Problema resolvido do Cap. 1: Evolução temporal de um pacote gaussiano de partícula livre

1. A função de onda de uma partícula que se move ao longo do eixo x é, em $t = 0$, dada por

$$\psi(x,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{\psi}(k)\, e^{ikx} dk,$$

onde

$$\overline{\psi}(k) = \left(\frac{a^2}{2\pi}\right)^{1/4} e^{-a^2(k-k_0)^2/4} e^{-i\alpha(k-k_0)}.$$

(a) Faça a integral e encontre a densidade de probabilidade da posição da partícula. Use

$$\int_{-\infty}^{\infty} e^{-b(k-z)^2} dk = \sqrt{\frac{\pi}{b}} \quad (\text{se } \mathrm{Re}[b] > 0 \text{ e } b, z \in \mathbb{C}). \tag{1}$$

(b) Confirme que a densidade de probabilidade está normalizada a 1.
(c) Assumindo que a partícula é livre, mostre que $\psi(x,t)$ é dada por

$$\psi(x,t) = \left(\frac{2}{\pi a^2}\right)^{1/4} \left(1 + \frac{4\hbar^2 t^2}{m^2 a^4}\right)^{-1/4} e^{i\left[k_0\left(x - \frac{v_0}{2}t\right) - \theta/2\right]} \exp\left[-\frac{(x-\alpha-v_0 t)^2}{a^2\left(1+\frac{4\hbar^2t^2}{m^2a^4}\right)}(1 - i\tan\theta)\right],$$

onde

$$\begin{aligned} v_0 &= \frac{\hbar k_0}{m}, \\ \tan\theta &= \frac{2\hbar t}{ma^2}, \end{aligned}$$

e que a densidade de probabilidade da posição no instante t é

$$P(x) = |\psi(x,0)|^2 = \frac{\sqrt{2/\pi}}{a\sqrt{1+\frac{4\hbar^2t^2}{m^2a^4}}} \exp\left[-\frac{2(x-\alpha-v_0t)^2}{a^2\left(1+\frac{4\hbar^2t^2}{m^2a^4}\right)}\right].$$

Solução:
(a) Precisamos calcular

$$\psi(x,0) = \frac{1}{\sqrt{2\pi}}\left(\frac{a^2}{2\pi}\right)^{1/4} \int_{-\infty}^{\infty} e^{-a^2(k-k_0)^2/4} e^{-i\alpha(k-k_0)} e^{ikx} dk.$$

Completando o quadrado

$$\begin{aligned} -\frac{a^2}{4}(k-k_0)^2 - i\alpha(k-k_0) + ikx &= -\frac{a^2}{4}(k-k_0)^2 + i(k-k_0)(x-\alpha) + ik_0x \\ &= -\frac{a^2}{4}\left[k - k_0 + \frac{2i}{a^2}(x-\alpha)\right]^2 - \frac{(x-\alpha)^2}{a^2} + ik_0x. \end{aligned}$$

Assim, definindo $z_0 = k_0 - \frac{2i}{a^2}(x-\alpha)$,

$$\begin{aligned} \psi(x,0) &= \frac{1}{\sqrt{2\pi}}\left(\frac{a^2}{2\pi}\right)^{1/4} \int_{-\infty}^{\infty} e^{-a^2(k-z_0)^2/4} e^{-(x-\alpha)^2/a^2} e^{ik_0x} dk \\ &= \frac{1}{\sqrt{2\pi}}\left(\frac{a^2}{2\pi}\right)^{1/4} e^{-(x-\alpha)^2/a^2} e^{ik_0x} \int_{-\infty}^{\infty} e^{-a^2(k-z_0)^2/4} dk \\ &= \frac{1}{\sqrt{2\pi}}\left(\frac{a^2}{2\pi}\right)^{1/4} e^{-(x-\alpha)^2/a^2} e^{ik_0x} \sqrt{\frac{4\pi}{a^2}} \\ &= \left(\frac{2}{\pi a^2}\right)^{1/4} e^{-(x-\alpha)^2/a^2} e^{ik_0x}, \end{aligned}$$

onde usamos a Eq. (1) entre a segunda e a terceira linha. A densidade de probabilidade é

$$P(x) = |\psi(x,0)|^2 = \left(\frac{2}{\pi a^2}\right)^{1/2} e^{-2(x-\alpha)^2/a^2}.$$

(b) A normalização de $P(x)$ é dada por

$$\begin{aligned}\int_{-\infty}^{\infty} P(x)\,dx &= \left(\frac{2}{\pi a^2}\right)^{1/2} \int_{-\infty}^{\infty} e^{-2(x-\alpha)^2/a^2} dx \\ &= \left(\frac{2}{\pi a^2}\right)^{1/2} \left(\frac{\pi a^2}{2}\right)^{1/2} = 1,\end{aligned}$$

onde usamos a Eq. (1) novamente.

(c) Para obtermos a evolução temporal fazemos

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}} \left(\frac{a^2}{2\pi}\right)^{1/4} \int_{-\infty}^{\infty} e^{-a^2(k-k_0)^2/4} e^{-i\alpha(k-k_0)} e^{ikx} e^{-i\hbar k^2 t/(2m)} dk.$$

Mais uma vez completamos o quadrado no argumento da exponencial, definindo por conveniência $q = k - k_0$ e $\beta = \hbar t/(2m)$,

$$\begin{aligned}-\frac{a^2}{4}(k-k_0)^2 - i\alpha(k-k_0) + ikx - i\beta k^2 &= -\frac{a^2}{4} q^2 + iq(x-\alpha) + ik_0 x - i\beta(q+k_0)^2 \\ &= -z_1 q^2 + iq(x-\alpha) - 2i\beta k_0 q - i\beta k_0^2 + ik_0 x \\ &= -z_1 q^2 + iCq + iD \\ &= -z_1 (q - z_2)^2 - \frac{C^2}{4z_1} + iD\end{aligned}$$

onde definimos

$$\begin{aligned}z_1 &= \frac{a^2}{4} + i\beta \in \mathbb{C}, \text{ com } \mathrm{Re}(z_1) > 0, \\ C &= x - \alpha - 2\beta k_0 \in \mathbb{R}, \\ D &= k_0(x - \beta k_0) \in \mathbb{R}, \\ z_2 &= i\frac{C}{2z_1} \in \mathbb{C}.\end{aligned}$$

Assim,

$$\begin{aligned}\psi(x,t) &= \frac{1}{\sqrt{2\pi}} \left(\frac{a^2}{2\pi}\right)^{1/4} \int_{-\infty}^{\infty} \exp\left[-z_1(q-z_2)^2 - \frac{C^2}{4z_1} + iD\right] dq. \\ &= \frac{1}{\sqrt{2\pi}} \left(\frac{a^2}{2\pi}\right)^{1/4} \sqrt{\frac{\pi}{z_1}} \exp\left(-\frac{C^2}{4z_1} + iD\right) \\ &= \frac{1}{\sqrt{2z_1}} \left(\frac{a^2}{2\pi}\right)^{1/4} \exp\left(-\frac{C^2}{4z_1} + iD\right)\end{aligned}$$

Podemos simplificar um pouco explicitando

$$\begin{aligned}z_1 &= \frac{a^2}{4} + i\frac{\hbar t}{2m}, \\ C &= x - \alpha - \frac{\hbar k_0}{m} t \equiv x - \alpha - v_0 t, \\ D &= k_0(x - \beta k_0) = k_0\left(x - \frac{\hbar k_0}{2m} t\right) = k_0\left(x - \frac{v_0}{2} t\right).\end{aligned}$$

Temos também que

$$z_1 = \rho_1 e^{i\theta},$$

onde

$$\begin{aligned} \rho_1 &= \frac{a^2}{4}\sqrt{1+\frac{4\hbar^2 t^2}{m^2 a^4}}, \\ \tan\theta &= \frac{2\hbar t}{m a^2}. \end{aligned}$$

Assim,

$$\frac{1}{\sqrt{2z_1}}\left(\frac{a^2}{2\pi}\right)^{1/4} = \left(\frac{a^2}{2\pi}\right)^{1/4}\frac{\sqrt{2}}{a}\left(1+\frac{4\hbar^2 t^2}{m^2 a^4}\right)^{-1/4} e^{-i\theta/2} = \left(\frac{2}{\pi a^2}\right)^{1/4}\left(1+\frac{4\hbar^2 t^2}{m^2 a^4}\right)^{-1/4} e^{-i\theta/2}.$$

Além disso,

$$-\frac{C^2}{4z_1} = -\frac{\left(x-\alpha-v_0 t\right)^2}{a^2\sqrt{1+\frac{4\hbar^2 t^2}{m^2 a^4}}} e^{-i\theta} = -\frac{\left(x-\alpha-v_0 t\right)^2}{a^2\sqrt{1+\frac{4\hbar^2 t^2}{m^2 a^4}}}\left(\cos\theta - i\sin\theta\right).$$

Mas

$$\begin{aligned} \cos\theta &= \frac{1}{\sqrt{1+\tan^2\theta}} = \frac{1}{\sqrt{1+\frac{4\hbar^2 t^2}{m^2 a^4}}}, \\ \sin\theta &= \frac{\tan\theta}{\sqrt{1+\tan^2\theta}} = \frac{\frac{2\hbar t}{m a^2}}{\sqrt{1+\frac{4\hbar^2 t^2}{m^2 a^4}}}, \end{aligned}$$

e

$$-\frac{C^2}{4z_1} = -\frac{\left(x-\alpha-v_0 t\right)^2}{a^2\left(1+\frac{4\hbar^2 t^2}{m^2 a^4}\right)}\left(1 - i\tan\theta\right).$$

Finalmente,

$$\psi\left(x,t\right) = \left(\frac{2}{\pi a^2}\right)^{1/4}\left(1+\frac{4\hbar^2 t^2}{m^2 a^4}\right)^{-1/4} e^{i\left[k_0\left(x-\frac{v_0}{2}t\right)-\theta/2\right]}\exp\left[-\frac{\left(x-\alpha-v_0 t\right)^2}{a^2\left(1+\frac{4\hbar^2 t^2}{m^2 a^4}\right)}\left(1-i\tan\theta\right)\right].$$

A densidade de probabilidade é

$$P\left(x\right) = \left|\psi\left(x,0\right)\right|^2 = \frac{\sqrt{2/\pi}}{a\sqrt{1+\frac{4\hbar^2 t^2}{m^2 a^4}}}\exp\left[-\frac{2\left(x-\alpha-v_0 t\right)^2}{a^2\left(1+\frac{4\hbar^2 t^2}{m^2 a^4}\right)}\right].$$