FI 193 – Teoria Quântica de Sistemas de Muitos Corpos

2º Semestre de 2023

09/11/2023

Aula 24

Aula passada

Formalismo a temperatura finita: funções de Green ordenadas temporalmente a temperatura finita.

$$iG_{\alpha\beta}\left(\boldsymbol{r}_1 t_1; \boldsymbol{r}_2 t_2\right) = \mathrm{Tr}\left\{\frac{\hat{\rho}}{Z} T_t\left[\psi_{\alpha H}\left(\boldsymbol{r}_1 t_1\right)\psi^{\dagger}_{\beta H}\left(\boldsymbol{r}_2 t_2\right)\right]\right\}$$

Formalismo a temperatura finita: funções de Green retardada e avançada a temperatura finita.

$$iG^{R,A}_{\alpha\beta}\left(\boldsymbol{r}_1 t_1; \boldsymbol{r}_2 t_2\right) = \pm\theta\left(t_1 - t_2\right)\mathrm{Tr}\left\{\frac{\hat{\rho}}{Z}\left[\psi_{\alpha H}\left(\boldsymbol{r}_1 t_1\right), \psi^{\dagger}_{\beta H}\left(\boldsymbol{r}_2 t_2\right)\right]_{\zeta}\right\}$$

$$\hat{H} \rightarrow \hat{H} - \mu\hat{N}$$

$$\hat{\rho} = e^{-\beta\hat{H}}$$

$$Z = \mathrm{Tr}\left(e^{-\beta\hat{H}}\right)$$

Aula passada

Formalismo a temperatura finita: funções de Green de Matsubara

$$-\mathcal{G}_{\alpha\beta}\left(\boldsymbol{r}_1\tau_1;\boldsymbol{r}_2\tau_2\right)=\mathrm{Tr}\left\{\frac{\hat{\rho}}{Z}T_\tau\left[\psi_{\alpha M}\left(\boldsymbol{r}_1\tau_1\right)\psi^\dagger_{\beta M}\left(\boldsymbol{r}_2\tau_2\right)\right]\right\}$$

$$\begin{aligned}\psi_{\alpha M}\left(\boldsymbol{r}_1\tau_1\right) &= e^{\hat{H}\tau_1}\psi_{\alpha S}\left(\boldsymbol{r}_1\right)e^{-\hat{H}\tau_1}\\ \psi^\dagger_{\beta M}\left(\boldsymbol{r}_2\tau_2\right) &= e^{\hat{H}\tau_2}\psi^\dagger_{\beta S}\left(\boldsymbol{r}_2\right)e^{-\hat{H}\tau_2}\end{aligned}$$

Propriedades das funções de Green de Matsubara:

a)

$$\mathcal{G}_{\alpha\beta}\left(\boldsymbol{r}_1\tau_1;\boldsymbol{r}_2\tau_2\right)=\mathcal{G}_{\alpha\beta}\left(\boldsymbol{r}_1,\boldsymbol{r}_2,\tau_1-\tau_2\right)=\mathcal{G}_{\alpha\beta}\left(\boldsymbol{r}_1,\boldsymbol{r}_2,\tau=\tau_1-\tau_2\right)\ \ (\tau\in[-\beta,\beta])$$

b)

$$\mathcal{G}_{\alpha\beta}\left(\boldsymbol{r}_1,\boldsymbol{r}_2,\tau\right)=\zeta\mathcal{G}_{\alpha\beta}\left(\boldsymbol{r}_1,\boldsymbol{r}_2,\tau+\beta\right)\ \ (\tau\in[-\beta,0])$$

Aula passada

Funções de Green de Matsubara: frequências de Matsubara

$$\mathcal{G}_{\alpha\beta}\left(\boldsymbol{r}_1,\boldsymbol{r}_2,\tau\right) = T\sum_{\omega_n} e^{-i\omega_n\tau}\mathcal{G}_{\alpha\beta}\left(\boldsymbol{r}_1,\boldsymbol{r}_2,\omega_n\right),$$

$$\mathcal{G}_{\alpha\beta}\left(\boldsymbol{r}_1,\boldsymbol{r}_2,\omega_n\right) = \int_0^\beta d\tau e^{i\omega_n\tau}\mathcal{G}_{\alpha\beta}\left(\boldsymbol{r}_1,\boldsymbol{r}_2,\tau\right),$$

$$\omega_n = 2\pi nT \;\; (n=0,\pm1,\pm2,\ldots) \;\; (\text{bósons}),$$

$$\omega_n = (2n+1)\,\pi T \;\; (n=0,\pm1,\pm2,\ldots) \;\; (\text{férmions}).$$

Funções de Green de Matsubara: caso não interagente

$$\widehat{H}_0 = \sum_{\mathbf{k}\sigma}\epsilon_{\mathbf{k}}a^\dagger_{\mathbf{k},\sigma}a_{\mathbf{k},\sigma}$$

$$\mathcal{G}^{(0)}_{\alpha\beta}\left(\mathbf{k},i\omega_n\right) = \frac{\delta_{\alpha,\beta}}{i\omega_n-\epsilon_{\mathbf{k}}}$$

válida para férmions ou bósons.

Aula passada

$$\rho(\mathbf{k},\omega) \;=\; \frac{1}{(2s+1)\,Z}\sum_{m,n}\left\{e^{-\beta K_m}(2\pi)^4\,\delta\left[\omega-(K_n-K_m)\right]\delta^{(3)}\left[\mathbf{k}-(\mathbf{P}_n-\mathbf{P}_m)\right]\right.$$

$$\left.\left(1-\zeta e^{-\beta\omega}\right)\left|\langle m|\,\psi_\alpha(0)\,|n\rangle\right|^2\right\} \qquad (K_n = E_n-\mu N_n)$$

$$G^R(\mathbf{k},\omega) \;=\; \int_{-\infty}^{+\infty}\frac{d\omega'}{2\pi}\frac{\rho(\mathbf{k},\omega')}{\omega-\omega'+i\eta}$$

$$G^A(\mathbf{k},\omega) \;=\; \int_{-\infty}^{+\infty}\frac{d\omega'}{2\pi}\frac{\rho(\mathbf{k},\omega')}{\omega-\omega'-i\eta}$$

$$\boxed{\begin{aligned}\zeta &= 1 \text{ (bósons)}\\ \zeta &= -1 \text{ (férmions)}\end{aligned}}$$

$$\mathcal{G}(\mathbf{k},i\omega_n) \;=\; \int_{-\infty}^{+\infty}\frac{d\omega'}{2\pi}\frac{\rho(\mathbf{k},\omega')}{i\omega_n-\omega'}$$

$$f_\zeta(\omega) = \frac{1}{e^{\beta\omega}-\zeta}$$

$$G(\mathbf{k},\omega) = \left[1+\zeta f_\zeta(\omega)\right]G^R(\mathbf{k},\omega)-\zeta f_\zeta(\omega)\,G^A(\mathbf{k},\omega)$$

$$\Gamma(k,z) = \int_{-\infty}^{+\infty}\frac{d\omega'}{2\pi}\frac{\rho(\vec{k},\omega')}{z-\omega'}$$

Teoria de perturbação à temperatura finita

NOTE A SEGUINTE ANALOGIA: $\rho = e^{-\beta H}$

$$\Rightarrow \frac{\partial \rho}{\partial \beta} = -H \rho$$

COMPARE COM A EQ. DE SCHRÖDINGER: $i\partial_t |\Psi\rangle = H|\Psi\rangle$

$$it \rightarrow \beta$$

$$H \rightarrow H - \mu N$$

$$|\Psi\rangle \rightarrow \rho = e^{-\beta H}$$

ALGUNS PASSOS QUE PERMITEM DEFINIR A EXPANSÃO PERTURBATIVA A TEO:

Versão de interação

i) DEFINIMOS A VERSÃO DE INTERAÇÃO (NO TEMPO IMAGINÁRIO)

$K = K_0 + K_1$

$O_I(\tau) = e^{K_0 \tau} O_S e^{-K_0 \tau}$ ONDE $K_0 = H_0 - \mu N$ $K_1 = H_1$

$\Rightarrow O_H(\tau) = e^{K\tau} O_S e^{-K\tau} = \underbrace{e^{K\tau} e^{-K_0 \tau}}_{\tilde{U}(0,\tau)} O_I(\tau) \underbrace{e^{K_0 \tau} e^{-K\tau}}_{\tilde{U}(\tau,0)}$ (1)

Operador "evolução temporal"

ii) DEFINIMOS O OPERADOR EVOLUÇÃO TEMPORAL NO TEMPO IMAGINÁRIO:

$$\tilde{U}(\tau,\tau_0) = e^{K_0\tau} e^{-K(\tau-\tau_0)} e^{-K_0\tau_0} \Rightarrow \tilde{U}(\tau,0) = e^{K_0\tau} e^{-K\tau}$$

<u>PROPRIEDADES</u>:

a) NÃO É UNITÁRIO!

b) SATISFAZ A PROPRIEDADE DE GRUPO: $\tilde{U}(\tau_1,\tau_2)\tilde{U}(\tau_2,\tau_3) = \tilde{U}(\tau_1,\tau_3)$

c) CONDIÇÃO INICIAL: $\tilde{U}(\tau_0,\tau_0) = \mathbb{1}$

d) EQ. DIFERENCIAL:

$$\partial_\tau \tilde{U}(\tau,\tau_0) = e^{K_0\tau} \underbrace{(K_0 - K)}_{-K_1 = -H_1} e^{-K(\tau-\tau_0)} e^{-K_0\tau_0}$$

$$= -\underbrace{e^{K_0\tau} K_1 e^{-K_0\tau}}_{K_{1I}(\tau)} \underbrace{e^{K_0\tau} e^{-K(\tau-\tau_0)} e^{-K_0\tau_0}}_{\tilde{U}(\tau,\tau_0)} = -K_{1I}(\tau)\tilde{U}(\tau,\tau_0)$$

Solução perturbativa do operador de evolução "temporal"

(c) E (d) PERMITEM ESCREVER UMA EQ. INTEGRAL PARA $\tilde{U}(\tau,\tau_0)$ QUE ADMITE UMA SOLUÇÃO PERTURBATIVA EM TOTAL ANALOGIA AO CASO DE TEMPO REAL:

$$\tilde{U}(\tau,\tau_0) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{\tau_0}^{\tau} d\tau_1 d\tau_2 \cdots d\tau_n \, T_\tau\left[K_{1I}(\tau_1)\cdots K_{1I}(\tau_n)\right]$$

iii) DE: $e^{-K\tau} = e^{-K_0\tau}\,\tilde{U}(\tau,0)$

SEGUE QUE: $e^{-\beta K} = e^{-\beta K_0}\,\tilde{U}(\beta,0) \Rightarrow \rho = \rho_0\,\tilde{U}(\beta,0)$

FINALMENTE:

$$Z = \mathrm{Tr}\left[e^{-\beta K}\right] = \mathrm{Tr}\left[e^{-\beta K_0}\,\tilde{U}(\beta,0)\right]$$

A função de Green de Matsubara

SEJA $\tau_1 > \tau_2$ (O OUTRO CASO É ANÁLOGO):

$$-\mathcal{G}(x_1, x_2) = \frac{1}{\mathcal{Z}} \mathrm{Tr}\left[e^{-\beta K} \psi_H(\bar{x}_1, \tau_1) \psi_H^\dagger(\bar{x}_2, \tau_2)\right]$$

$$= \frac{1}{\mathcal{Z}} \mathrm{Tr}\Big[e^{-\beta K_0} \underbrace{\tilde{U}(\beta,0)\, \hat{U}(0,\tau_1)}_{\tilde{U}(\beta,\tau_1)} \psi_I(\bar{x}_1, \tau_1) \underbrace{\tilde{U}(\tau_1,0)\, \tilde{U}(0,\tau_2)}_{\tilde{U}(\tau_1,\tau_2)} \psi_I^\dagger(\bar{x}_2, \tau_2) \tilde{U}(\tau_2, 0)\Big]$$

$$= \frac{\mathrm{Tr}\left[e^{-\beta K_0} T_\tau\left[\tilde{U}(\beta,0)\, \Psi_I(x_1) \Psi_I^\dagger(x_2)\right]\right]}{\mathrm{Tr}\left[e^{-\beta K_0} \tilde{U}(\beta,0)\right]}$$

DE MANEIRA GERAL:

$$-\mathcal{G}(x_1, x_2) = \frac{\mathrm{Tr}\left\{e^{-\beta K_0} T_\tau\left[\tilde{U}(\beta,0)\, \Psi_I(x_1) \Psi_I^\dagger(x_2)\right]\right\}}{\mathrm{Tr}\left[e^{-\beta K_0} \tilde{U}(\beta,0)\right]}$$

$$= \frac{\mathrm{Tr}\left\{e^{-\beta K_0} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^\beta d\tau_1 \cdots d\tau_n\, T_\tau\left[K_{1I}(\tau_1) \cdots K_{1I}(\tau_n) \Psi_I(x_1) \Psi_I^\dagger(x_2)\right]\right\}}{\mathrm{Tr}\left[e^{-\beta K_0} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^\beta d\tau_1 \cdots d\tau_n\, T_\tau\left[K_{1I}(\tau_1) \cdots K_{1I}(\tau_n)\right]\right]}$$

O teorema de Wick

MESMO A $T \neq 0$, EXISTE UMA GENERALIZAÇÃO DO TEOREMA DE WICK. ELE ENVOLVE O TRAÇO COM PESO $e^{-\beta K_0}$ DE UMA CADEIA DE OPERADORES DE CRIAÇÃO E DESTRUIÇÃO NA VERSÃO DE INTERAÇÃO NO TEMPO IMAGINÁRIO.

ELE TEM A MESMA ESTRUTURA QUE A $T=0$ (EMBORA SÓ EXISTA COMO IGUALDADE DE TRAÇOS, NÃO DE OPERADORES) E AS "CONTRAÇÕES" DE $\psi_I(x_1)$ E $\psi_I^\dagger(x_2)$ SÃO:

$$-\mathcal{G}^{(0)}(x_1, x_2) = \frac{1}{Z_0} \mathrm{Tr}\left[e^{-\beta K_0} T_\tau\left[\psi_I(x_1)\psi_I^\dagger(x_2)\right]\right]$$

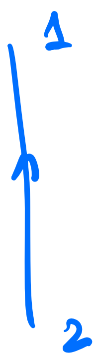
DADA ESSA ANALOGIA, AS REGRAS SÃO PRATICAMENTE AS MESMAS QUE A $T=0$, COM PEQUENAS MODIFICAÇÕES.

Expansão diagramática

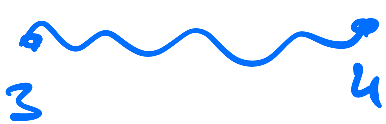
PRIMEIRAMENTE:

$$U(\bar{x}_1 - \bar{x}_2, \tau_1 - \tau_2) = V(\bar{x}_1 - x_2)\,\delta(\tau_1 - \tau_2)$$

ALÉM DISSO:



$$-\mathcal{G}^{(0)}(x_1, x_2)$$



$$-U(x_3 - x_4)$$

Regras diagramáticas

1. Draw all topologically distinct diagrams containing n interaction lines and $2n+1$ directed particle lines.
2. Associate a factor $\mathscr{G}^0_{\alpha\beta}(1,2)$ with each directed particle line running from 2 to 1.
3. Associate a factor $\mathscr{V}_0(1,2)$ with each interaction line joining points 1 and 2.
4. Integrate all internal variables: $\int d^3x_i \int_0^{\beta\hbar} d\tau_i$.
5. The indices form a matrix product along any continuous particle line. Evaluate all spin sums.
6. Multiply each nth-order diagram by $(-1/\hbar)^n(-1)^F$, where F is the number of closed fermion loops.
7. Interpret any temperature Green's function at equal values of τ as

$$\mathscr{G}^0(\mathbf{x}_i\,\tau_i, \mathbf{x}_j\,\tau_i) = \lim_{\tau_j \to \tau_i^+} \mathscr{G}^0(\mathbf{x}_i\,\tau_i, \mathbf{x}_j\,\tau_j). =$$

$$= -\frac{1}{Z_0}\, \mathrm{Tr}\left[e^{-\beta(H_0-\mu N)}\psi^\dagger(x_j,\tau_i)\psi(x_i,\tau_i)\right]$$

COMO μ É DO SISTEMA INTERAGENTE, ISSO NÃO É $\langle n \rangle(T)$ DO SISTEMA NÃO INTERAGENTE

Espaço de Fourier

PARA SISTEMAS HOMOGÊNEOS É CONVENIENTE TRANSFORMAR FOURIER DE $\vec{r}_1 - \vec{r}_2 \longrightarrow \vec{k}$

TAMBÉM TRANSFORMAREMOS FOURIER DE $\tau \longrightarrow \omega_n$

É PRECISO CUIDADO AQUI:

i) NAS INTERAÇÕES:

$$U(\vec{r},\tau) = V(\vec{r})\,\delta(\tau) = V(\vec{r})\, T \sum_{\substack{n \\ n\ \text{par}}} e^{-i\omega_n \tau}$$

$\longrightarrow$ PORTANTO, AS "CORRINAS" ATUAM COMO SE FOSSEM BÓSONS.

ii) NAS FUNÇÕES DE GREEN:

$$\mathcal{G}^{(0)}(\omega_n,\vec{k}) = \frac{1}{i\omega_n - (\epsilon_{\vec{k}} - \mu)} \longrightarrow \begin{cases} \omega_n = 2\pi n T & \text{(BÓSONS)} \\ \omega_n = (2n+1)\pi T & \text{(FÉRMIONS)} \end{cases}$$

DAQUI, AS REGRAS DOS DIAGRAMAS SEGUEM ANÁLOGAS A $T=0$

Regras diagramáticas (espaço $\mathbf{k},\omega$)

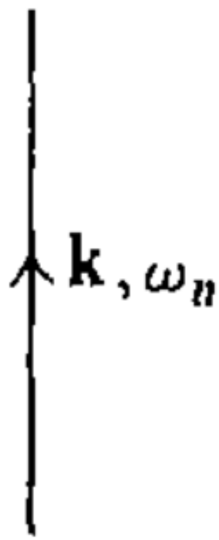
1. Draw all topologically distinct connected graphs with n interaction lines and $2n+1$ directed particle lines.
2. Assign a direction to each interaction line. Associate a wave vector and discrete frequency with each line and conserve each quantity at every vertex.
3. With each particle line associate a factor

$$\mathscr{G}^0_{\alpha\beta}(\mathbf{k},\omega_m) = \frac{\delta_{\alpha\beta}}{i\omega_m - \hbar^{-1}(\epsilon^0_k - \mu)} \tag{25.28}$$

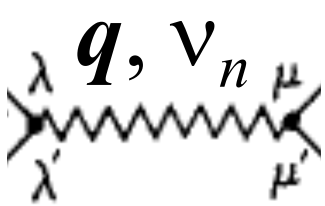
   where ω_m contains even (odd) integers for bosons (fermions).
4. Associate a factor $\mathscr{V}_0(\mathbf{k},\omega_m) \equiv V(\mathbf{k})$ with each interaction line.
5. Integrate over all n independent internal wave vectors and sum over all n independent internal frequencies.
6. The indices form a matrix product along any continuous particle line. Evaluate all matrix sums.
7. Multiply by $[-\beta\hbar^2(2\pi)^3]^{-n}(-1)^F$, where F is the number of closed fermion loops.
8. Whenever a particle line either closes on itself or is joined by the same interaction line, insert a convergence factor $e^{i\omega_m\eta}$.

Regras diagramáticas (espaço $\mathbf{k},\omega$)

Elementos dos diagramas

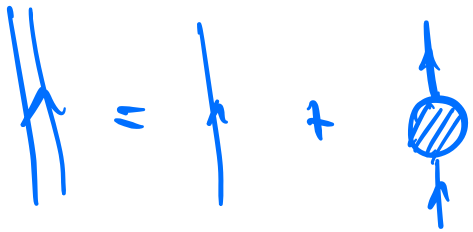


$$-\mathcal{G}^{(0)}_{\alpha\beta}\left(\mathbf{k}, i\omega_n\right) = -\frac{\delta_{\alpha,\beta}}{i\omega_n - \epsilon_{\mathbf{k}}}$$

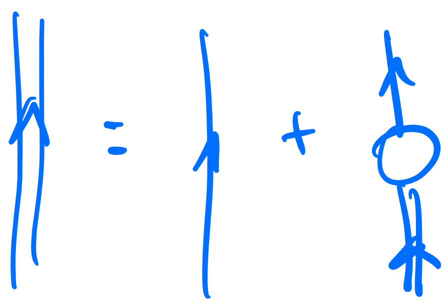


$$-U\left(\mathbf{q}\right)\delta_{\lambda,\lambda'}\delta_{\mu,\mu'}$$

Auto-energia

(1) 

ONDE  $= -\tilde{\Sigma}(\omega_m,\vec{k})$ AUTO-ENERGIA IMPRÓPRIA

(2) 

ONDE  $= -\Sigma(\omega_m,\vec{k})$ AUTO-ENERGIA PRÓPRIA

(1): $-\mathcal{G}(\omega_m,\vec{k}) = -\mathcal{G}^{(0)}(\omega_m,\vec{k}) + \left[-\mathcal{G}^{(0)}(\omega_m,\vec{k})\right]\left[-\tilde{\Sigma}(\omega_m,\vec{k})\right]\left[-\mathcal{G}^{(0)}(\omega_m,\vec{k})\right]$

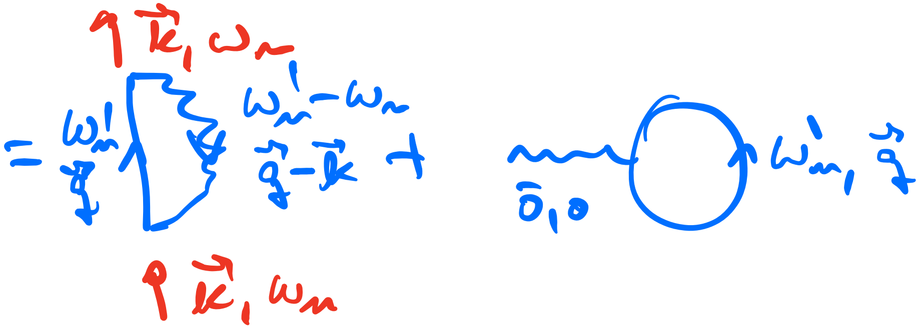
$$\Rightarrow \mathcal{G} = \mathcal{G}^{(0)} + \left[\mathcal{G}^{(0)}\right]^2\tilde{\Sigma}$$

(2): $\mathcal{G} = \mathcal{G}^{(0)} + \mathcal{G}^{(0)}\Sigma\mathcal{G} \Rightarrow (1-\mathcal{G}^{(0)}\Sigma)\mathcal{G} = \mathcal{G}^{(0)} \Rightarrow \mathcal{G} = \dfrac{\mathcal{G}^{(0)}}{1-\mathcal{G}^{(0)}\Sigma}$

$$= \frac{1}{\left[\mathcal{G}^{(0)}\right]^{-1} - \Sigma} \Rightarrow \mathcal{G}(\omega_m,\vec{k}) = \frac{1}{i\omega - \epsilon_{\vec{k}} - \Sigma(\omega_m,\vec{k})}$$

EM PRIMEIRA ORDEM: PARTÍCULAS DE SPIN S

$$-\Sigma^{(1)}(\omega_m,\vec{k}) =$$



$$\Rightarrow \Sigma^{(1)}(\omega_m,\vec{k}) = -T\sum_{\omega_m'}\int\frac{d^3q}{(2\pi)^3}\,e^{i\omega_m'\eta}\,g^{(0)}(\omega_m',\vec{q})\times\Big[V(\vec{q}-\vec{k})$$

$$+\,\zeta\,V(\vec{0})\,(2s+1)\Big]$$

SOMA SOBRE FREQUÊNCIAS ω_m':

$$H(x) = T\sum_{\omega_m}\frac{e^{i\omega_m\eta}}{i\omega_m - x}$$

ONDE:

$$\omega_m = 2\pi m T$$

OU

$$(2m+1)\pi T$$

ONDE QUEREMOS: $H(\epsilon_{\vec{q}})$