Problema 2.9

a) De acordo com (2.15) a dens. de carga induzida na esfera é:

$$\sigma_0 = 3\epsilon_0 E_0 \cos\theta$$

A força sobre um elemento dS de área é:

$$\frac{\sigma_0^2}{2\epsilon_0} dS \quad \text{pois} \qquad \theta \rightarrow E_n \Rightarrow E_n = \frac{\sigma}{\epsilon_0}$$

mas o campo devido ao elemento é $\frac{\sigma}{2\epsilon_0}$, logo o campo do resto é $E_n^{ext} = \frac{\sigma}{2\epsilon_0}$

$$\text{e} \quad F = E_n^{ext} \sigma \, dS = \frac{\sigma^2}{2\epsilon_0} dS$$

Assim:

$$F_1 = 2\pi \int_0^{\pi/2} \sin\theta \, d\theta \, (a^2)(F\cos\theta) \qquad \mu = \cos\theta, \quad \mu \in (0,1)$$

$$= 2\pi a^2 \int_0^1 d\mu \, \mu \frac{\sigma^2}{2\epsilon_0} = \frac{\pi a^2}{\epsilon_0} \int_0^1 \mu \, 9\epsilon_0^2 E_0^2 \mu^2 \, d\mu$$

$$= 9\pi\epsilon_0 a^2 E_0^2 \int_0^1 \mu^3 d\mu = \frac{9\pi}{4} \epsilon_0 a^2 E_0^2$$

$$F_2 = F_1 \qquad F_2 = 2\pi a^2 \int_{\pi/2}^{\pi} \sin\theta \, d\theta \, F(\cos\theta) \rightarrow \int_{-1}^{0} \Rightarrow F_2 = F_1$$

b) Agora há uma σ adicional, $\sigma' = \frac{Q}{4\pi a^2}$

$$\sigma = \sigma_0 + \sigma' \Rightarrow \sigma^2 = \sigma_0^2 + 2\sigma_0\sigma' + \sigma'^2 = \sigma_0^2 + 2(3\epsilon_0 E_0)\cos\theta \frac{Q}{4\pi a^2} + \frac{Q^2}{16\pi^2 a^4}$$

$$= \sigma_0^2 + \frac{3\epsilon_0 E_0 Q}{2\pi a^2}\cos\theta + \frac{Q^2}{16\pi^2 a^4}$$

b) (cont.) $\frac{\sigma^2}{2\epsilon_0} = \frac{F}{A} = P = \frac{(3\epsilon_0 E_0 \cos\theta + \sigma')^2}{2\epsilon_0}$

$$F_1 = \int_0^{\pi/2} P \cos\theta \, dS \qquad F_2 = \int_{\pi/2}^{\pi} P(-\cos\theta)\, dS$$

Note que F_1 e F_2 são ambos positivos, pois $P > 0$ e $\pm\cos\theta > 0$ se $\theta \in (0,\pi/2)$, $(\frac{\pi}{2},\pi)$.

$$dS = 2\pi a^2 \sin\theta \, d\theta \longrightarrow 2\pi a^2 d\mu$$

$$F_1 = \int_0^1 2\pi a^2 \frac{(3\epsilon_0 E_0 \overbrace{\cos\theta}^{\mu} + \sigma')^2}{2\epsilon_0} \mu \, d\mu$$

$$F_2 = \int_{-1}^{0} 2\pi a^2 \frac{(3\epsilon_0 E_0 \mu + \sigma')^2}{2\epsilon_0} (-\mu)\, d\mu = \int_0^1 2\pi a^2 \frac{(3\epsilon_0 E_0 \mu - \sigma')^2}{2\epsilon_0} \mu \, d\mu$$

$$F_{1,2} = \frac{\pi a^2}{\epsilon_0} \int_0^1 (3\epsilon_0 E_0 \mu \pm \sigma')^2 \mu \, d\mu \qquad y = 3\epsilon_0 E_0 \mu$$

$$= \frac{\pi a^2}{\epsilon_0} \frac{1}{(3\epsilon_0 E_0)^2} \int_0^{3\epsilon_0 E_0} (y \pm \sigma')^2 y \, dy$$

$$= \frac{\pi a^2}{\epsilon_0} \frac{1}{9\epsilon_0^2 E_0^2} \left[\frac{(3\epsilon_0 E_0)^4}{4} \pm 2\sigma' \frac{(3\epsilon_0 E_0)^3}{3} + \sigma'^2 \frac{(3\epsilon_0 E_0)^2}{2} \right]$$

$$F_{1,2} = \frac{\pi a^2}{\epsilon_0} \left[\frac{9\epsilon_0^2 E_0^2}{4} \pm 2\sigma' \epsilon_0 E_0 + \frac{\sigma'^2}{2} \right]$$

$$= \frac{\pi a^2}{\epsilon_0} \left[\frac{9\epsilon_0^2 E_0^2}{4} \pm \frac{2Q\,\epsilon_0 E_0}{\cancel{2}4\pi a^2} + \frac{Q^2}{32\pi^2 a^4} \right]$$

$$= \frac{9\pi \epsilon_0 a^2 E_0^2}{4} \pm \frac{Q E_0}{2} + \frac{Q^2}{32\pi\epsilon_0 a^2}$$