Supplemental Material for "Strong correlations protect T_c against disorder'

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A. The gap equation in AG theory

Within the $U(1)$ slave boson theory for the $t-J$ model of the cuprates, the superconducting transition *in the over-doped regime* is described by the usual BCS equation for a d-wave superconductor in which it is the spinons (the auxiliary f-fermions) which pair to form the condensate [1]. In other words, the transition is signaled by a non-zero value of the d-wave pairing order parameter $\Delta_{ij} = \left\langle f_{i\uparrow} f_{j\downarrow} - f_{i\downarrow} f_{j\uparrow} \right\rangle$. We remind the reader that at the transition the other auxiliary fields (r_i, λ_i, χ_{ij}) are well formed. The effects of non-magnetic impurities on T_c can then be treated by using the AG theory [2]. Within that theory, the linearized gap equation is written as [3]

$$\Delta_0 \;=\; \frac{2\widetilde{J}kT_c}{d} \sum_{i\omega_n} \int \frac{d^2\mathbf{k}}{(2\pi)^2} G^f(\mathbf{k}, -i\omega_n)\, G^f(-\mathbf{k}, i\omega_n)\, \Lambda(\mathbf{k}, i\omega_n)\, \Gamma_d(\mathbf{k}), \tag{1}$$

where Δ_0 is the superconducting gap amplitude, $\omega_n = (2n+1)\pi T_c$, and $\Lambda(\mathbf{k}, i\omega_n)$ is the vertex correction function, which satisfies

$$\Lambda(\mathbf{k}, i\omega_n) = \Delta_0 \Gamma_d(\mathbf{k}) + n \int \frac{d^2\mathbf{k}'}{(2\pi)^2} \left|\langle \mathbf{k}| T^f |\mathbf{k}'\rangle\right|^2 G^f(\mathbf{k}', -i\omega_n)\, G^f(-\mathbf{k}', i\omega_n)\, \Lambda(\mathbf{k}', i\omega_n). \tag{2}$$

In the last equation, $\langle \mathbf{k}| T^f |\mathbf{k}'\rangle$ is the single-impurity scattering T-matrix for the spinons. The disorder-averaged spinon Green's function in Eqs. (1) and (2) is [2]

$$G^f(\mathbf{k}, i\omega_n) = \frac{1}{i\omega_n\left(1 + \frac{1}{2\tau_{\mathbf{k}}|\omega_n|}\right) - \widetilde{h}(\mathbf{k}) - \nu_0}, \tag{3}$$

where $\widetilde{h}(\mathbf{k}) = -\left(tx + \widetilde{J}\chi\right)\Gamma_s(\mathbf{k})$ is the renormalized dispersion, $\nu_0 = \lambda_0 - \mu_0$ is the negative value of the renormalized chemical potential, which controls the doping level, and

$$\frac{1}{\tau_{\mathbf{k}}} \equiv 2\pi n \int \frac{d^2\mathbf{k}'}{(2\pi)^2} \left|\langle \mathbf{k}| T^f |\mathbf{k}'\rangle\right|^2 \delta\left[\widetilde{h}(\mathbf{k}) - \widetilde{h}(\mathbf{k}')\right], \tag{4}$$

is the quasiparticle scattering rate.

Following the arguments in [2, 3], Eq. (2) can be solved to first order to give

$$\Lambda(\mathbf{k}, i\omega_n) = \Delta_0 \left[\Gamma_d(\mathbf{k}) + \frac{1}{2\tau^d_{\mathbf{k}}|\omega_n|\left(1 + \frac{1}{2\tau_{\mathbf{k}}|\omega_n|}\right)}\right], \tag{5}$$

where

$$\frac{1}{\tau^d_{\mathbf{k}}} \;\equiv\; 2\pi n \int \frac{d^2\mathbf{k}'}{(2\pi)^2} \left|\langle \mathbf{k}| T^f |\mathbf{k}'\rangle\right|^2 \delta\left(\widetilde{h}(\mathbf{k}) - \widetilde{h}(\mathbf{k}')\right)\Gamma_d(\mathbf{k}'), \tag{6}$$

is a different scattering rate. We will show later that [see Eq. (33)], when calculated at the approximately circular Fermi surface $\mathbf{k} = k_F\hat{\mathbf{k}}$, the quasiparticle scattering time is essentially isotropic $\tau_{\mathbf{k}} \approx \tau$, whereas

$$\frac{1}{\tau^d_{\mathbf{k}}} = \frac{1}{\tau^d}\Gamma_d(\mathbf{k}). \tag{7}$$

We can thus write

$$\Lambda(\mathbf{k},i\omega_n)=\Delta_0\Gamma_d(\mathbf{k})\left[1+\frac{1}{2\tau^d|\omega_n|\left(1+\frac{1}{2\tau|\omega_n|}\right)}\right]. \tag{8}$$

Eq. (2) can now be solved to all orders by noting that

$$\begin{aligned}\Lambda(\mathbf{k},i\omega_n) &= \Delta_0\Gamma_d(\mathbf{k})\left\{1+\frac{1}{2\tau^d|\omega_n|\left(1+\frac{1}{2\tau|\omega_n|}\right)}+\left[\frac{1}{2\tau^d|\omega_n|\left(1+\frac{1}{2\tau|\omega_n|}\right)}\right]^2+\ldots\right\}\\ &= \Delta_0\Gamma_d(\mathbf{k})\left[1-\frac{1}{2\tau^d|\omega_n|\left(1+\frac{1}{2\tau|\omega_n|}\right)}\right]^{-1}\\ &= \Delta_0\Gamma_d(\mathbf{k})\frac{|\omega_n|+\frac{1}{2\tau}}{|\omega_n|+\frac{1}{2\tau}-\frac{1}{2\tau^d}}. \end{aligned}\tag{9}$$

Plugging this result into Eq. (1), we find that T_c is determined by:

$$1=\frac{2\widetilde{J}kT_c}{d}\sum_{i\omega_n}\int\frac{d^2\mathbf{k}}{(2\pi)^2}G^f(\mathbf{k},-i\omega_n)G^f(-\mathbf{k},i\omega_n)\frac{|\omega_n|+\frac{1}{2\tau}}{|\omega_n|+\frac{1}{2\tau}-\frac{1}{2\tau^d}}\Gamma_d^2(\mathbf{k}). \tag{10}$$

The momentum integral is, as usual, dominated by the region close to the renormalized Fermi surface, which we assume to be approximately circular in the over-doped region. We thus get, using polar coordinates in the (k_x,k_y)-plane,

$$1=\frac{\widetilde{J}kT_c}{d}\sum_{i\omega_n}\int\frac{d\theta}{2\pi}\frac{1}{|\omega_n|+\frac{1}{2\tau}-\frac{1}{2\tau^d}}\Gamma_d^2\left(k_F\hat{\mathbf{k}}\right). \tag{11}$$

Using now $\Gamma_d\left(k_F\hat{\mathbf{k}}\right)=2(\cos k_x-\cos k_y)\approx k_x^2-k_y^2\approx k_F^2\cos(2\theta)$ we get

$$1=\frac{\widetilde{J}kT_cm^*k_F^2}{d}\sum_{n\geq 0}\frac{1}{\omega_n+\frac{1}{2\tau}-\frac{1}{2\tau^d}}. \tag{12}$$

As usual, the integral is formally divergent, but by comparing with the equally divergent expression for the clean transition temperature T_{c0}, we can get the ratio of clean (T_{c0}) to dirty (T_c) transition temperatures [2]

$$\ln\frac{T_{c0}}{T_c}=\psi\left(\frac{1}{2}+\frac{\alpha}{2}\right)-\psi\left(\frac{1}{2}\right), \tag{13}$$

where $\alpha=\frac{1}{2\pi T_c}\left(\frac{1}{\tau}-\frac{1}{\tau^d}\right)\equiv\frac{1}{2\pi T_c\tau_{pb}}$. The leading behavior is

$$T_c = T_{c0}-\frac{\pi}{8\tau_{pb}}. \tag{14}$$

The relevant scattering rates τ and τ_d will be calculated in the next Section.

B. T-matrix for the spinons and the pair-breaking scattering rate

As seen in Section A, the suppression of the superconducting transition temperature T_c by disorder requires the determination of the scattering T-matrix of the f fermions. We will find it to first order in the disorder. In other words, the fields (r_i,λ_i) will be calculated to $\mathscr{O}(\varepsilon_i)$. We define the renormalized site energy for the f electrons as $v_i\equiv\varepsilon_i-\mu_0+\lambda_i$, whose clean value limit is $v_0=\lambda_0-\mu_0$. Clean and disordered f-fermion Green's functions are given by, respectively,

$$\mathbf{G}_0^{f-1} = \left[i\omega_n\mathbf{1} + r_0^2 t\boldsymbol{\Gamma}_s - v_0\mathbf{1} + \widetilde{J}\chi\boldsymbol{\Gamma}_s\right], \tag{15}$$

$$\mathbf{G}^{f-1} = \left[i\omega_n\mathbf{1} - \mathbf{v} + t\mathbf{r}\boldsymbol{\Gamma}_s\mathbf{r} + \widetilde{J}\chi\boldsymbol{\Gamma}_s\right], \tag{16}$$

where we have used boldface to denote matrices in the lattice site basis, whose elements are $\mathbf{1}_{ij} = \delta_{ij}$, $\mathbf{r}_{ij} = r_i\delta_{ij}$, $\mathbf{v}_{ij} = v_i\delta_{ij}$, and $(\boldsymbol{\Gamma}_s)_{ij}$ is equal to 1 if sites i and j are nearest neighbors and zero otherwise. We remind the reader that we are neglecting spatial fluctuations of the χ_{ij} field. The spinon T-matrix is defined through

$$\mathbf{G}^f = \mathbf{G}_0^f + \mathbf{G}_0^f\mathbf{T}^f\mathbf{G}_0^f = \mathbf{G}_0^f\left(\mathbf{1} + \mathbf{T}^f\mathbf{G}_0^f\right), \tag{17}$$

from which we obtain

$$\mathbf{G}^{f-1} = \left(\mathbf{1} + \mathbf{T}^f\mathbf{G}_0^f\right)^{-1}\mathbf{G}_0^{f-1}, \tag{18}$$

and, to first order in the disorder,

$$\mathbf{G}^{f-1} \approx \left(\mathbf{1} - \mathbf{T}^f\mathbf{G}_0^f\right)\mathbf{G}_0^{f-1} = \mathbf{G}_0^{f-1} - \mathbf{T}^f. \tag{19}$$

Thus, again to first order,

$$\begin{aligned}\mathbf{T}^f &= \mathbf{G}_0^{f-1} - \mathbf{G}^{f-1}\\ &= (\mathbf{v} - v_0\mathbf{1}) - t\mathbf{r}\boldsymbol{\Gamma}_s\mathbf{r} + r_0^2 t\boldsymbol{\Gamma}_s\\ &= \delta\mathbf{v} - tr_0\left(\delta\mathbf{r}\boldsymbol{\Gamma}_s + \boldsymbol{\Gamma}_s\delta\mathbf{r}\right),\end{aligned} \tag{20}$$

where $\delta\mathbf{r} \equiv (\mathbf{r} - r_0\mathbf{1})$ and $\delta\mathbf{v} \equiv (\mathbf{v} - v_0\mathbf{1})$. Defining $\delta v_i \equiv \varepsilon_i + \lambda_i - \lambda_0$ and $\delta r_i = r_i - r_0$, we have, in components,

$$\mathbf{T}_{ij}^f = \delta v_i\delta_{ij} - r_0\left(\delta r_i + \delta r_j\right)t_{ij}. \tag{21}$$

All we need now is to find δr_i and δv_i to first order in ε_i. This was already obtained in reference [4] (see, in particular, the Supplemental Material). After setting in those equations the gap and its fluctuations to zero $\boldsymbol{\Delta} = \delta\boldsymbol{\Delta} = 0$ (normal state) and $\delta\boldsymbol{\chi} = 0$ (as being negligible), we obtain in $\mathbf{k}$-space

$$\Pi^a(\mathbf{k})\,\delta v(\mathbf{k}) + r_0\left[1 + \Pi^b(\mathbf{k})\right]\delta r(\mathbf{k}) = 0, \tag{22}$$

$$\left[\lambda_0 - \frac{\lambda_0}{2d}\Gamma_s(\mathbf{k})\right]\delta r(\mathbf{k}) + r_0\delta v(\mathbf{k}) = r_0\varepsilon(\mathbf{k}), \tag{23}$$

where

$$\Pi^a(\mathbf{k}) = \frac{1}{V}\sum_{\mathbf{q}}\frac{f\left[\widetilde{h}(\mathbf{q}+\mathbf{k})\right] - f\left[\widetilde{h}(\mathbf{q})\right]}{\widetilde{h}(\mathbf{q}+\mathbf{k}) - \widetilde{h}(\mathbf{q})}, \tag{24}$$

$$\Pi^b(\mathbf{k}) = \frac{1}{V}\sum_{\mathbf{q}}\frac{f\left[\widetilde{h}(\mathbf{q}+\mathbf{k})\right] - f\left[\widetilde{h}(\mathbf{q})\right]}{\widetilde{h}(\mathbf{q}+\mathbf{k}) - \widetilde{h}(\mathbf{q})}\left[h(\mathbf{q}+\mathbf{k}) + h(\mathbf{q})\right], \tag{25}$$

where d is the lattice dimension ($d = 2$, for our purposes), $f(x)$ is the Fermi-Dirac function, $\Gamma_s(\mathbf{k}) = 2(\cos k_x + \cos k_y)$, and $h(\mathbf{k}) = -t\Gamma_s(\mathbf{k})$ is the bare energy dispersion. Solving Eqs. (22)-(23) for $\delta r(\mathbf{k})$ and $\delta v(\mathbf{k})$

$$\delta v(\mathbf{k}) = -x\frac{\Pi(\mathbf{k})\,\varepsilon(\mathbf{k})}{\lambda_0 - \frac{\lambda_0}{2d}\Gamma_s(\mathbf{k}) - x\Pi(\mathbf{k})}, \tag{26}$$

$$\delta r(\mathbf{k}) = r_0\frac{\varepsilon(\mathbf{k})}{\lambda_0 - \frac{\lambda_0}{2d}\Gamma_s(\mathbf{k}) - x\Pi(\mathbf{k})}, \tag{27}$$

where we used $x = r_0^2$ and defined

$$\Pi(\mathbf{k}) \equiv \frac{1+\Pi^b(\mathbf{k})}{\Pi^a(\mathbf{k})}. \tag{28}$$

Therefore

$$\begin{aligned} \langle \mathbf{k}| T^f |\mathbf{k}'\rangle &= \delta v\left(\mathbf{k}'-\mathbf{k}\right) + r_0\left[h(\mathbf{k}) + h\left(\mathbf{k}'\right)\right]\delta r\left(\mathbf{k}'-\mathbf{k}\right) & (29)\\ &= x\left[\frac{h(\mathbf{k})+h(\mathbf{k}')-\Pi(\mathbf{k}'-\mathbf{k})}{\lambda_0 - \frac{\lambda_0}{2d}\Gamma_s(\mathbf{k}'-\mathbf{k}) - x\Pi(\mathbf{k}'-\mathbf{k})}\right]\varepsilon\left(\mathbf{k}'-\mathbf{k}\right). & (30)\end{aligned}$$

The result in Eq. (30) is general. For the T_c calculation within the AG theory, we only need it for a single impurity [see Eq. (2)]. We therefore set $\varepsilon_i = t\delta_{i,0}$ or $\varepsilon(\mathbf{k}) = t$. We must now plug Eq. (30) into Eqs. (4) and (6). Since the superconducting pairing is mostly affected by the scattering near Fermi surface, we can set $\mathbf{k} = k_F\hat{\mathbf{k}}$ in Eq. (30). For computations, k_F is taken as the magnitude of the Fermi momentum averaged over the approximately circular Fermi surface. We can thus make the following simplifications: $h(\mathbf{k}) + h(\mathbf{k}') = 2E_F$, $\Pi^b(\mathbf{k}) = 2E_F\Pi^a(\mathbf{k})$, $\Pi(\mathbf{k}) = [\Pi^a(\mathbf{k})]^{-1} + 2E_F$, and $\Pi^a(\mathbf{k}-\mathbf{k}') = -\rho^* g_L(y)$, where $y = \frac{|\mathbf{k}-\mathbf{k}'|}{2k_F} = \left|\sin\left(\frac{\varphi}{2}\right)\right|$, $\varphi = \theta-\theta'$ is the angle between $\mathbf{k}$ and $\mathbf{k}'$, E_F is the bare Fermi energy (obtained by solving the mean-field equations with the constraint of an electron filling of $1-x$), the function $g_L(y)$ is defined as [5]

$$g(y) \equiv \begin{cases} 1 & y \le 1, \\ 1-\sqrt{1-y^{-2}} & y > 1, \end{cases} \tag{31}$$

and $\rho^* = \frac{m^*}{2\pi}$ is the renormalized (spinon) density of states. Finally, defining

$$g(y) \equiv \frac{t^2}{\left\{\rho^*\lambda_0 k_F^2 y^2 g_L(y) + x\left[1-2\rho^* E_F g_L(y)\right]\right\}^2}, \tag{32}$$

we obtain

$$\begin{aligned} \frac{1}{\tau^d(\theta)} &= x^2\frac{nm^*}{2\pi}\int_0^{2\pi} d\theta' g\left[\left|\sin\left(\frac{\theta-\theta'}{2}\right)\right|\right]\cos\left(2\theta'\right) \\ &= x^2\frac{nm^*}{2\pi}\int_0^{2\pi} du g\left[\left|\sin\left(\frac{u}{2}\right)\right|\right]\cos\left[2(\theta-u)\right] \\ &= \cos 2\theta\left[x^2\frac{nm^*}{2\pi}\int_0^{2\pi} du g\left[\left|\sin\left(\frac{u}{2}\right)\right|\right]\cos 2u\right] & (33)\\ &\equiv \cos 2\theta\frac{1}{\tau^d} = \Gamma_d(\mathbf{k})\frac{1}{\tau_d}. \end{aligned}$$

In the last step we dropped the term in $\sin 2\theta$ since this term vanishes after integration in Eq.(1). Analogously,

$$\frac{1}{\tau} = x^2\frac{nm^*}{2\pi}\int_0^{2\pi} d\theta g\left[\left|\sin\left(\frac{\theta}{2}\right)\right|\right], \tag{34}$$

and

$$\frac{1}{\tau_{pb}} = x^2\frac{nm^*}{2\pi}\int_0^{2\pi} d\theta g\left[\left|\sin\left(\frac{\theta}{2}\right)\right|\right](1-\cos 2\theta). \tag{35}$$

C. T-matrix for the physical electrons and the normal state transport scattering rate

In order to describe transport in the normal state, we must analyze the physical electron scattering T-matrix. The calculation is analogous to the one in Section B. The bare and renormalized Green's functions for the physical electrons are given by

$$\begin{aligned} \mathbf{G}_0^{e-1} &= r_0^{-2}\left[i\omega_n\mathbf{1} + r_0^2 t\boldsymbol{\Gamma}_s - v_0\mathbf{1} + \widetilde{J}\chi\boldsymbol{\Gamma}_s\right], & (36)\\ \mathbf{G}^{e-1} &= \mathbf{r}^{-1}\left[i\omega_n\mathbf{1} - \mathbf{v} + t\mathbf{r}\boldsymbol{\Gamma}_s\mathbf{r} + \widetilde{J}\chi\boldsymbol{\Gamma}_s\right]\mathbf{r}^{-1}. & (37)\end{aligned}$$

Proceeding to first order in the disorder as before yields

$$\begin{aligned}\mathbf{T}^{e} &= \mathbf{G}_{0}^{e-1}-\mathbf{G}^{e-1}\\ &= r_{0}^{-2}\delta\mathbf{v}-2v_{0}r_{0}^{-3}\delta\mathbf{r}+\widetilde{J}\chi\left(r_{0}^{-2}\boldsymbol{\Gamma}_{s}-\mathbf{r}^{-1}\boldsymbol{\Gamma}_{s}\mathbf{r}^{-1}\right)\\ &= x^{-1}\left[\delta\mathbf{v}-2v_{0}r_{0}^{-1}\delta\mathbf{r}+\widetilde{J}\chi r_{0}^{-1}\left(\delta\mathbf{r}\boldsymbol{\Gamma}_{s}+\boldsymbol{\Gamma}_{s}\delta\mathbf{r}\right)\right].\end{aligned} \tag{38}$$

$$\left\langle\mathbf{k}\right|T^{e}\left|\mathbf{k}'\right\rangle = -\left\{\frac{\Pi\left(\mathbf{k}'-\mathbf{k}\right)+\frac{2v_{0}}{x}+\frac{\widetilde{J}\chi}{tx}\left[h\left(\mathbf{k}\right)+h\left(\mathbf{k}'\right)\right]}{\lambda_{0}-\frac{\lambda_{0}}{2d}\Gamma_{s}\left(\mathbf{k}'-\mathbf{k}\right)-x\Pi\left(\mathbf{k}'-\mathbf{k}\right)}\right\}\varepsilon\left(\mathbf{k}'-\mathbf{k}\right). \tag{39}$$

This scattering T-matrix can now be used to calculate the transport scattering rate that enters the expression for the conductivity in the normal state

$$\frac{1}{\tau_{\mathbf{k}}^{tr}}\equiv 2\pi xn\int\frac{d^{2}\mathbf{k}'}{\left(2\pi\right)^{2}}\left|\left\langle\mathbf{k}\right|T^{e}\left|\mathbf{k}'\right\rangle\right|^{2}\delta\left[\widetilde{h}\left(\mathbf{k}\right)-\widetilde{h}\left(\mathbf{k}'\right)\right]\left(1-\cos\varphi\right), \tag{40}$$

where, as before, φ is the angle between $\mathbf{k}$ and $\mathbf{k}'$ and the x factor in front of the integral comes from the quasiparticle weight of the physical electron Green's function. We now focus again on wave vectors close to the approximately circular Fermi surface and implement the same approximations used in the previous Section. We first note that the numerators of the fractions is Eqs. (30) and (39), though seemingly different, are actually the same, since

$$\frac{2v_{0}}{x}+\frac{\widetilde{J}\chi}{tx}\left[h\left(\mathbf{k}\right)+h\left(\mathbf{k}'\right)\right]=-2\frac{tx+\widetilde{J}\chi}{tx}E_{F}+2\frac{\widetilde{J}\chi}{tx}E_{F}=-2E_{F}, \tag{41}$$

where the negative of the renormalized chemical potential $v_{0}=-\frac{m}{m^{*}}E_{F}=-\frac{tx+\widetilde{J}\chi}{t}E_{F}$. Like the quasiparticle scattering rate, the transport scattering rate does not depend on the wave vector direction within the assumed approximations and we get

$$\frac{1}{\tau^{tr}}=\frac{xnm^{*}}{2\pi}\int_{0}^{2\pi}d\theta g\left[\left|\sin\left(\frac{\theta}{2}\right)\right|\right]\left(1-\cos\theta\right). \tag{42}$$

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