Universidade Estadual de Campinas
Instituto de Física Gleb Wataghin
F 609 – Tópicos de Ensino de Física I

# Relatório Final

Harmonógrafo – Registrador do Movimento Harmônico de Pêndulos


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## Introdução

Um exemplo de um pêndulo é um fio com um pesinho numa ponta e a outra presa em algum lugar (uma viga, por exemplo) de modo que deixe o resto do fio com o pesinho livre para se movimentar. Dando um pequeno empurrão, o peso passa a se movimentar sempre no mesmo lugar (isto é, reto, ou fazendo círculos, elipses...) e num certo ritmo até parar. Este tipo de movimento chama-se movimento harmônico. Imagine agora dois pêndulos. Um movendo em uma direção e outro numa diferente, o que aconteceria se juntássemos estes dois movimentos em um único? Formaria um movimento circular? Reto? Elipse? Ou um outro que não dá nem para definir um nome? Seria caótico?

Este projeto consiste em construir um aparato com dois pêndulos, sendo que em um deles está fixa uma caneta que permite reproduzir tais movimentos em uma folha de papel presa no outro pêndulo, possibilitando a visualização destes movimentos como figuras geométricas. Este aparato tem o nome de harmonógrafo, isto é, um instrumento que registra movimentos harmônicos.

## Montagem Experimental

Para este experimento foi montada uma mesa de em torno de 1,0 m de altura para ser colocado dois cabos de vassoura com pesos nas extremidades inferiores de cada um.

Fez-se dois furos de diâmetro na mesa a uma distância de 30 cm entre os centros dos furos.

Na oficina e com a ajuda de técnicos de outros laboratórios foram feitas peças que permitem o movimento de 360° dos pêndulos. São concêntricas de diâmetros diferentes, sendo uma delas, que servirá de base com duas cavidades triangulares diametralmente opostas onde são encaixadas por uma segunda peça com pequenos braços triangulares, também com cavidades na parte superior para uma terceira peça (a qual será fixada o cabo de vassoura) se encaixar. Esta descrição é visível na figura 2.

Na parte superior de um dos pêndulos (a da direita pela figura 1), foi colocada uma peça de madeira que foi encaixada na vassoura, de modo a servir de suporte para uma placa de vidro onde é colocada a folha de papel.

No outro pêndulo (a da esquerda pela figura 1) foi encaixada uma pequena roda de fácil deslize e nela foi fixa um pedaço de madeira (figura 3) que serve de braço para permitir que a caneta alcance o centro da folha de papel (veja figura 4).

A pilha que foi colocada na outra ponta do braço na figura 3 serve como contra-peso à caneta, para que este permita um menor contato da caneta com a folha, o necessário para traçar as figuras, mas com um mínimo de atrito possível.

Foram usadas clipes para prender a folha na placa de vidro e para segurar a caneta perpendicular ao papel.

Nas extremidades inferiores dos pêndulos foram colocados pesos, sendo que estes são possíveis de variar. Foram usadas discos de metal e pilhas grandes, conforme figura 5.

*Figura 1 - A montagem do experimento*

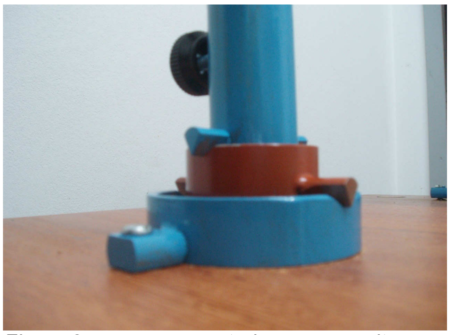


*Figura 2: as peças montadas que permitem o movimento de 360° de cada pêndulo.*

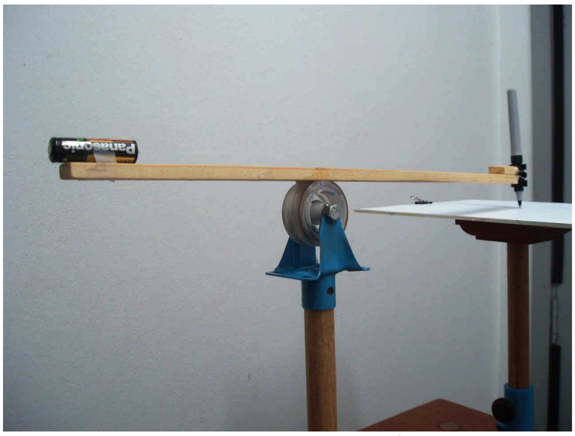


*Figura 3: Roda com o braço de madeira fixo com a caneta na ponta e uma pilha na outra ponta.*

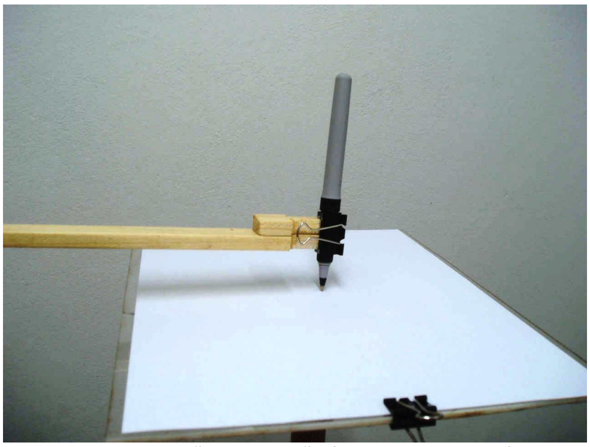


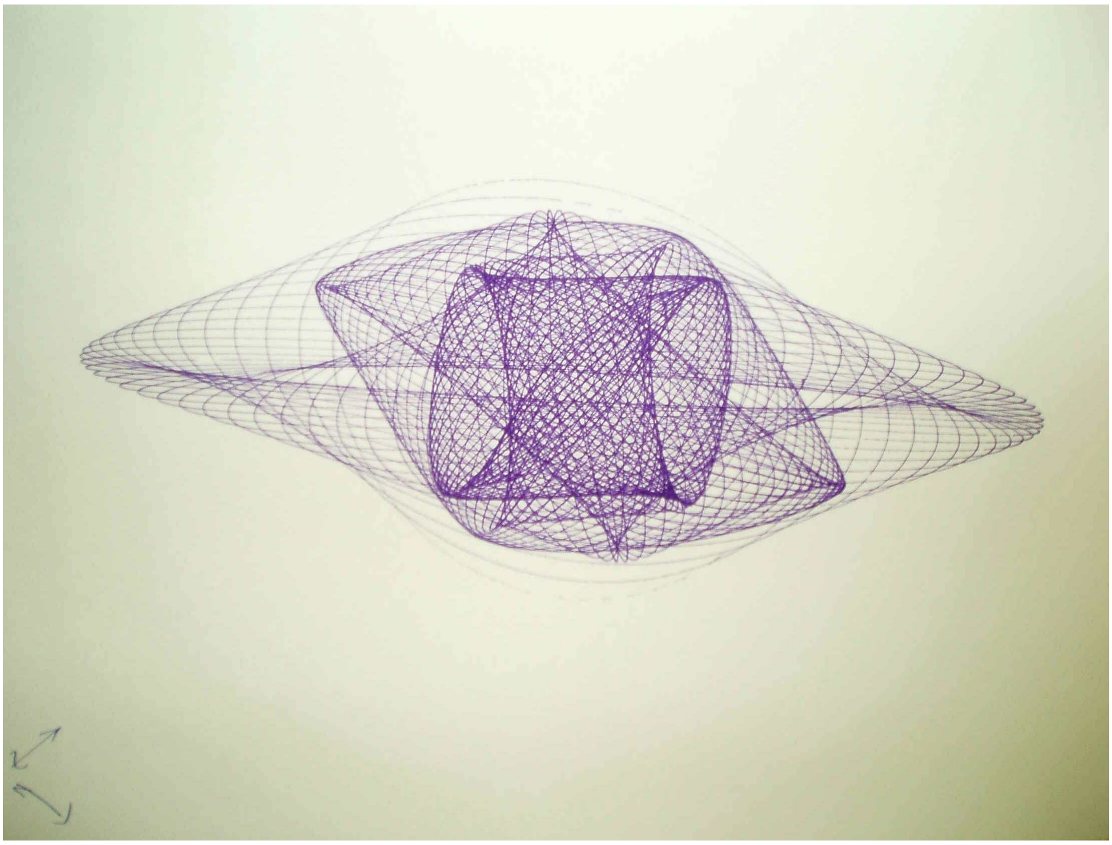
*Figura 4: Visualização da posição final da caneta na folha de papel.*

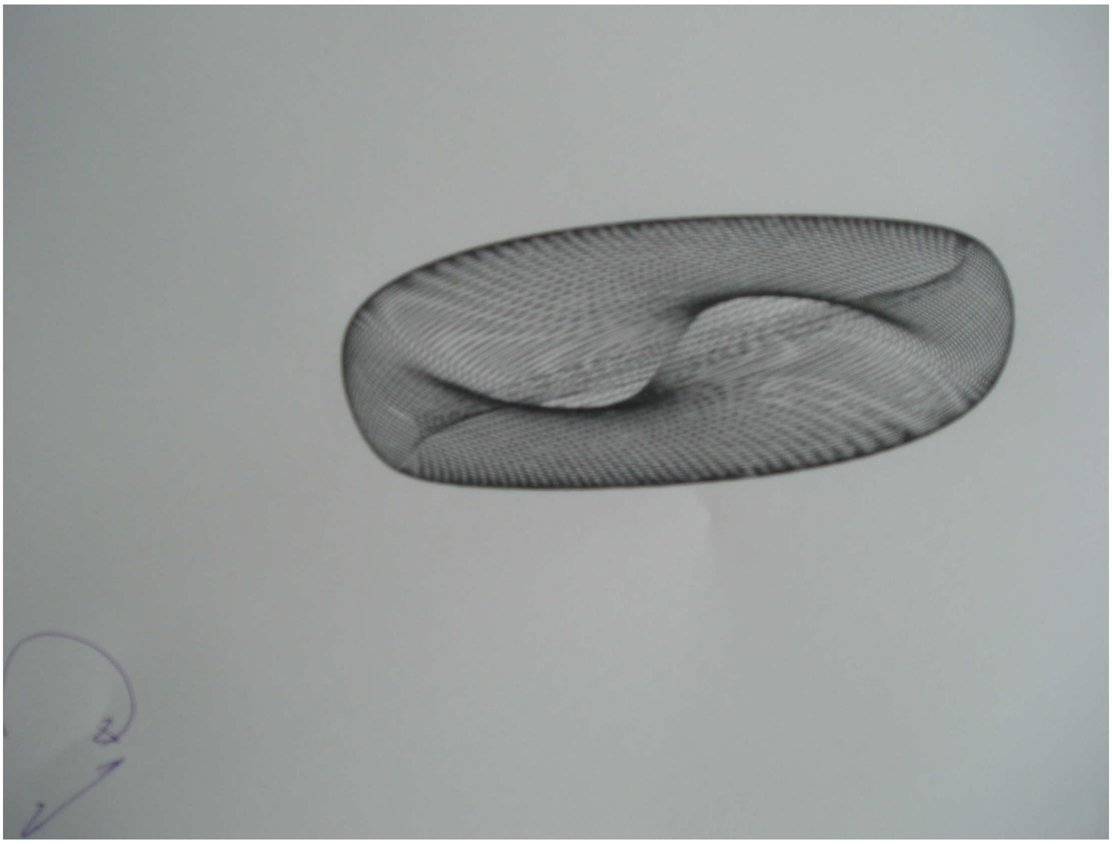
Agora, com o experimento montado, oscila-se pelo menos um dos pêndulos. Desta

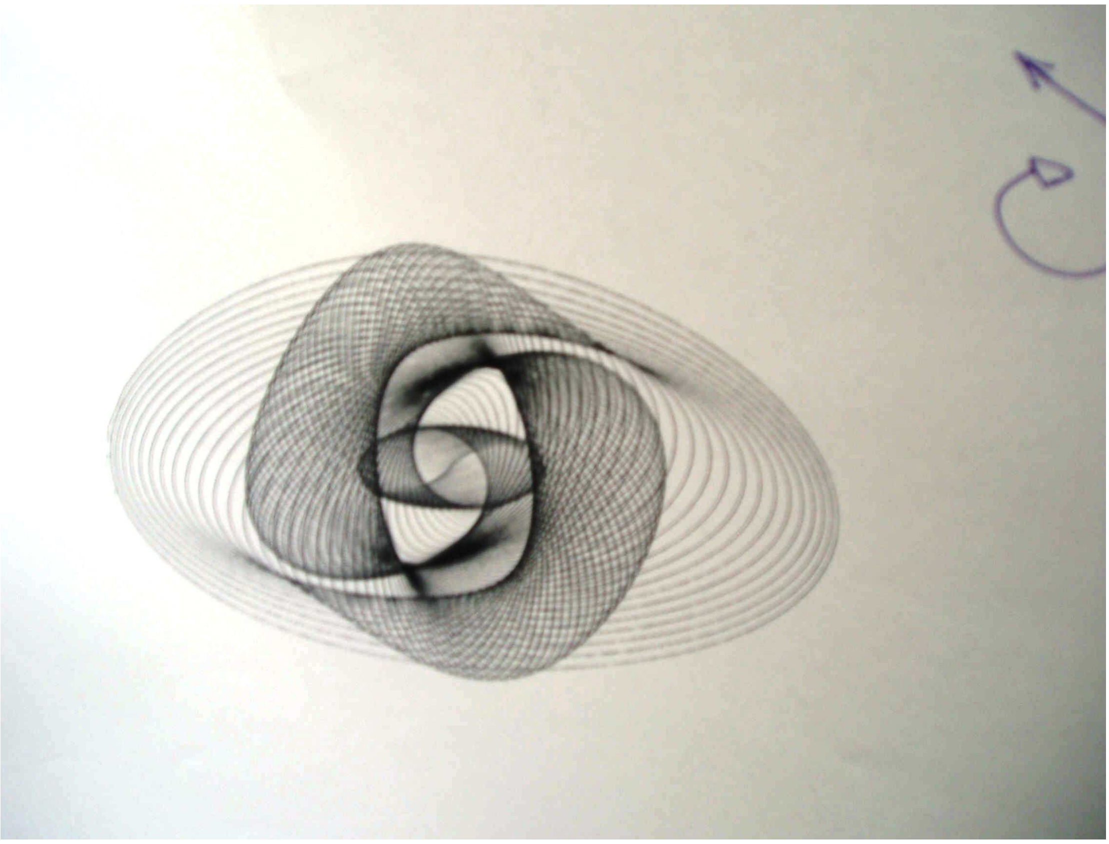
*Figura 5: Os pesos dos pêndulos*

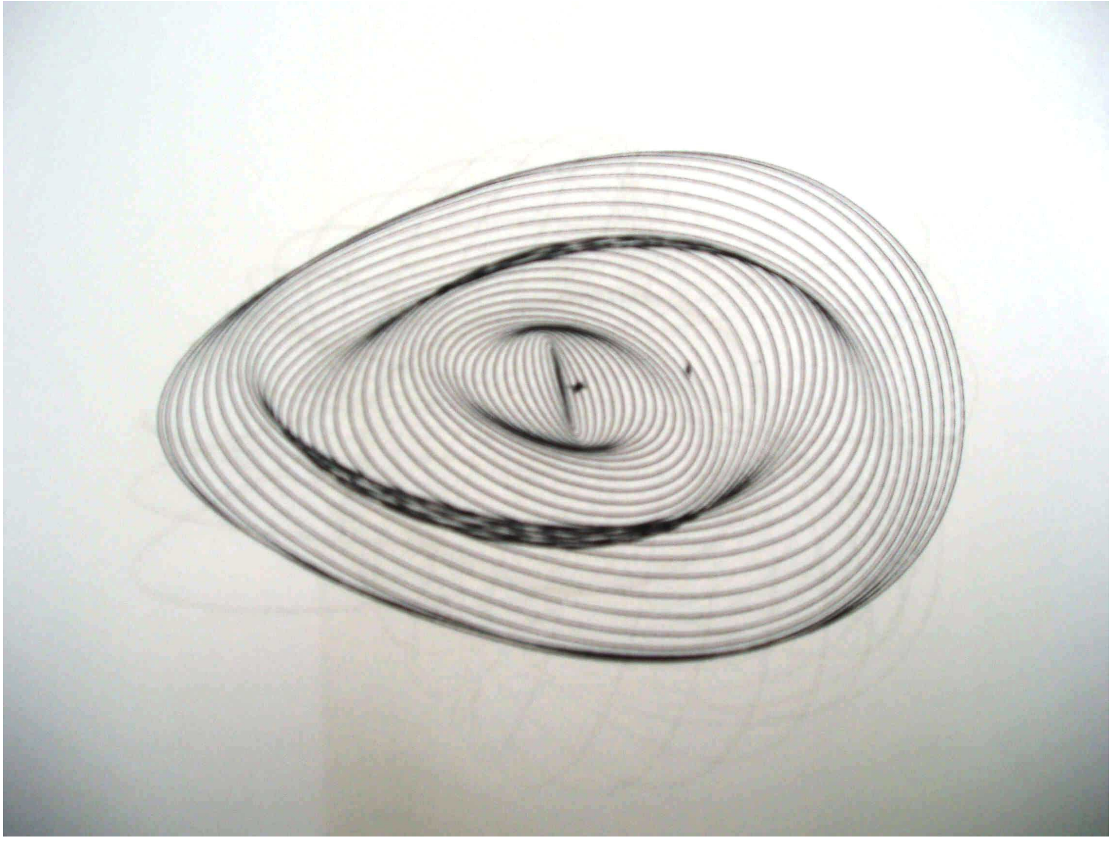
forma, a caneta irá desenhar no papel a figura resultante de ambas (ou de uma delas, se fizer oscilar apenas um dos pêndulos). Alguns resultados obtidos estão a seguir.

## Resultados Obtidos









## Problemas Encontrados

Como estamos trabalhando com pêndulos, qualquer agente externo que realize uma força, mesmo pequena e momentânea, afeta no movimento dos pêndulos, afetando, desta forma, no resultado.

Além disso, mesmo tentando minimizar o atrito entre a caneta e o papel, ainda é suficiente para afetar no movimento de um pêndulo com outro. Por exemplo, começa-se o movimento de um pêndulo como retilíneo e outro circular. Através do ponto de contato da caneta com o papel, passa um pouco do movimento de um para o outro, resultando, por exemplo, num movimento meio elíptico naquele pêndulo que estava com movimento retilíneo.

E como é um aparato mecânico, é difícil de determinar exatamente as condições iniciais de movimento de ambos os pêndulos, de modo que pode-se pensar que está realizando um movimento circular, mas que no fato é uma elipse quase próxima a uma circunferência.

Há um outro problema devido à própria montagem do experimento: como a mesa ficou pequena, a distância entre um pêndulo e outro ficou menor, de modo que, dependendo da amplitude dos movimentos, um pode bater no outro, influenciando muito nos resultados, pois estaria mudando os vetores das oscilações de modo brusco.

## Parte Teórica

Este experimento produz movimentos caóticos, isto é, mesmo que tentando igualar as condições iniciais, os resultados não são iguais, há diferenças devido ao atrito, ao erro sistemático, das condições do ambiente. Desta forma é difícil equacionar tais movimentos. Assim, vamos mostrar aqui uma situação na qual um dos pêndulos esteja parando e o outro em movimento.

### Oscilador Harmônico Bidimensional com Amortecimento

Consideremos o caso em que o referencial esteja no papel e este está parado, ou seja, somente o pêndulo com a caneta está em movimento oscilatório.

A equação geral da oscilação é a seguinte:

$$\vec{F} = -b\dot{\vec{r}} - k\vec{r}$$

Em duas dimensões, x e y, temos:

$$F_x = -b\dot{x} - kx \Rightarrow m\ddot{x} = -b\dot{x} - kx$$
$$F_y = -b\dot{y} - ky \Rightarrow m\ddot{y} = -b\dot{y} - ky$$

Resolvendo para x:

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = 0$$

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0 \qquad \text{sendo} \quad \omega_0^2 = \frac{k}{m} \quad \text{e} \quad 2\gamma = \frac{b}{m}$$

Fazendo o mesmo para y:

$\ddot{y}+2\gamma\dot{y}+\omega_0^2 y=0$

As soluções para ambas as equações são:

$$x(t)=A e^{-\gamma t}\cos(\omega t-\alpha)\Rightarrow\cos(\omega t-\alpha)=\frac{x}{A}e^{-\gamma t}$$
$$y(t)=B e^{-\gamma t}\cos(\omega t-\beta)$$

Resolvendo este sistema de equações:

$$y(t)=B e^{-\gamma t}\cos(\omega t-\alpha+(\alpha-\beta))=B e^{-\gamma t}\cos(\omega t-\alpha)\cos(\alpha-\beta)-B e^{-\gamma t}\sin(\omega t-\alpha)\sin(\alpha-\beta)$$

Fazendo $\delta\equiv\alpha-\beta$ :

$$y(t)=\frac{B}{A}x e^{-2\gamma t}\cos\delta-B e^{-\gamma t}\sqrt{1-\frac{x^2}{A^2}e^{-2\gamma t}}\sin\delta\Rightarrow$$

$$\Rightarrow Ay-Bx e^{-2\gamma t}\cos\delta=-B e^{-\gamma t}\sqrt{A^2-x^2 e^{-2\gamma t}}\sin\delta\Rightarrow$$
$$A^2-2\mathrm{ABxy}\,e^{-2\gamma t}\cos\delta+B^2x^2e^{-4\gamma t}\cos\delta=B^2e^{-2\gamma t}(A^2-x^2e^{-2\gamma t})\sin\delta\Rightarrow$$
$$\Rightarrow B^2x^2e^{-4\gamma t}-2\mathrm{ABxy}\,e^{-2\gamma t}\cos\delta+A^2y^2=A^2B^2e^{-2\gamma t}\sin\delta$$

Supondo uma situação ideal na qual não amortece ( $\gamma=0\Rightarrow e^{-a\gamma t}=1$ , com a = constante), podemos encontrar as figuras formadas por um oscilador.

Se $\delta=\pm\pi/2$ e $A\neq B$ : $B^2x^2+A^2y^2=A^2B^2\Rightarrow\frac{x^2}{A^2}+\frac{y^2}{B^2}=1$ (equação de uma elipse)

Se $\delta=\pm\pi/2$ e $A=B$ : $x^2+y^2=A^2$ (equação de uma circunferência)

Se $\delta=0$ : $B^2x^2-2\mathrm{ABxy}+A^2y^2=0\Rightarrow(Bx-Ay)^2=0\Rightarrow Bx=Ay\Rightarrow y=\frac{B}{A}x$

(equação de uma reta)

Se $\delta=\pm\pi$ : $B^2x^2+2\mathrm{ABxy}+A^2y^2=0\Rightarrow(Bx+Ay)^2=0\Rightarrow Bx=-Ay\Rightarrow y=\frac{-B}{A}x$

(equação de uma reta)

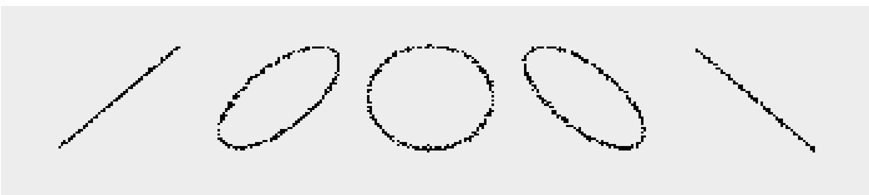


Figura 6: figuras geométricas formadas pela oscilação de um pêndulo.

A figura 6 representa o resultado que pode ser obtido pela oscilação de apenas um pêndulo sem amortecimento.

### Oscilações Não-Lineares

A equação de movimento para osciladores amortecidos em uma dimensão pode ser escrita como:

$$m\ddot{\vec{r}}+f(\dot{\vec{r}})+g(\vec{r})=h(t)$$

Se $f(\dot{\vec{r}})$ ou $g(\vec{r})$ conter uma potência maior que a linear para $\dot{\vec{r}}$ ou $\vec{r}$ então o sistema físico é não-linear. Soluções completas não existem para a equação acima, e algumas vezes considerações especiais são necessárias para resolvê-las.

A natureza aparenta ser caótica, e neste caso consideramos um caos determinístico, cujo movimento do sistema tem um tempo de evolução com dependência sensível das condições iniciais.

### Bibliografia


(1) Thornton; Marion; "*Classical Dynamics of Particles and Systems*"; 5th Edition.
(2) Anthony, Ashton; "*Harmonograph: A Visual Guide to the Mathematics of Music*"; editora Walker; original de Wooden Books.
(3) http://local.wasp.uwa.edu.au/~pbourke/surfaces_curves/harmonograph/
Site sobre harmonógrafo no qual este projeto foi baseado.
(4) http://www.airbrushmagic.com/harmonograph.htm
Site com outro tipo de montagem do harmonógrafo e com várias fotos das figuras formadas.
(5) http://members.pcug.org.au/~apurdam/harmonograph/harmonograph.html
Site que é possível fazer uma simulação online de um harmonógrafo.
(6) http://www.technorama.ch/Harmonograph.181.0.html?&L=1
Site simples que faz uma simulação com dois pêndulos interligados.
(7) http://www.fisica.net/mecanicaclassica/mhs_movimento_harmonico_simples.pdf
Arquivo pdf que explica o MHS (Movimento Harmônico Simples) e Pêndulos.

**Anexos**

(3) http://local.wasp.uwa.edu.au/~pbourke/surfaces_curves/harmonograph/

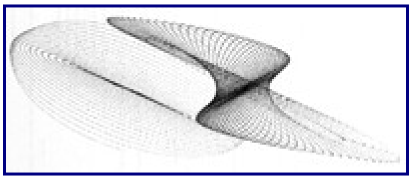
# Harmonograph

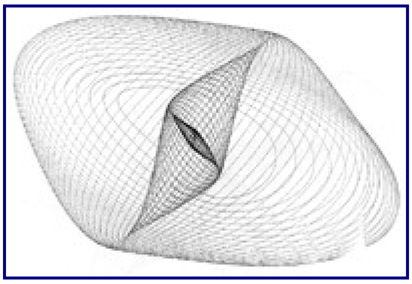
Written by Paul Bourke
Contribution by Rick and Richard Speir

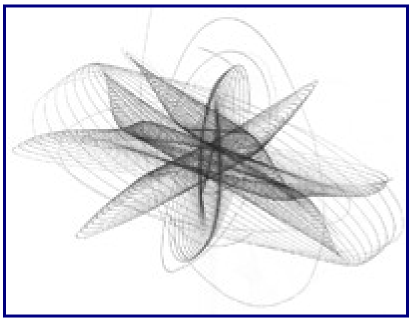
August 1999, Updated July 2002

The two pendulum **harmonograph** draws attractive patterns that arise from drawing the relative path traversed by two swinging masses as their motion is slowly damped. The resulting figures are called **harmonograms** or sometimes a **Lissajous curve**.

The harmongraph was pioneered by the French physicist, Jules Antoine Lissajous in 1857. The first harmonograph actually used a light beam on a screen instead of the pens on paper that are used today. Following the invention of the harmonograph it became a very popular device and was found in many homes. After the early 1900s it decreased in popularity and is rarely seen today.







Another device also called an harmonograph since it produces the same essential motion is based upon a platform suspended by each corner. The platform can be swung and twisted and a stationary pen draws a trace on some paper attached to the platform. Weights are often located at various positions on the table to produce different oscillatory patterns. One of the largest harmonographs of this kind can be found in the Science Centre and Planetarium, Wollongong, Australia.

Andrew Purdam has produced equations that allow one to explore the beauty of the harmongraph without building one....not that building one is any less satisfying. His equations are:

$$x(t) = A_x(t)\sin(w_x t + p_x) + A_s(t)\sin(w_s t + p_s)$$

$$y(t) = A_y(t)\sin(w_y t + p_y)$$

All initial amplitudes, frequencies (w) and phases (p) should be different and not integer multiples for the most complicated (interesting) patterns.

In order for the amplitude to decay (not necessary but occurs in the real harmonograph) the amplitudes can decay as follows, where d is typically a suitable small positive number. This gives an exponential decay function.

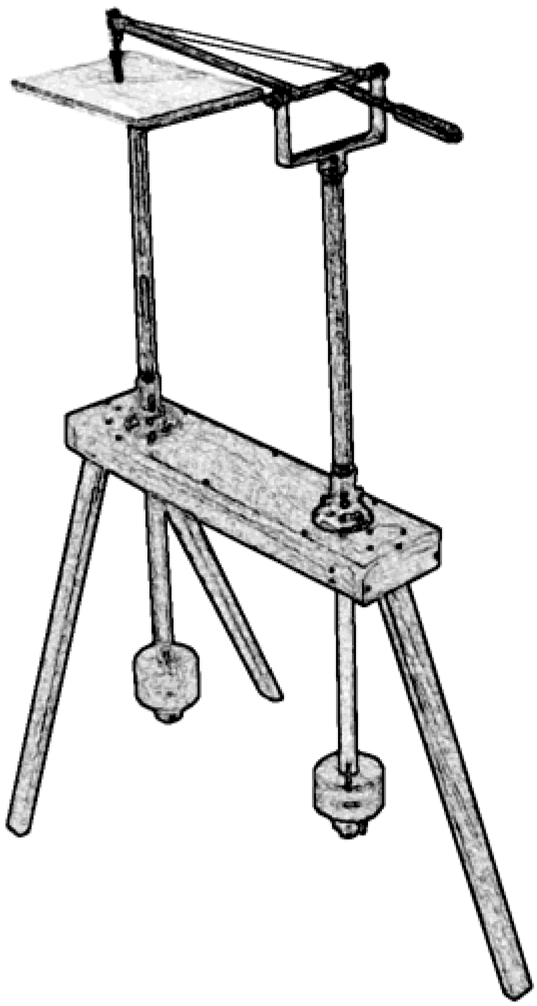
$$A(t) = A(t-1)\ (1 - d)$$

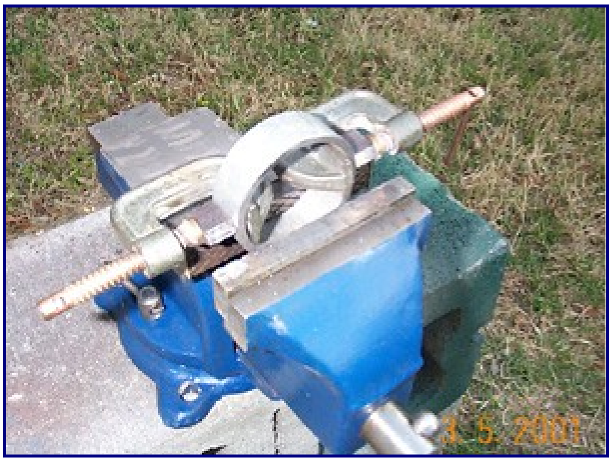
These images illustrate a physical harmonograph located at the Swinburne University School of Biophysical Sciences and Electrical Engineering.

**Contribution by Rick and Richard Speir**

My son, Richard, became interested in harmonographs about 3 or 4 years ago. I found him in the basement one evening, attempting to suspend a piece of plywood on 4 strings attached to the floor joists above. He'd seen a diagram or photo of a harmonograph that worked in that manner on the internet. I didn't really understand what he was trying to do, but a friend who was visiting had seen a harmonograph when he was a child, and was enthusiastic. The board-on-a-string attempt fizzled, and harmonographs were kind of forgotten. Some time later, Richard found a photo of a museum's harmonograph on a web site and showed it to me, along with some patterns traced by mathematical harmonograph models (Lissajous curves). The photograph he showed me was the "board-on-a-string" type, but I think he had seen Paul Bourke's web site photos, and attempted to describe it to me. He could not find the site again, however. But the patterns looked interesting to me, and convinced me he was seriously interested in harmonographs--it was not just a passing fancy.

I began looking around the internet, but found little of practical interest. I looked on Amazon.com for books about harmonographs. I found virtually nothing. Then one day, I found Paul Bourke's web site, and the photographs that I suspect are the only ones of their type on the internet. I liked the "old-fashioned", mechanical nature of that instrument, and e-mailed Mr. Bourke about it. Unfortunately, he no longer had access to the harmonograph, so my questions about measurements, materials, and so forth could not be answered. He generously provided what information he could, and that got me started. At first, I thought I could "cheat", and make a simple gimbal. I sawed a section of 1-1/2 inch square steel thin-wall tubing, used some thin sheet metal to make supports for this gimbal. It didn't work very well at all, for a number of reasons.





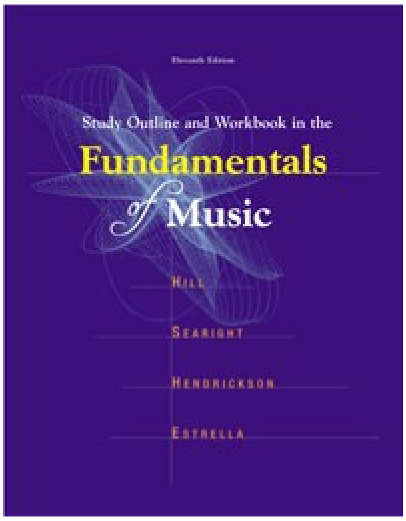
Gimbal ring ready for welding

The results can be seen in the photos. (I was even inspired by the old-looking wooden case, with "Harmonograph" stenciled on the lid, in the background of one of Paul's photos, and built a similar case for ours. We've made several refinements since the first iteration; you'll notice in one of the photos, we're using barbell weights

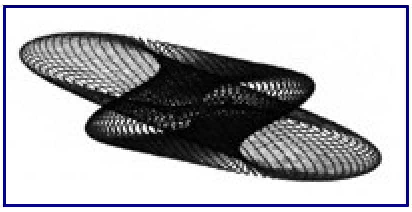
So I decided to scale the photograph in Mr. Bourke's web site as best I could, to get some measurements for building the thing. I printed a large photo of the harmonograph, and using the basic measurement of the pen I intended to use for our harmonograph, scaled the entire instrument photograph to the pen's measurements. I was not trying for a perfect replication, merely a functional one. I used mahogany for the body, and as much brass hardware as I could, to make it look old and Victorian, and as attractive as possible. I'd love to use brass rods for the pendulum shafts, brass billets for the weights, and brass turnings for the gimbals and base rings, but have to draw the line somewhere! Brass stock is expensive around here. And, of course, who wants to keep all that brass polished up?

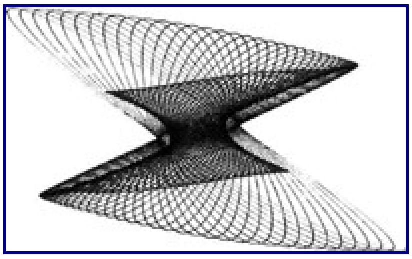
I had most of the instrument completed for Richard's 18th birthday, and I think he was surprised and pleased with his gift. He and I worked together to complete the instrument.

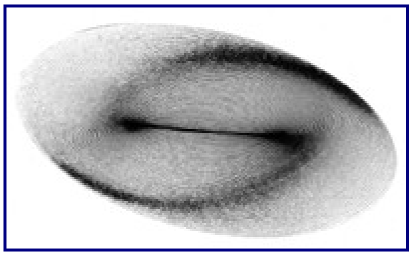
for the pendulums. (or is it pendulii?) The barbell weights worked fine as far as the weight went, however, their diameter was too large, and in some cases would strike the legs, when the pendulum was swung too vigorously. We subsequently cast lead weights from waste tire balance weights. We also have experimented with different lengths of pipe for the top and bottom halves of the pendulum shafts.



Appeared on the cover of "Fundamentals of Music"







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The Harmonograph, Mathematical Models, 3rd ed. Stradbroke, England: Tarquin Pub., pp. 244-248, 1989.

Wells, D.
The Penguin Dictionary of Curious and Interesting Geometry. London: Penguin, pp. 92-93, 1991.

(4) http://www.airbrushmagic.com/harmonograph.htm

# The Amazing Harmonograph

## Pendulum Drawing Machine

Close up photos of two home made harmonograph machines.

Instructions for making a basic harmonograph which can be easily used in your home.

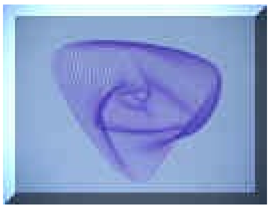
The machine consists of two weighted pendulums swinging freely. An attached pen traces the path of the upper pendulum, drawing a picture on the paper.

This model requires an anchor point in the wall 8 to 10 feet above the floor.

Shorter pendulums produce smaller pictures. The anchor point must be very strong, stable, and safe. Enough to hold up to 8 swinging bricks.

Here's a harmonograph which uses two large stones as weights on its upper pendulum.

click on the one picture below to see more



Here are six adjustments you can make to this harmonograph machine change the pictures.

This harmonograph machine is based on a design invented and patented in the early sixties by Mr. Edward Lias, of West Nyack NY.. In 1968, I worked with a small company in Eugene, Oregon operated by Mr. and Mrs. Lee Trippett. They developed and manufactured about 500 harmonographs to be sold as a toy in a big box for $29.95. The venture was not successful.

The original instruction book for this toy contained a fine explanation of the scientific principles written by Lee Trippett. At length were discussed such matters as ratios, effective length and pendulum weights.

But my last remaining copy of the instruction book was lost when my family moved from Seattle to Cincinnati in 1970. Other adventures became more important. But I have never forgotten the machine, and the hypnotic power of watching it create pictures. I thought it would be interesting to include these instructions on my web

The anchor can be a hook, a big nail, a "c" clamp, or a high beam. Attached to the anchor are chains holding four hooks in the top of a triangle board.

The chain has to suspend the pendulums securely, and the chain has to hold the triangle board steady against the wall. There's a hole in the triangle board.

The hole has to be bigger than the shaft of the upper pendulum, and it has to be located so as to hold the upper pendulum as far away from the wall as possible. Even though the picture above shows that I used a broomstick for a pendulum, it turned out to be a bit too flimsy. I recommend using a dowel or something stronger like the one below.

site.

I'd like here to recommend a little book called "Harmonographs". It is one of the wooden series of books. This one is a joy to read, exploring all about the pictures and the harmonic principles which produce them.

While staying at the Haven resort on Gabriola Island, BC (www.haven.ca), I decided to build a very simple harmonograph machine to put in the games room. I wanted it to be so simple that even young children could operate it to produce elaborate pictures.

At the Annual Eliot Unitarian Family Conference at Seabeck, (www.seabeck.com) Washington. I built a much larger, more sturdy machine. It too was a great success. Unsupervised children can play with it for hours and hours, producing quite amazing pictures. This web page is largely in response to the interest and questions that people asked while these two machines have been in place.

There are several other websites with information about harmonograph machines. But I think this is the only one using this type of free swinging double pendulum. This machine is easily moved from place to place. Relatively simple adjustments to the chain length and the distribution of weights will produce an infinite variety of three (triangular) or four sided (square) pictures.

It is essentially a very simple machine, easy to operate, and

For detailed plans click here

For an interpretation of the
philosophy of harmonograph
pictures and what they might
mean, click here and go to
the bottom of the page

very sensitive. Every picture is unique. Several simple adjustments can be made to the machine to alter characteristics of the pictures. It helps to have some idea of what kind of picture you want before you set the machine in motion by swinging it gently.

To see sketches and diagrams how to build your own harmonograph machine Click Here

•index page

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To contact Peter Joyes, email to

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