4.1:

a) $\rho(\vec{r}) = \dfrac{q}{r^2 \sin\theta}\,\delta(r-a)\,\delta\left(\theta-\frac{\pi}{2}\right)\left[\delta(\phi)+\delta\left(\phi-\frac{\pi}{2}\right)-\delta(\phi-\pi)-\delta\left(\phi-\frac{3\pi}{2}\right)\right]$

$= \dfrac{q}{a^2}\,\delta(r-a)\,\delta\left(\theta-\frac{\pi}{2}\right)[\quad]$

$q_{\ell m} = \displaystyle\int Y^*_{\ell m}(\theta,\phi)\, r^\ell \rho(\vec{r})\, r^2 dr \sin\theta\, d\theta\, d\phi$

$= q a^\ell \left[Y^*_{\ell m}\left(\frac{\pi}{2},0\right)+Y^*_{\ell m}\left(\frac{\pi}{2},\frac{\pi}{2}\right)-Y^*_{\ell m}\left(\frac{\pi}{2},\pi\right)-Y^*_{\ell m}\left(\frac{\pi}{2},\frac{3\pi}{2}\right)\right]$

$= \sqrt{\dfrac{(2\ell+1)}{4\pi}\dfrac{(\ell-m)!}{(\ell+m)!}}\; q a^\ell P_\ell^m(0)\,\underbrace{\left[1+e^{-im\frac{\pi}{2}}-e^{-im\pi}-e^{-i3m\pi/2}\right]}_{(*)}$

$(*) = 1-(-1)^m + e^{-im\frac{\pi}{2}}\left[1-(-1)^m\right]$

$= \underbrace{\left[1-(-1)^m\right]}_{\begin{array}{l} 0 \text{ se } m \text{ par} \\ 2 \text{ se } m \text{ ímpar}\end{array}}\left(1+e^{-im\frac{\pi}{2}}\right) \Rightarrow m \to \text{ímpar}$

Se m ímpar, $e^{-im\frac{\pi}{2}} = (-1)^{(m+1)/2}\, i$

$\Rightarrow (*) = 2\left[1+(-1)^{(m+1)/2}\, i\right]$ se m ímpar

0 se m par

$P_\ell^m(x) \propto (1-x^2)^{m/2}\dfrac{d^{(\ell+m)}}{dx^{(\ell+m)}}(x^2-1)^\ell$ é ímpar se $\ell+m$ é ímpar $\Rightarrow P_\ell^m(0)=0$ se $\ell+m$ é ímpar

Portanto, devemos ter $(\ell+m)$ par. Como m deve ser ímpar, ℓ também deve ser ímpar.

$q_{11} = \sqrt{\dfrac{3}{4\pi}\times\dfrac{1}{2}}\; q a \underbrace{P_1^1(0)}_{-1} \times 2(1-i) = \sqrt{\dfrac{3}{2\pi}}\, q a (i-1) = -q^*_{1,-1}$

$$q_{31} = \sqrt{\frac{7}{4\pi}\frac{2}{24}}\, qa^3 \underbrace{P_3^1(0)}_{3/2} \times 2(1-i) = \sqrt{\frac{21}{\pi}}\frac{qa^3}{4}(1-i) = -q_{3,-1}^*$$

$$q_{33} = \sqrt{\frac{7}{4\pi}\frac{1}{6.5.4.3.2}}\, qa^3 \underbrace{P_3^3(0)}_{-15} \times 2(1+i) = -\sqrt{\frac{35}{\pi}}\frac{qa^3}{4}(i+1) = -q_{3,-3}^*$$

Onde usamos $P_1^1(0) = -1$, $P_3^1(0) = \frac{3}{2}$ e $P_3^3(0) = -15$

b) $\rho(\bar{r}) = \rho_1(\bar{r}) + \rho_2(\bar{r})$

$$\rho_1(\bar{r}) = -2q\,\delta^{(3)}(\bar{r}) \Rightarrow q_{\ell m}^1 = \int Y_{\ell m}^*(\theta,\phi)\, r^\ell (-2q)\,\delta^{(3)}(\bar{r})\, d^3x$$

$$= \int Y_{\ell m}^*(\theta,\phi)\, r^\ell (-2q)\frac{\delta(r)}{4\pi r^2}\, r^2 dr\, d\Omega$$

$$= \delta_{\ell,0}(-2q)\, Y_{00}^*(\theta,\phi) = \frac{-2q}{\sqrt{4\pi}}\delta_{\ell,0}$$

$$\rho_2(\bar{r}) = q\left[\delta^{(3)}(\bar{r}-a\hat{z}) + \delta^{(3)}(\bar{r}+a\hat{z})\right]$$

$$\int Y_{\ell m}^*(\theta,\phi)\, r^\ell\, \delta^{(3)}(\bar{r} \mp a\hat{z})\, d^3x = a^\ell\, Y_{\ell m}^*(\theta = \{{0 \atop \pi}\}, \phi)$$

$Y_{\ell m}(\theta = \{{0 \atop \pi}\}, \phi) \propto \delta_{m,0}$ pois $P_\ell^m(\pm 1) = 0$ se $m \neq 0$
por (3.50)

$$Y_{\ell 0}(\theta = \{{0 \atop \pi}\}, \phi) = \sqrt{\frac{2\ell+1}{4\pi}}\, P_\ell(\pm 1) = \sqrt{\frac{2\ell+1}{4\pi}}\left\{{1 \atop (-1)^\ell}\right\}$$

$$\int Y_{\ell m}^*(\theta,\phi)\, \rho_2(\bar{r})\, d^3x = q a^\ell \delta_{m,0}\sqrt{\frac{2\ell+1}{4\pi}}\left[1 + (-1)^\ell\right] =$$

$$= \begin{cases} 0 & \text{se } \ell \text{ é ímpar} \\ qa^\ell \sqrt{\frac{2\ell+1}{\pi}}\,\delta_{m,0} & \text{se } \ell \text{ é par} \end{cases}$$

$$q_{\ell m} = q\left[a^{\ell}\sqrt{\frac{2\ell+1}{\pi}}\,\delta_{m,0}\,S_{\ell} - \frac{\delta_{\ell,0}\,\delta_{m,0}}{\sqrt{\pi}}\right]$$

$$= \frac{q}{\sqrt{\pi}}\left[a^{\ell}\sqrt{2\ell+1}\,\delta_{m,0}\,S_{\ell} - \delta_{\ell,0}\,\delta_{m,0}\right]$$

onde $S_{\ell} = \begin{cases} 0 & \text{se } \ell \text{ é ímpar} \\ 1 & \text{se } \ell \text{ é par} \end{cases}$

$$q_{00} = 0$$

$$q_{20} = q a^2 \sqrt{\frac{5}{\pi}} \qquad q_{40} = \sqrt{\frac{9}{\pi}}\, q a^4$$

(c) $\ell = 2n \Rightarrow q_{2n,0} = \sqrt{\frac{4n+1}{\pi}}\, a^{2n}\, q$

$$\Phi(\bar{x}) = \frac{q}{\epsilon_0}\sum_{n=1}^{\infty}\frac{1}{4n+1}\sqrt{\frac{4n+1}{\pi}}\,\frac{a^{2n}}{r^{2n+1}}\,Y_{2n,0}(\theta,\phi)$$

$$Y_{2n,0}(\theta,\phi) = \sqrt{\frac{4n+1}{4\pi}}\,P_{2n}(\cos\theta)$$

$$\Rightarrow \boxed{\Phi(\bar{x}) = \frac{q}{2\pi\epsilon_0}\sum_{n=1}^{\infty}\frac{a^{2n}}{r^{2n+1}}\,P_{2n}(\cos\theta)}$$

$$\Phi(\bar{x}) \simeq \frac{q a^2}{(2\pi\epsilon_0)\, r^3}\,P_2(\cos\theta) = \frac{q}{4\pi\epsilon_0}\,\frac{a^2}{r^3}\,(3\cos^2\theta - 1)$$

xy plane $\rightarrow \cos\theta = 0 \Rightarrow \Phi(\bar{x}) \simeq -\frac{q}{4\pi\epsilon_0}\,\frac{a^2}{r^3}$

d) $\Phi(\bar{x}) = \frac{1}{4\pi\epsilon_0}\left[-\frac{2q}{r} + \frac{2q}{\sqrt{r^2+a^2}}\right]$ no plano xy

$$\Phi(x) = \frac{q}{2\pi\epsilon_0}\left[\frac{1}{\sqrt{r^2+a^2}} - \frac{1}{r}\right]$$

$\frac{1}{\sqrt{r^2+a^2}} \simeq \frac{1}{r} + \frac{1}{r^3}\left(-\frac{1}{2}\right)a^2 \Rightarrow \Phi(x) \cong -\frac{q}{4\pi\epsilon_0}\frac{a^2}{r^3}$ como antes