2) $V(r) = \alpha\,\delta(r-a)$ ONDA-S $(\ell=0)$

$$\left[-\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2}+V(r)\right]u_{k,0}(r) = \frac{\hbar^2k^2}{2\mu}u_{k0}(r)$$

$$\left[\frac{d^2}{dr^2}+k^2-\underbrace{\frac{2\mu}{\hbar}V(r)}_{U(r)}\right]u_{k,0}(r)=0$$

$V(r) = 0$ SE $0<r<a$ E $r>a$

$$u_{k,0}(r) = (kr)\left[A_0 j_0(kr)+B_0 n_0(kr)\right]$$

$r<a$: $u_{k,0}(r) \propto r$ $r\to 0$

$$u_{k,0}(r) = A_0\,(kr)\,j_0(kr) \to A_0 \sin(kr) \quad r<a$$

$$u_{k,0}(r) = (kr)\left[C_0 j_0(kr)+D_0 n_0(kr)\right] \quad r>a$$

CONTINUIDADE DE $u_0(kr)$ EM $r=a$

$$u(a^-) = A_0\,(\cancel{ka})\,j_0(ka) = u(a^+) = (\cancel{ka})\left[C_0 j_0(ka)+\right.$$

$$\left.+D_0 n_0(ka)\right]$$

$$(A_0-C_0)\,j_0(ka) = D_0 n_0(ka) \qquad (1)$$

$$(A_0-C_0)\sin ka = -D_0\cos ka \qquad (1)$$

$r>a$: $u_{k,0}(r) = C_0\sin(kr) - D_0\cos(kr)$

$$\left[\frac{d^2}{dr^2} + k^2 - \frac{2\mu\alpha}{\hbar^2}\delta(r-a)\right] u(r) = 0$$

INTEGRANDO ENTRE $a-\varepsilon$ E $a+\varepsilon$: $(\varepsilon \to 0^+)$

$$\left[u'(a^+) - u'(a^-)\right] = \frac{2\mu\alpha}{\hbar^2} u(a) = \frac{2\mu\alpha}{\hbar^2} A_0 (ka) j_0 (ka)$$

$$k^2 \int_{a-\varepsilon}^{a+\varepsilon} u(r)\, dr \underset{\varepsilon \to 0}{\xrightarrow{\quad}} \simeq k^2 u(a)(2\varepsilon) \longrightarrow 0$$

$$k\left[C_0 \cos(ka) + D_0 \sin(ka) - A_0 \cos(ka)\right] = \frac{2\mu\alpha}{\hbar^2} A_0 \sin(ka) \quad (2)$$

$$(A_0 - C_0)\sin ka = -D_0 \cos ka \quad (1)$$

$$(C_0 - A_0)\cos ka = \left[\underbrace{\frac{2\mu\alpha}{\hbar^2 k}}_{\lambda} A_0 - D_0\right] \sin ka \quad (2)$$

$(1)/(2)$: $\quad C_0' = \frac{C_0}{A_0}$; $D_0' = \frac{D_0}{A_0}$

$$\frac{A_0 - C_0}{\lambda A_0 - D_0} = \frac{-D_0}{C_0 - A_0} \Rightarrow \frac{1 - C_0'}{\lambda - D_0'} = \frac{-D_0'}{C_0' - 1}$$

$$\Rightarrow |C_0' - 1| = \sqrt{D_0'(\lambda - D_0')}$$

$$\cancel{-}(C_0' - 1)^2 = \cancel{-}D_0'(\lambda - D_0') \Rightarrow C_0' = 1 + \sqrt{D_0'(\lambda - D_0')} \quad (3)$$

$$(1) \Rightarrow \tan ka = -\frac{D_0}{A_0 - C_0} = -\frac{D_0'}{1 - C_0'} \quad (4) \Rightarrow \tan ka = \frac{-D_0'}{s\sqrt{D_0'(\lambda - D_0')}}$$

EM BAIXAS ENERGIAS:

APENAS $\ell=0$ CONTRIBUI E $\tan(ka) \simeq ka \ll 1$

$$ka = \frac{-D_0'}{S\left[D_0'(n-D_0')\right]^{1/2}}$$

$$n = \frac{2\mu\alpha}{\hbar^2 k} \gg 1 \qquad k \ll 1$$

$$ka = \frac{-D_0'}{S\sqrt{D_0'}\sqrt{n}} = -\frac{\sqrt{D_0'}}{S\sqrt{\frac{2\mu\alpha}{\hbar^2 k}}} = -\frac{\sqrt{D_0'}}{S}\frac{\hbar\sqrt{k}}{\sqrt{2\mu\alpha}}$$

$$\sqrt{D_0'} = -S\frac{\sqrt{2\mu\alpha}}{\hbar}\sqrt{k}\,a$$

$$D_0' = \frac{2\mu\alpha}{\hbar^2}ka^2$$

$$C_0' = 1+\sqrt{D_0' n} = 1+\sqrt{\frac{2\mu\alpha}{\hbar^2}ka^2\frac{2\mu\alpha}{\hbar^2 k}}$$

$$C_0' = 1+\frac{2\mu\alpha}{\hbar^2}a$$

$$\tan[\delta_0(k)] = -\frac{D_0}{C_0} = -\frac{D_0'}{C_0'} = -\frac{\frac{2\mu\alpha}{\hbar^2}ka^2}{1+\frac{2\mu\alpha}{\hbar^2}a}$$

$$\delta_0(k) = \frac{-ka}{\hbar^2/(2\mu\alpha)+1}$$

$$\sigma(k) = \frac{4\pi}{k^2} \sin^2 \delta_0(k) \simeq \frac{4\pi}{k^2} \delta_0^2(k)$$

$$\sigma(k) \simeq \frac{4\pi a^2}{\left[1 + \frac{\hbar^2}{2\mu\alpha a}\right]^2}$$