Exercise Class 1

Consider a particle of mass m subject to the potential

$$V(x) = \begin{cases} \infty & x \leq 0, \\ -V_0\delta(x-a) & x > 0, \end{cases} \tag{1}$$

where $V_0 > 0$.

a) Show that the derivative of the eigenfunction $\psi(x)$ presents a discontinuity at $x = a$ and determine it in terms of V_0 , m and $\psi(a)$.

b) Discuss the existence of bound states in terms of the size of a.

Solution

a) Let's integrate the Schrodinger equation for $x > 0$ from $a - \varepsilon$ to $a + \varepsilon$:

$$-\frac{\hbar^2}{2m}\int_{a-\varepsilon}^{\varepsilon+a} dx \left[\frac{d^2\psi(x)}{dx^2}\right] - V_0 \int_{a-\varepsilon}^{a+\varepsilon} dx\delta(x-a)\psi(x) = E\int_{a-\varepsilon}^{a+\varepsilon} dx\psi(x),$$
$$-\frac{\hbar^2}{2m}\left[\frac{d\psi(a+\varepsilon)}{dx} - \frac{d\psi(a-\varepsilon)}{dx}\right] - V_0\psi(a) = E\int_{a-\varepsilon}^{a+\varepsilon} dx\psi(x),$$

taking $\varepsilon \to 0$ we obtain

$$\frac{d\psi(a^+)}{dx} - \frac{d\psi(a^-)}{dx} = -\frac{2mV_0}{\hbar^2}\psi(a). \tag{2}$$

b) For $x > 0$ the Schrodinger equation is written as

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} - V_0\delta(x-a)\psi(x) = E\psi(x). \tag{3}$$

For $x > a$ we have

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_R(x)}{dx^2} = E\psi_R(x), \tag{4}$$
$$\frac{d^2\psi_R(x)}{dx^2} = k\psi_R(x), \tag{5}$$

and for $x < a$:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_L(x)}{dx^2} = E\psi_L(x), \tag{6}$$
$$\frac{d^2\psi_L(x)}{dx^2} = k^2\psi_L(x). \tag{7}$$

Note that in both equations, (5) and (7),

$$k = \frac{\sqrt{-2mE}}{\hbar}, \tag{8}$$

and as we are interested in the bound states of the system, we can write

$$E = -|E|, \tag{9}$$

and therefore,

$$k = \frac{\sqrt{2m|E|}}{\hbar}. \tag{10}$$

Based on (5) and (7) the solutions of the equations are of the form:

$$\psi(x) = \begin{cases} \psi_R(x) = A_R e^{kx} + B_R e^{-kx}, & x > a \\ \psi_L(x) = A_L e^{kx} + B_L e^{-kx}, & x < a \end{cases}. \tag{11}$$

Let's apply the boundary conditions to obtain the eigenvalues of the problem.
(i) $x \to \infty \implies \psi(x) \to 0 \implies \psi_R(x) \to 0$:

$$\Rightarrow A_R = 0 \tag{12}$$

(ii) $\psi(0) = 0 \implies \psi_L(0) = 0$:

$$\Rightarrow B_L = -A_L \tag{13}$$

Using (12) and (13) in (11), we can write

$$\begin{cases} \psi_R(x) = B_R e^{-kx}, & x > a \\ \psi_L(x) = A_L \left(e^{kx} - e^{-kx}\right), & x < a \end{cases}. \tag{14}$$

(iii) $\psi_L(a) = \psi_R(a)$:

$$\Rightarrow B_R e^{-ka} = A_L \left(e^{ka} - e^{-ka}\right) \tag{15}$$

(iv) Descontinuity of the derivative calculated in (2):

$$\frac{d\psi_R(x)}{dx} = -kB_R e^{-kx}, \tag{16}$$

$$\frac{d\psi_L(x)}{dx} = kA_L \left(e^{kx} + e^{-kx}\right). \tag{17}$$

Substituting (16) and (17) in (2) we obtain

$$B_R k e^{-ka} + kA_L \left(e^{ka} + e^{-ka}\right) = \frac{2mV_0}{\hbar^2} B_R e^{-ka}. \tag{18}$$

Substituting (15) in (18) we find that

$$\begin{aligned} kA_L \left(e^{ka} - e^{-ka}\right) + kA_L \left(e^{ka} + e^{-ka}\right) &= \frac{2mV_0}{\hbar^2} A_L \left(e^{ka} - e^{-ka}\right) \\ kA_L e^{ka} &= \frac{mV_0}{\hbar^2} A_L \left(e^{ka} - e^{-ka}\right) \end{aligned}$$

$$\Rightarrow A_L \left[k - \frac{mV_0}{\hbar^2}\left(1 - e^{-2ka}\right)\right] = 0. \tag{19}$$

We are looking for nontrivial solutions, i.e., $A_L \neq 0$:

$$\Rightarrow k = \frac{mV_0}{\hbar^2}\left(1 - e^{-2ka}\right) \tag{20}$$

The equation (20) is a transcedental equation (very common in the solution of physical problems!!). The solutions of this equation is given by the intersection points between the function $f(k) = k$ and the function $g(k) = \frac{mV_0}{\hbar^2}\left(1 - e^{-2ka}\right)$. One way of determining the solutions (in most problems we can just solve this kind of equation numerically) is note that this equation will just have a solution if at the point $x = 0$:

$$\left[\frac{dg(k)}{dk}\right]_{k=0} > \left[\frac{df(k)}{dk}\right]_{k=0}$$
$$\Rightarrow a > \frac{\hbar^2}{2mV_0} \tag{21}$$

Thus, using (20) and the condition (21), we obtain that the energy of the bound state is given by

$$\Rightarrow E = -\frac{m{V_0}^2}{2\hbar^2}\left(1 - e^{-2ka}\right)^2, \quad a > \frac{\hbar^2}{2mV_0} \tag{22}$$