# Spin-Liquid Behavior in Electronic Griffiths Phases

D. Tanasković,[1] V. Dobrosavljević,[1] and E. Miranda[2]

[1]Department of Physics and National High Magnetic Field Laboratory, Florida State University, Tallahassee, Florida 32306, USA
[2]Instituto de Física Gleb Wataghin, Unicamp, Caixa Postal 6165, Campinas, SP, CEP 13083-970, Brazil

(Received 7 December 2004; published 13 October 2005)

We examine the interplay of the Kondo effect and the RKKY interactions in electronic Griffiths phases using extended dynamical mean-field theory methods. We find that sub-Ohmic dissipation is generated for sufficiently strong disorder, leading to the suppression of Kondo screening on a finite fraction of spins, and giving rise to universal spin-liquid behavior.

DOI: 10.1103/PhysRevLett.95.167204 PACS numbers: 75.20.Hr, 71.27.+a, 75.10.Nr

Disorder induced non-Fermi liquid (NFL) behavior has remained an important focus of study in heavy fermion systems [1]. According to the ''magnetic Griffiths phase'' scenario [2], this behavior reflects the formation of rare magnetic clusters with large susceptibilities, similarly as in disordered insulating magnets [3]. Another approach focused on the interplay of disorder and the Kondo effect far away from any magnetic ordering [4]. More recent work [5,6] has demonstrated that such ''electronic Griffiths phases'' are a generic feature of strongly correlated electronic systems with disorder. Neither picture, however, seems satisfactory for the following key reason: in both scenarios, the resulting NFL behavior is characterized by power law anomalies, with nonuniversal, rapidly varying powers. In contrast, most experimental data seem to show reasonably weak anomalies, close to marginal Fermi liquid behavior [1].

Physically, it is clear what is missing from the theory. Similarly as magnetic Griffiths phases, the electronic Griffiths phase is characterized [4–6] by a broad distribution $P(T_K) \sim (T_K)^{\alpha-1}$ of local energy scales (Kondo temperatures), with the exponent $\alpha \sim W^{-2}$ rapidly decreasing with disorder $W$. At any given temperature, the local moments with $T_K(i) < T$ remain unscreened. As disorder increases, the number of such unscreened spins rapidly proliferates. Within the existing theory [4–6] these unscreened spins act essentially as free local moments and provide a very large contribution to the thermodynamic response. In a more realistic description, however, even the Kondo-unscreened spins are *not* completely free, since the metallic host generates long-ranged Ruderman-Kittel-Kasuya-Yosida (RKKY) interactions even between relatively distant spins. As the RKKY interaction has an oscillatory character, the effective interactions between the (randomly located) unscreened spins will be random in magnitude and sign. In this Letter, we use an extended dynamical mean-field formulation to examine the role of such RKKY interactions within the electronic Griffiths phase scenario.

Our main results are as follows: (a) for disorder $W$ weaker than a critical value $W_c$ we find Fermi liquid behavior, but for $W > W_c$ the Kondo effect is suppressed on a finite fraction of spins, resulting in a spin-liquid phase displaying (universal) marginal Fermi liquid behavior; (b) the spins that remain screened are still characterized by a power law distribution of (renormalized) energy scales $P(T^*) \sim (T^*)^{\alpha^*-1}$, but the exponent $\alpha^*$ acquires a universal value $\alpha^* \approx 1/2$ throughout the spin-liquid phase; (c) the spin-liquid phase is unstable to spin glass ordering at the lowest temperatures, but we find robust marginal Fermi liquid behavior in a broad temperature window above the freezing temperature.

We consider the disordered Kondo lattice model as given by the Hamiltonian

$$H = -t\sum_{\langle ij\rangle\sigma}(c^\dagger_{i\sigma}c_{j\sigma} + \text{H.c.}) + \sum_{i\sigma}\upsilon_i c^\dagger_{i\sigma}c_{i\sigma} + J_K\sum_i \mathbf{S}_i\cdot\mathbf{s}_i + \sum_{\langle ij\rangle}J_{ij}\mathbf{S}_i\cdot\mathbf{S}_j, \quad (1)$$

where the exchange couplings $J_{ij}$ between localized spins, and site energies $\upsilon_i$ are distributed according to Gaussian distributions, $P_J(J_{ij}) \sim \exp(-J_{ij}^2/2J^2)$ and $P_W(\upsilon_i) \sim \exp(-\upsilon_i^2/2W^2)$ [6]. In this expression, $\mathbf{S}_i$ and $\mathbf{s}_i = \frac{1}{2}\sum_{\alpha\beta}c^\dagger_{i\alpha}\boldsymbol{\sigma}_{\alpha\beta}c_{i\beta}$ represent a localized spin and the conduction electron spin density at site $i$, respectively. We concentrate on the paramagnetic phase. Applying the standard procedure to average over disorder in the $J_{ij}$ couplings [7] and taking the limit of infinite coordination $z \to \infty$ [8], the local effective action assumes the form

$$\mathcal{A}_j = \sum_\sigma\int_0^\beta d\tau\int_0^\beta d\tau' c^\dagger_{j\sigma}(\tau)[(\partial_\tau - \mu + \upsilon_j)\delta(\tau-\tau') - t^2G_c(\tau-\tau')]c_{j\sigma}(\tau') + J_K\int_0^\beta d\tau\mathbf{S}_j(\tau)\cdot\mathbf{s}_j(\tau) - \frac{J^2}{2}\int_0^\beta d\tau\int_0^\beta d\tau'\chi(\tau-\tau')\mathbf{S}_j(\tau)\cdot\mathbf{S}_j(\tau'). \quad (2)$$

The local action of Eq. (2) describes the so-called Bose-Fermi Kondo model [9,10], which describes a Kondo spin interacting with both a fermionic bath of conduction electrons and a bosonic bath of spin fluctuations. For a disordered Kondo lattice, we must consider an ensemble of such impurity models supplemented by

 

following self-consistency conditions. The bosonic spin bath $\chi(\tau) = \overline{\langle T_\tau \mathbf{S}_j(\tau)\cdot\mathbf{S}_j(0)\rangle_{\mathcal{A}_j}} = \int d\upsilon_j P_W(\upsilon_j)\times \langle T_\tau \mathbf{S}_j(\tau)\cdot\mathbf{S}_j(0)\rangle_{\mathcal{A}_j}$ and the conduction electron bath $G_c(\tau) = \overline{G_{cj}(\tau)} = -\overline{\langle T_\tau c_{j\sigma}(\tau)c^\dagger_{j\sigma}(0)\rangle_{\mathcal{A}_j}}$ are obtained by appropriate disorder averaging [6], and for simplicity we use a simple semicircular model density of states for conduction electrons.

*Destruction of the Kondo effect.*—The presence of RKKY interactions introduces a qualitative modification in the dynamics of the Kondo spins, through the presence of a dissipative bosonic bath of spin fluctuations. This behavior depends crucially on the precise spectral form of the bosonic bath, allowing for the destruction of the Kondo effect in the presence of sub-Ohmic dissipation [9,10]. For a spectrum of the form

$$\chi(i\omega_n) \sim \chi(0) - C|\omega_n|^{1-\varepsilon},$$

Fermi liquid behavior is recovered for $\varepsilon = 0$, but for $\varepsilon > 0$ (sub-Ohmic dissipation), and for sufficiently small bare Kondo temperature $T_K$, the spin decouples from the conduction electrons. Within an electronic Griffiths phase, however, the disordered Kondo lattice has a very broad distribution of local Kondo temperatures $P(T_K) \sim (T_K)^{\alpha-1}$. Therefore, for $\varepsilon > 0$ and arbitrarily weak coupling to the bosonic bath (i.e., weak RKKY interaction), a fraction of the spins will decouple.

To obtain a sufficient condition for decoupling, we examine the stability of the Fermi liquid solution by considering the limit of infinitesimal RKKY interactions. To leading order we replace $\chi(\tau) \to \chi_o(\tau) \equiv \chi(\tau; J = 0)$, and the calculation reduces to the "bare model" of Ref. [6]. The resulting bosonic bath, which is an average over the site-dependent local dynamic spin susceptibility, $\chi_o(i\omega_n) = \int dT_K P(T_K)\chi(T_K, i\omega_n)$, has a Fermi liquid form in the presence of weak disorder. However, for stronger randomness, $W > W^* \approx \sqrt{t^2\rho_c J_K}/2$ corresponding to $\alpha < 2$ (here, $\rho_c$ is the density of states for conduction electrons) [6], the power law distribution of energy scales within a Griffiths phase produces sub-Ohmic dissipation, corresponding to $\varepsilon = 2 - \alpha > 0$. Note that the estimate based on the bare theory sets an *upper bound* for the true critical disorder strength, i.e., $W_c < W^* = W_{\mathrm{NFL}}/\sqrt{2}$ [here, $W_{\mathrm{NFL}} \approx \sqrt{t^2\rho_c J_K/2}$ is the threshold for NFL behavior in the bare model [6], corresponding to $\alpha = 1$]. We emphasize that within the electronic Griffiths phase, such decoupling emerges for $W > W_c$ even for arbitrarily small $J$, in contrast to the clean case [11] where much stronger RKKY interactions ($J > J_c \approx 10T_K$) [12] are required to destroy the Kondo effect.

*The spin-liquid phase.*—For finite $J$, the actual value of $\varepsilon$ has to be self-consistently determined as follows. For $W > W_c$, the spins break up into two groups: the decoupled spins and those that remain Kondo screened. Since the self-consistent bosonic bath function $\chi(i\omega_n)$ is an algebraic average over all spins, it is an additive function of the contributions from each fluid

$$\chi(i\omega_n) = n\chi_{dc}(i\omega_n) + (1-n)\chi_s(i\omega_n). \tag{3}$$

Here, $n$ is the fraction of spins in the decoupled phase. As we shall see, the functions $\chi_{dc}(i\omega_n)$ and $\chi_s(i\omega_n)$ both have a singular, non-Fermi liquid form characterized by exponents $\varepsilon_{dc}$ and $\varepsilon_s$, respectively. Deferring for a moment the study of the critical region (infinitesimally small $n$), we first examine the solution deep within the spin-liquid phase. The first step in the self-consistent procedure is computing $\varepsilon_{dc}$ and $\varepsilon_s$ for a given value of the bath exponent $\varepsilon$. The spin autocorrelation function in the decoupled phase assumes the form [9,10] $\chi_{dc}(\tau) = \langle T_\tau \mathbf{S}(\tau)\cdot\mathbf{S}(0)\rangle \sim 1/\tau^\varepsilon$, a result valid to all orders in $\varepsilon$ [10]. Since $\varepsilon_{dc}$ is defined by $\chi_{dc}(\tau) \sim 1/\tau^{2-\varepsilon_{dc}}$, we find

$$\varepsilon_{dc}(\varepsilon) = 2 - \varepsilon. \tag{4}$$

The nonanalytic part of $\chi_s(i\omega_n)$ comes from the spins with the smallest (renormalized) Kondo temperatures $T^*$ ("barely screened spins")

$$\chi_{bs}(i\omega_n) = \int_0^\Lambda dT^* P(T^*)\chi_{bs}(T^*, i\omega_n). \tag{5}$$

Here $P(T^*)$ is the distribution of *renormalized* Kondo temperatures (local Fermi liquid coherence scales), and $\chi_{bs}(T^*, i\omega_n)$ is the local dynamic susceptibility for a given $T^*$. Properties of the Bose-Fermi Kondo model in the critical region of the decoupling transition have been extensively studied within renormalization group (RG) [9,10] and large-$N$ approaches [11], and we use these results to calculate $\chi_{bs}$. In particular, $T^* \sim (\delta J_K)^\nu \sim (\delta T_K)^\nu$, which gives $dT^*/dT_K \sim (T^*)^{1-1/\nu}$. Therefore, $P(T^*) = P[T_K(T^*)]\frac{dT_K}{dT^*} \sim \frac{dT_K}{dT^*} \sim (T^*)^{1/\nu-1}$. From scaling arguments [9,10], $\chi_{bs}(T^*, \omega) = (T^*)^{\eta-1}\phi(\omega/T^*)$, where $\eta$ is the anomalous dimension, which is known to be exactly $\varepsilon$ [10]. Performing the integration in Eq. (5), we find at low frequencies $\chi_{bs}(i\omega_n) = \chi_{bs}(0) - C''|\omega_n|^{\eta+(1/\nu)-1}$, or, equivalently, at large times $\chi_{bs}(\tau) \sim 1/\tau^{\eta+(1/\nu)}$. By definition, $\chi_s(\tau) \sim 1/\tau^{2-\varepsilon_s}$, which gives

$$\varepsilon_s(\varepsilon) = 2 - \eta - 1/\nu. \tag{6}$$

Since $\eta = \varepsilon$ and $\nu > 0$ (as the relevant eigenvalue at the unstable fixed point), Eqs. (4) and (6) imply that $\varepsilon_{dc} > \varepsilon_s$. Therefore, $\varepsilon = \max\{\varepsilon_{dc}, \varepsilon_s\} = \varepsilon_{dc}$, and from Eq. (4) we find that the self-consistent bath is characterized by the exponent $\varepsilon = 1$, as in the spin-liquid model of Sachdev and Ye [13], producing a logarithmic divergence of the average local dynamic susceptibility. Note that, in contrast to the bare ($J = 0$) model of the electronic Griffiths phase, the renormalized distribution $P(T^*)$ of local energy scales now assumes a *universal* form characterized by an exponent $\alpha^*(\varepsilon) = 1/\nu(\varepsilon) \approx \varepsilon/2 = 1/2$ within the spin-liquid phase. More work is needed to determine the behavior of the uniform susceptibility, as well as the behavior of the specific heat.

*Transport in the spin-liquid phase.*—Although the renormalized Kondo coupling scales to zero for the decoupled spins, the precise form of the RG flows (scaling dimension of "irrelevant operators") near the spin-liquid fixed point still determines the finite frequency (or finite temperature) corrections. To leading order, the contribution from decoupled spins scales as $\rho(\omega) \sim [J_K^*(\omega)]^2$, while $J_K^*(\omega) \sim \omega^{1/\nu}$. To compute the appropriate exponent at the spin-liquid fixed point we have used the $\varepsilon$-expansion approach of Ref. [10], and we find $\nu = 2/\varepsilon + O(\varepsilon^3)$. From our self-consistent solution for the spin-liquid phase ($\varepsilon = 1$), we obtain $1/\nu \approx 1/2$, producing again a marginal Fermi liquid correction to the resistivity $\delta\rho_{dc}(\omega) \sim \omega$, or at $\omega = 0$ and finite temperature

$$\delta\rho_{dc}(T) \sim T.$$

*Numerical results.*—As an illustration of our analytical predictions, and to obtain quantitative results, we proceed to the numerical solution of our equations in the large-$N$ limit [11,13]. Introducing site-dependent slave boson parameters $r_j$ and $\varepsilon_{fj}$, and minimizing the local free energy, we come to the following saddle-point equations [11]

$$\frac{1}{\beta}\sum_{\omega_n} e^{i\omega_n 0^+} G_{fj}(i\omega_n) = \frac{1}{2}, \tag{7}$$

$$\frac{1}{\beta}\sum_{\omega_n} G_{fj}(i\omega_n)\Delta_{fj}(i\omega_n) = -\frac{1}{J_K}. \tag{8}$$

The local $f$-pseudo-fermion Green's function, $G_{fj}(\tau) = -\langle T_\tau f_{j\sigma}(\tau) f_{j\sigma}^\dagger(0)\rangle$, is given by $G_{fj}^{-1}(i\omega_n) = i\omega_n - \varepsilon_{fj} - \sum_j(i\omega_n) - r_j^2\Delta_{fj}(i\omega_n)$. The self-energy is equal to $\sum_j(\tau) = J^2\chi(\tau)G_{fj}(\tau)$, and $\Delta_{fj}^{-1}(i\omega_n) = i\omega_n + \mu - v_j - t^2 G_c(i\omega_n)$. Self-consistency requires $\chi(\tau) = -\overline{G_{fj}(\tau)G_{fj}(-\tau)}$, and $G_c(i\omega_n) = \overline{G_{cj}}(i\omega_n)$, where $G_{cj}^{-1}(i\omega_n) = \Delta_{fj}^{-1}(i\omega_n) - r_j^2/[i\omega_n - \varepsilon_{fj} - \sum_j(i\omega_n)]$.

These equations were solved on the imaginary axis at $T = 0$ using fast Fourier transform methods. The total average local dynamic susceptibility $\chi$ together with the contributions coming from Kondo screened $\chi_s$ and decoupled spins $\chi_{dc}$ is shown in Fig. 1(a). At low frequencies, the contribution from Kondo screened spins saturates to a constant, while the decoupled spins produce a logarithmic divergence. A comparison with the bare model illustrates how the strong power law divergence of $\chi$ found for $J = 0$ is suppressed by the dynamical RKKY interactions. Figure 1(b) shows how $\chi$ evolves with the change of disorder. Note that marginal Fermi liquid behavior persists up to a crossover scale $\omega_{sl} \sim 0.1T_K(v_j = 0)$ which has very weak dependence on the disorder strength.

*Critical behavior.*—Near the critical point the arguments which followed Eq. (3) have to be modified since the relative importance of the various contributions to the average local susceptibility changes. First, we concentrate on the contribution from the barely screened spins given by

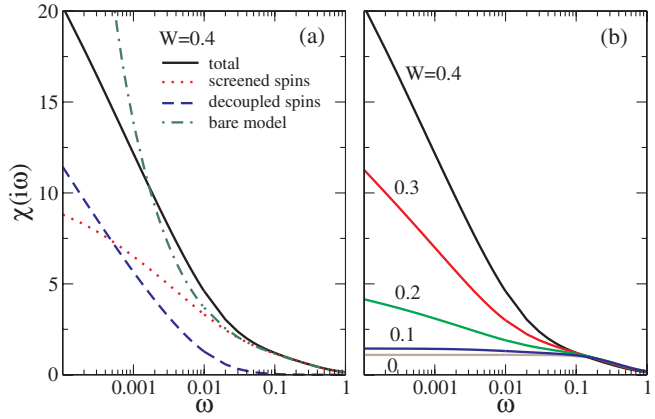


FIG. 1 (color online). Local dynamic magnetic susceptibility. (a) At low frequencies, the decoupled spins (dashed line) provide the leading logarithmic dependence of the total averaged susceptibility (full line). For the values of the parameters used ($J_K = 0.8$, $J = 0.05$, $\mu = -0.1$ in units of the half bandwidth, corresponding to $T_K(v_j = 0) = 0.1$ and $W_c \approx 0.1$), there are $n = 8\%$ of decoupled spins at $W = 0.4$. The bare model ($J = 0$) leads to a stronger nonuniversal power law singularity (dash-dotted line). (b) $\chi(i\omega)$ for the disorder strength ranging from 0 to 0.4.

Eq. (5). As before, $P(T^*) = P[T_K(T^*)]dT_K/dT^*$, but close to the transition $P(T_K)$ is small and cannot be replaced by a constant prefactor of order 1. Since $P(T_K) \approx P(T_{Kc}) \sim (T_{Kc})^{\alpha-1}$, we find $n = \int_0^{T_{Kc}} dT_K P(T_K) \sim (T_{Kc})^\alpha$, where $T_{Kc}$ is the bare Kondo temperature at the site energy $v_c$ at which the spins start to decouple. Therefore $P(T_K) \sim n^{(\alpha-1)/\alpha}$. From the bare model, we know that (for small $J$) $\alpha \approx 2$ near the critical point. Now we are in a position to write down the general form of the total bosonic bath at low frequencies

$$\chi(i\omega_n) = \chi_o - C_1|\omega_n| - C_2 n^{1/2}|\omega_n|^{\eta+(1/\nu)-1} - C_3 n\ln|\omega_n|. \tag{9}$$

The first two terms come from the well-screened spins and have the Fermi liquid form. The third term is due to the "barely screened" spins and the last term is the contribution from the decoupled spins. The crucial point is that the nonanalytic term from the barely screened spins, being proportional to $\sqrt{n}$, is much larger than the logarithmic term due to the $n$ decoupled spins, except at *exponentially small* frequencies. Therefore, we can neglect the last term in Eq. (9). Below the crossover frequency $\omega^* \sim n^{(1/2)/[2-\eta-(1/\nu)]}$, the nonanalytic term in Eq. (9) is dominant and close to the decoupling point, i.e., in the limit $\varepsilon_{fj} \to 0$ and $r_j^2 \to 0$, $G_f$ assumes the form $G_f(i\omega) \approx -iC\,\mathrm{sgn}\omega/|\omega|^{1-\varepsilon/2}$ for $|\omega| < \omega^*$, and $G_f(iw) \approx -i/w$ for $|\omega| > \omega^*$. The parameter $\varepsilon$ has to be self-consistently determined from the equation $1 - \varepsilon = \eta + \frac{1}{\nu} - 1$. Within the large-$N$ theory, $\nu = 2/\varepsilon$ which gives $\varepsilon = 4/5$. Inserting this expression into Eq. (8), we find the critical site energy for decoupling $v_c \sim \sqrt{|\ln n|}$. Since the number of decoupled spins is equal to $n = \int_{v_c}^\infty dv P_W(v)$, we have a

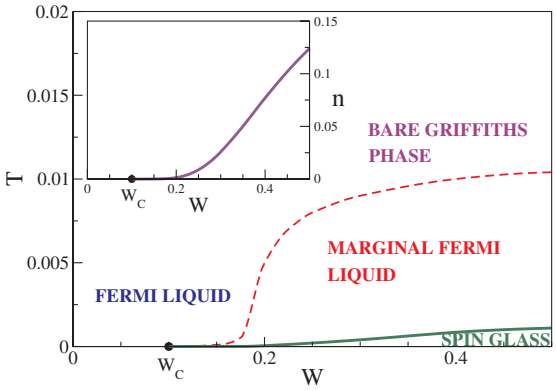


FIG. 2 (color online). Phase diagram obtained for the same values of parameters as in Fig. 1. The inset shows the fraction of decoupled spins as a function of disorder. Note that the decoupling sets in already in the presence of moderate disorder.

closed set of equations for $n(W)$, from which we find that the number of decoupled spins is exponentially small in the vicinity of the critical point

$$n \sim e^{-A/(W-W_c)}, \tag{10}$$

where $A$ is a positive constant. The numerical results for $n(W)$ are shown in the inset of Fig. 2. Interestingly, no precursors (vanishing coherence scale of the Fermi liquid) arise as the critical point is approached from the FL side, in contrast to what we have found by solving the same equations in absence of disorder [11]. This indicates a novel type of quantum critical behavior that has a character of an essential singularity, a feature that appears specific to quantum Griffiths phases.

*Spin glass instability and phase diagram.*—In this Letter we have concentrated on the paramagnetic solution of our model. However, the decoupled spins can be expected to form a spin glass (SG) at low temperatures in the presence of random intersite interactions [7]. For a rigorous treatment of the spin glass phase, one needs to go beyond the $N = \infty$ limit, but a rough estimate of the temperature for SG ordering [11] may be obtained by using the large-$N$ approach as an approximate theory for the considered $N = 2$ case. The spin glass instability criterion [7], as appropriately generalized to the case of additional site randomness, then reads

$$\sqrt{\overline{\chi_j^2}} J/\sqrt{2} = 1. \tag{11}$$

Figure 2 represents a generic phase diagram of our model. For weak disorder the system is in the Fermi liquid phase, while for $W > W_c$ the marginal Fermi liquid phase emerges. The crossover temperature (dashed line) delimiting this regime can be estimated from the frequency up to which the logarithmic behavior in $\chi(i\omega)$ is observed [14]. The spin glass phase, obtained from Eq. (11), appears only at the lowest temperatures, well below the marginal Fermi liquid boundary [14]. Interestingly, recent experiments have indeed found evidence of dynamical spin freezing in the milliKelvin temperature range for some Kondo alloys [15].

To summarize, we have introduced and solved a disordered Kondo lattice model with random intersite RKKY interactions. Our solution, valid within extended dynamical mean-field theory, illustrates how non-Ohmic dissipation arising from intersite RKKY interaction restores universality for non-Fermi liquid behavior of electronic Griffiths phases. Although considerably different in detail, this dissipative mechanism is reminiscent of the processes leading to dynamical freezing of droplets within magnetic Griffiths phases [16], suggesting a generic role of RKKY interactions in disordered heavy fermion systems.

The authors thank Antoine Georges, Daniel Grempel, Dirk Morr, and Qimiao Si for useful discussions. This work was supported by the NSF through Grant No. NSF-0234215 (V. D. and D. T.), FAPESP through Grant No. 01/00719-8 (E. M.), CNPq through Grant No. 302535/02-0 (E. M.), FAEP through Grant No. 0268/02 (E. M.), and the NHMFL.

[1] For a review of experiments on disordered heavy fermion systems, see: G. R. Stewart, Rev. Mod. Phys. **73**, 797 (2001); D. E. MacLaughlin *et al.*, J. Phys. Condens. Matter **16**, S4479 (2004).
[2] A. H. Castro Neto *et al.*, Phys. Rev. Lett. **81**, 3531 (1998); A. H. Castro Neto and B. A. Jones, Phys. Rev. B **62**, 014975 (2000).
[3] R. B. Griffiths, Phys. Rev. Lett. **23**, 17 (1969).
[4] E. Miranda, V. Dobrosavljević, and G. Kotliar, Phys. Rev. Lett. **78**, 290 (1997); J. Phys. Condens. Matter **8**, 9871 (1996).
[5] E. Miranda and V. Dobrosavljević, Phys. Rev. Lett. **86**, 264 (2001).
[6] D. Tanasković, E. Miranda, and V. Dobrosavljević, Phys. Rev. B **70**, 205108 (2004).
[7] A. J. Bray and M. A. Moore, J. Phys. C **13**, L655 (1980).
[8] A. Georges *et al.*, Rev. Mod. Phys. **68**, 13 (1996).
[9] J. L. Smith and Q. Si, Europhys. Lett. **45**, 228 (1999); A. M. Sengupta, Phys. Rev. B **61**, 4041 (2000).
[10] L. Zhu and Q. Si, Phys. Rev. B **66**, 024426 (2002); G. Zaránd and E. Demler, Phys. Rev. B **66**, 024427 (2002).
[11] S. Burdin, D. R. Grempel, and A. Georges, Phys. Rev. B **66**, 045111 (2002).
[12] We have numerically solved the equations of Ref. [11], and found that $J_c \approx 10T_K$, in agreement with an improved analytical estimate [D. Grempel (private communication)].
[13] S. Sachdev and J. Ye, Phys. Rev. Lett. **70**, 3339 (1993).
[14] For quantitative results, we need to solve the large-$N$ equations at finite temperature. However, for the purpose of obtaining a rough phase diagram, we have replaced temperature with frequency in $\chi_j(T)$, using the well-known $\omega/T$ scaling of the Bose-Fermi Kondo model.
[15] D. E. MacLaughlin *et al.*, Phys. Rev. Lett. **87**, 066402 (2001).
[16] V. Dobrosavljević and E. Miranda, Phys. Rev. Lett. **94**, 187203 (2005).