F-315 (Mecânica Geral I)
Aula 10

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Slides do prof. Antonio Vidiella Barranco:
http://www.ifi.unicamp.br/~vidiella/aulas.html

Princípio da Superposição:

Teorema:

Considere um conjunto de funções $x_n(t)$ de modo que

$$m\frac{d^2x_n}{dt^2}+b\frac{dx_n}{dt}+kx_n=F_n(t)$$

Se $F(t)=\sum_n F_n(t)$, a função $x(t)=\sum_n x_n(t)$

satisfaz a equação $$m\frac{d^2x}{dt^2}+b\frac{dx}{dt}+kx=F(t)$$

Princípio da Superposição: Séries de Fourier

Teorema de Fourier :

Qualquer função periódica, $F(t) = F(t+\tau)$ com período $\tau = 2\pi/\omega$ pode ser escrita como

$$F(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \cos(n\omega t) + b_n \sin(n\omega t)\right]$$ onde

$$a_n = \frac{2}{\tau}\int_{-\tau/2}^{\tau/2} F(t)\cos(n\omega t)dt \; ; \qquad b_n = \frac{2}{\tau}\int_{-\tau/2}^{\tau/2} F(t)\sin(n\omega t)dt$$

e $$a_0 = \frac{2}{\tau}\int_{-\tau/2}^{\tau/2} F(t)dt$$

Para demonstrar as fórmulas de Euler–Fourier para os coeficientes $\{a_0, a_n, b_n\}$, vamos utilizar as integrais para $\nu \in \mathbb{N}^*$:

$$\int_0^\tau \cos(\nu\omega t)\,\mathrm{d}t = \frac{1}{\nu\omega}\,\mathrm{sen}(\nu\omega t)\Big|_0^\tau = \frac{1}{\nu\omega}\,\mathrm{sen}(2\pi\nu) = 0, \text{ já que } \omega\tau = 2\pi$$

$$\int_0^\tau \mathrm{sen}(\nu\omega t)\,\mathrm{d}t = -\frac{1}{\nu\omega}\cos(\nu\omega t)\Big|_0^\tau = \frac{1}{\nu\omega}[1-\cos(2\pi\nu)] = 0$$

$$\int_0^\tau \begin{bmatrix}\cos^2(\nu\omega t)\\ \mathrm{sen}^2(\nu\omega t)\end{bmatrix}\mathrm{d}t = \frac{1}{2}\int_0^\tau [1\pm\cos(2\nu\omega t)]\,\mathrm{d}t = \left[\frac{t}{2} \pm \frac{1}{4\nu\omega}\,\mathrm{sen}(2\nu\omega t)\right]_0^\tau = \frac{\tau}{2}$$

Relações de ortogonalidade das funções trigonométricas:

$$\int_0^\tau \cos(n\omega t)\,\mathrm{sen}(m\omega t)\,\mathrm{d}t = \frac{1}{2}\int_0^\tau \mathrm{sen}[(n+m)\omega t]\,\mathrm{d}t - \frac{1}{2}\int_0^\tau \mathrm{sen}[(n-m)\omega t]\,\mathrm{d}t = 0,\ \ \forall(n,m)$$

$$\int_0^\tau \begin{bmatrix}\cos(n\omega t)\cos(m\omega t)\\ \mathrm{sen}(n\omega t)\,\mathrm{sen}(m\omega t)\end{bmatrix}\mathrm{d}t = \frac{1}{2}\int_0^\tau \cos[(n-m)\omega t]\,\mathrm{d}t \pm \frac{1}{2}\int_0^\tau \cos[(n+m)\omega t]\,\mathrm{d}t = \frac{\tau}{2}\,\delta_{nm}$$

Delta de Kronecker: $\delta_{nm} = \begin{cases} 0, & n \neq m, \\ 1, & n = m. \end{cases}$

Série de Fourier : TM eq. (3.89)

Multiplicando

$$F(t) = \tfrac{1}{2}a_0 + \sum_{m=1}^{\infty} \left[a_m \cos(m\omega t) + b_m \,\mathrm{sen}(m\omega t)\right]$$

por $\cos(n\omega t)$ e integrando o produto no período $0 \le t \le \tau$, obtemos

$$\int_0^\tau F(t)\cos(n\omega t)\,\mathrm{d}t = \tfrac{1}{2}a_0 \int_0^\tau \cos(n\omega t)\,\mathrm{d}t + \sum_{m=1}^{\infty} \Big[a_m \int_0^\tau \cos(n\omega t) \times \cos(m\omega t)\,\mathrm{d}t + b_m \int_0^\tau \cos(n\omega t)\,\mathrm{sen}(m\omega t)\,\mathrm{d}t\Big] = \tfrac{\tau}{2} \sum_{m=1}^{\infty} a_m \delta_{mn} = \tfrac{\tau}{2} a_n$$

Para o caso particular $n = 0$: $\int_0^\tau F(t)\,\mathrm{d}t = \tfrac{1}{2}a_0 \int_0^\tau \mathrm{d}t = \tfrac{\tau}{2}a_0$

De forma análoga, $\int_0^\tau F(t)\,\mathrm{sen}(n\omega t)\,\mathrm{d}t = \tfrac{\tau}{2} \sum_{m=1}^{\infty} b_m \delta_{mn} = \tfrac{\tau}{2} b_n$

Fórmulas de Euler–Fourier : TM eqs. (3.90)

$$a_0 = \frac{2}{\tau}\int_0^\tau F(t)\,\mathrm{d}t\,, \qquad \begin{bmatrix} a_n \\ b_n \end{bmatrix} = \frac{2}{\tau}\int_0^\tau F(t) \begin{bmatrix} \cos(n\omega t) \\ \mathrm{sen}(n\omega t) \end{bmatrix} \mathrm{d}t$$

Problema

TM exemplo 3.6, pg. 128/129

Calcular os coeficientes de Fourier da função "dente de serra"

$$F(t) = A\frac{t}{\tau} = \frac{A\omega}{2\pi}t \qquad -\frac{\tau}{2} < t < \frac{\tau}{2}$$

TM exemplo 3.6, pgs. 128–129

Série de Fourier da função dente de serra:

Função ímpar $F(-t) = -F(t) \rightarrow a_n = 0, \forall n$

$$b_n = \frac{2}{\tau}\int_{-\tau/2}^{\tau/2} F(t)\,\mathrm{sen}(n\omega t)\,\mathrm{d}t = \frac{\omega^2 A}{2\pi^2}\int_{-\pi/\omega}^{\pi/\omega} t\,\mathrm{sen}(n\omega t)\,\mathrm{d}t$$

$$= \frac{\omega^2 A}{2\pi^2}\left[-\frac{t\cos(n\omega t)}{n\omega} + \frac{\mathrm{sen}(n\omega t)}{n^2\omega^2}\right]_{-\pi/\omega}^{\pi/\omega}$$

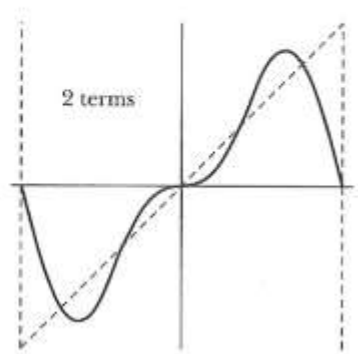
$$= \frac{\omega^2 A}{2\pi^2}\,\frac{2\pi}{n\omega^2}\,[-\cos(n\pi)] = \frac{A}{n\pi}(-1)^{n+1}$$

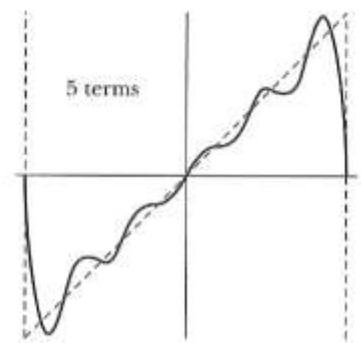
TM eq. (3.95)

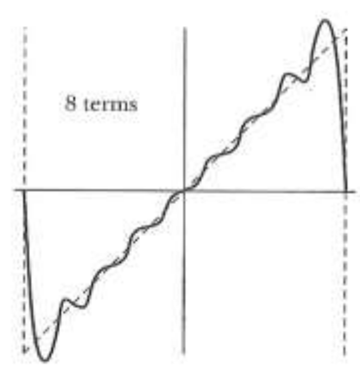
$$F(t) = \frac{A}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\,\mathrm{sen}(n\omega t)$$

Função “dente de serra”

$$F(t) = A\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n\pi}\sin(n\omega t)$$







Série de Fourier para uma força externa periódica:

$$F(t) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} [a_n \cos(n\omega t) + b_n \operatorname{sen}(n\omega t)]$$

$$= \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} c_n \overbrace{\cos(n\omega t - \theta_n)}^{\cos(n\omega t)\cos\theta_n + \operatorname{sen}(n\omega t)\operatorname{sen}\theta_n}$$

$$\left.\begin{aligned} a_n &= c_n \cos\theta_n \\ b_n &= c_n \operatorname{sen}\theta_n \end{aligned}\right\} \to \quad \begin{aligned} a_n + \mathrm{i}b_n &= c_n \mathrm{e}^{\mathrm{i}\theta_n}, \\ c_n = \sqrt{a_n^2 + b_n^2}, &\quad \theta_n = \arg(a_n + \mathrm{i}b_n) \end{aligned}$$

Solução particular: TM eqs. (3.86) e (3.87); S eq. (2.199)

$$x_\mathrm{p}(t) = \frac{a_0}{2m\omega_0^2} + \frac{1}{m}\sum_{n=1}^{\infty} \frac{c_n \cos(n\omega t - \theta_n - \varphi_n)}{\sqrt{(\omega_0^2 - n^2\omega^2)^2 + 4n^2\gamma^2\omega^2}}, \quad \operatorname{tg}\varphi_n = \frac{2n\gamma\omega}{\omega_0^2 - n^2\omega^2}$$

G problema 6.15, pg. 274

$F(t)$ é uma função qualquer (aperiódica) expressa como:

$F(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\omega) e^{i\omega t} d\omega$, $f(\omega)$ pode ser pensado como os coeficientes de uma expansão na variável contínua ω

Transformada de Fourier da função $F(t)$:

$$f(\omega) = \mathscr{F}[F(t)] \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(t') e^{-i\omega t'} dt'$$

Solução estacionária

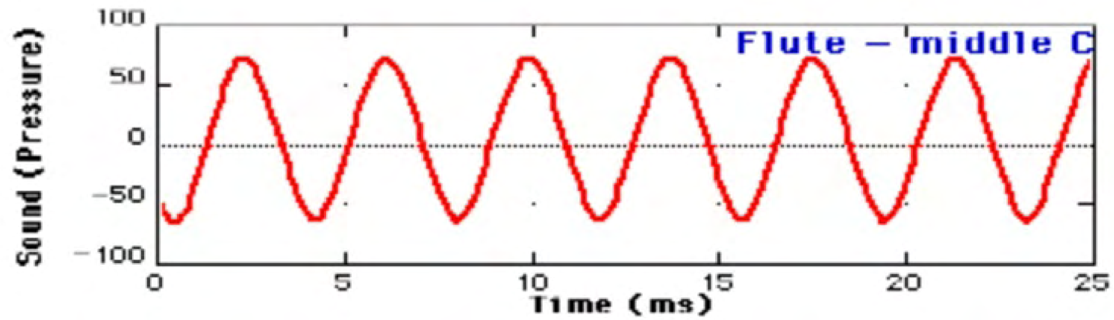
$$x_p(t) = \frac{1}{m\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{f(\omega) e^{i\omega t} d\omega}{\omega_0^2 - \omega^2 + 2i\gamma\omega}$$

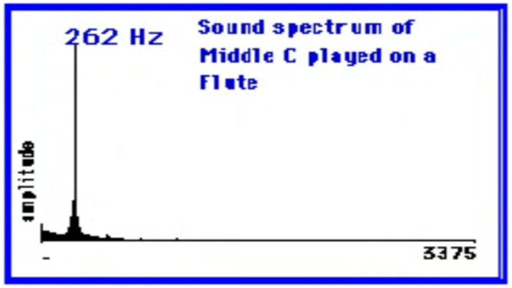
Função delta de Dirac:

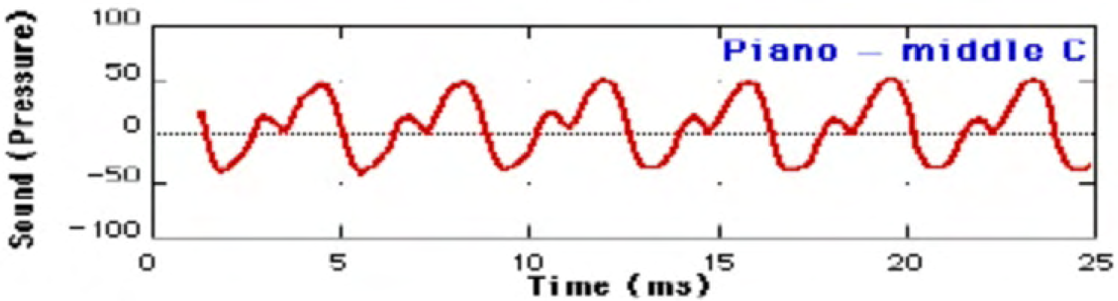
$$\delta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} d\omega$$

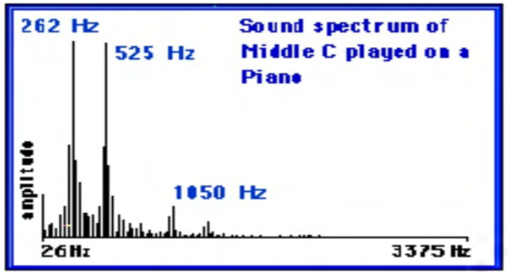
$$F(t) = \int_{-\infty}^{\infty} F(t')\delta(t'-t) dt'$$

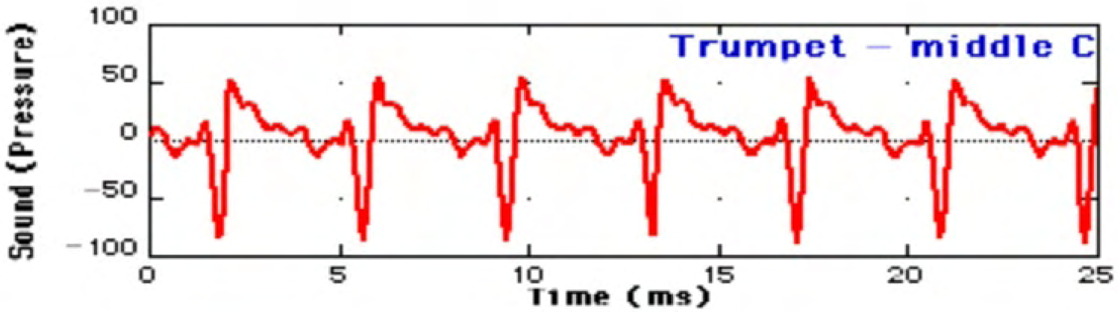
Timbre: análise de Fourier

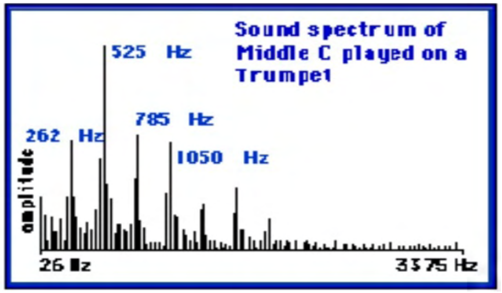




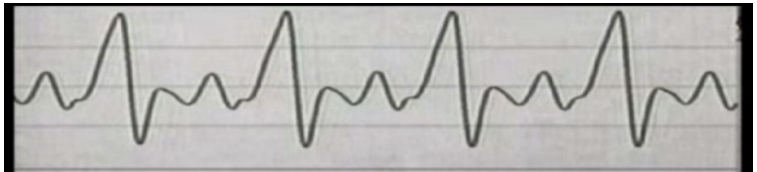




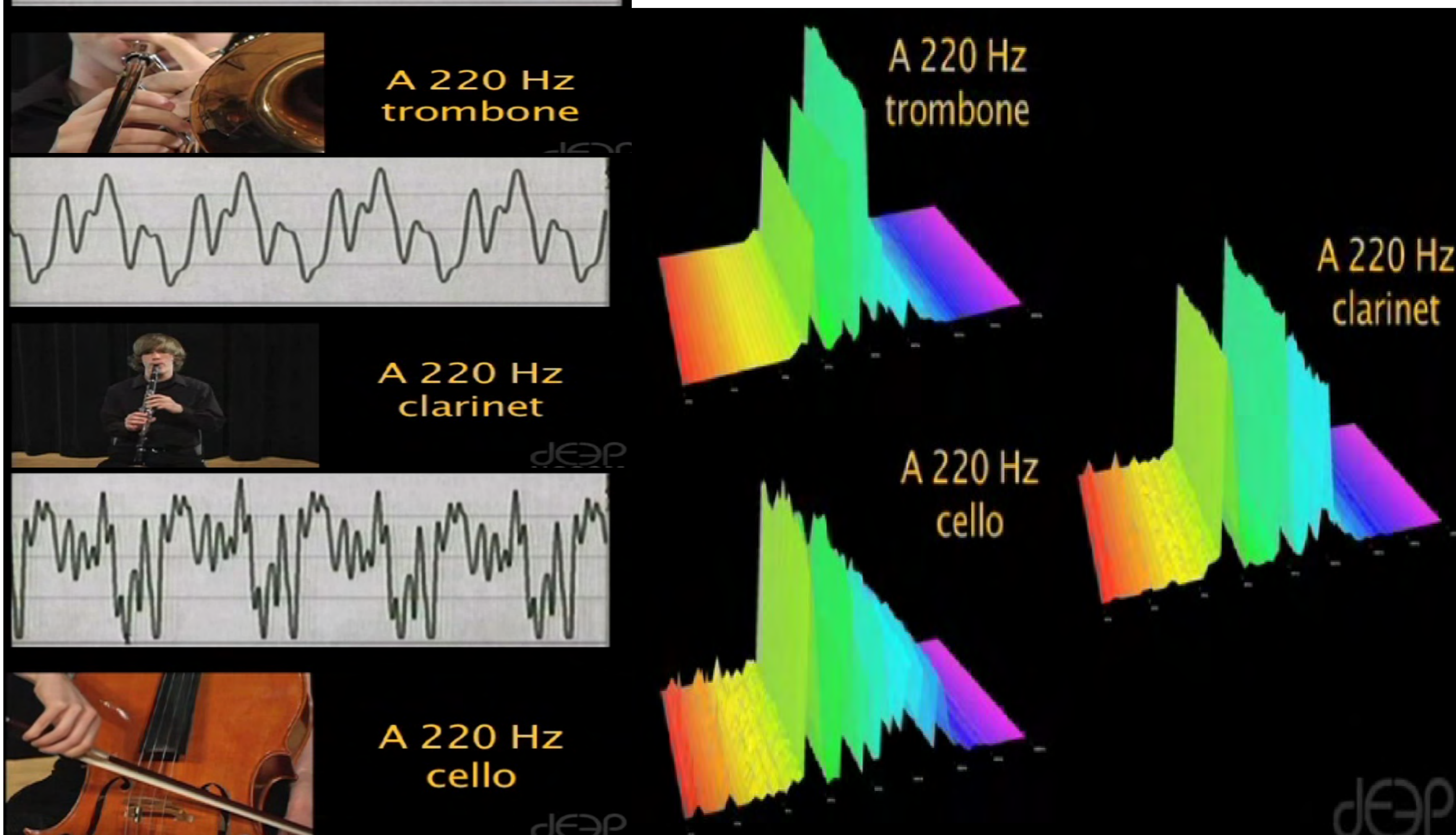


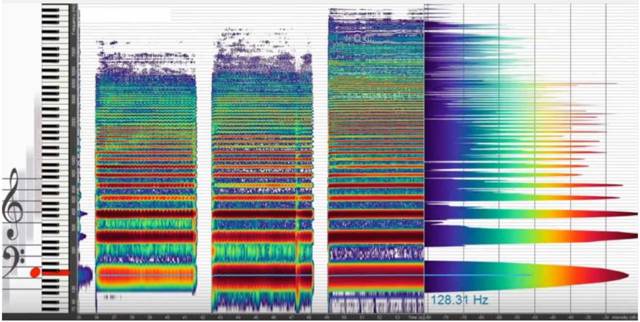


Timbre: análise de Fourier



https://www.youtube.com/watch?v=nlv5byIQDsE





Caso já tenha carregado o vídeo aqui, salve-o com o nome *musical_timbre.mp4* no mesmo diretório deste arquivo pdf. Da próxima vez, basta clicar na foto acima.