec{r}|\psi\rangle = x\psi(\vec{r})$$

$$\Rightarrow x\psi(\vec{r}) = \psi'(\vec{r}) = X\psi(\vec{r})$$

$$P_x|\psi\rangle = |\psi'\rangle \Rightarrow \langle\vec{r}|P_x|\psi\rangle = \langle\vec{r}|\psi'\rangle = \psi'(\vec{r})$$

$$\langle\vec{r}|P_x|\psi\rangle = \int d^3p\, \langle\vec{r}|P_x|\vec{p}\rangle\langle\vec{p}|\psi\rangle =$$

$$= \int d^3p\,(p_x)\langle\vec{r}|\vec{p}\rangle\langle\vec{p}|\psi\rangle$$

$$= \int d^3p\,(p_x)\,\frac{e^{i\vec{p}\cdot\vec{r}/\hbar}}{(2\pi\hbar)^{3/2}}\,\bar{\psi}(\vec{p})$$

$$= \int d^3p\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)\left(\frac{e^{i\vec{p}\cdot\vec{r}/\hbar}}{(2\pi\hbar)^{3/2}}\right)\bar{\psi}(\vec{p})$$

$$= \frac{\hbar}{i}\frac{\partial}{\partial x}\underbrace{\left[\int d^3p\,\frac{e^{i\vec{p}\cdot\vec{r}/\hbar}}{(2\pi\hbar)^{3/2}}\,\bar{\psi}(\vec{p})\right]}_{\int d^3p\,\langle\vec{r}|\vec{p}\rangle\langle\vec{p}|\psi\rangle = \langle\vec{r}|\psi\rangle = \psi(\vec{r})}$$

$$= \frac{\hbar}{i}\frac{\partial\psi(\vec{r})}{\partial x} = \psi'(\vec{r}) = P_x\,\psi(\vec{r})$$

$$P_x \bar{\Psi}(\bar{p}) = ?$$

$$\langle \vec{p} | P_x | \psi \rangle = \langle \vec{p} | \psi' \rangle = \bar{\Psi}'(\vec{p})$$

$$\shortparallel \qquad\qquad\qquad\qquad \shortparallel$$

$$p_x \langle \vec{p} | \psi \rangle = p_x \bar{\Psi}(\bar{p})$$

$$P_x \bar{\Psi}(\bar{p}) = p_x \bar{\Psi}(\bar{p})$$

$$X \bar{\Psi}(\vec{p}) = \bar{\Psi}'(\vec{p})$$

$$\langle \vec{p} | X | \psi \rangle = \langle \vec{p} | \psi' \rangle = \bar{\Psi}'(\vec{p})$$

$$\shortparallel$$

$$\int d^3 r \langle \vec{p} | X | \vec{r} \rangle \langle \vec{r} | \psi \rangle =$$

$$= \int d^3 r \, x \langle \vec{p} | \vec{r} \rangle \langle \vec{r} | \psi \rangle$$

$$= \int d^3 r \, x \frac{e^{-i\vec{p}\cdot\vec{r}/\hbar}}{(\quad)^{3/2}} \psi(\vec{r})$$

$$= \int d^3 r (i\hbar) \frac{\partial}{\partial p_x} \left[ \frac{e^{-i\vec{p}\cdot\vec{r}/\hbar}}{(\quad)^{3/2}} \right] \psi(\vec{r})$$

$$= i\hbar \frac{\partial}{\partial p_x} \int d^3 x \, \langle \vec{p} | \vec{x} \rangle \langle \vec{x} | \psi \rangle$$

$$= i\hbar \frac{\partial}{\partial p_x} \langle \vec{p} | \psi \rangle = i\hbar \frac{\partial}{\partial p_x} \bar{\psi}(\vec{p})$$

$$\Rightarrow X \bar{\psi}(\vec{p}) = i\hbar \frac{\partial}{\partial p_x} \bar{\psi}(\vec{p})$$

$$(XP_x - P_x X)\overline{\psi}(\overline{p}) = i\hbar \frac{\partial}{\partial p_x}\left[p_x \overline{\psi}(\overline{p})\right] - (p_x)\, i\hbar \frac{\partial \overline{\psi}(\overline{p})}{\partial p_x}$$

$$= i\hbar\, \overline{\psi}(\overline{p})$$

Resumindo:

Atuando em $\psi(\mathbf{r})$

$$X \rightarrow x$$

$$P_x \rightarrow \frac{\hbar}{i}\frac{\partial}{\partial x}$$

Atuando em $\overline{\psi}(\mathbf{p})$

$$X \rightarrow i\hbar \frac{\partial}{\partial p_x}$$

$$P_x \rightarrow p_x$$

$$[X, P_x] = i\hbar$$

$$(XP_x - P_x X)\varphi(\vec{r}) = x\,\frac{\hbar}{i}\frac{\partial \varphi(\vec{r})}{\partial x} - \frac{\hbar}{i}\frac{\partial}{\partial x}\left[x\,\varphi(\vec{r})\right]$$

$$= -\frac{\hbar}{i}\varphi(\vec{r}) = i\hbar\,\varphi(\vec{r})$$