F-315 (Mecânica Geral I)
Aula 18

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Slides do prof. Antonio Vidiella Barranco:
http://www.ifi.unicamp.br/~vidiella/aulas.html

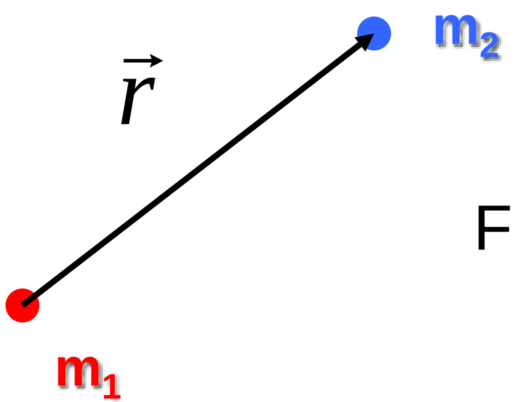
Força gravitacional

TM capítulo 5, S capítulo 6

Força Gravitacional

massas puntiformes

$$\vec{F}_{21} = -\frac{G m_1 m_2}{r^2}\hat{r}$$



TM eq. (5.1); S eq. (6.3)

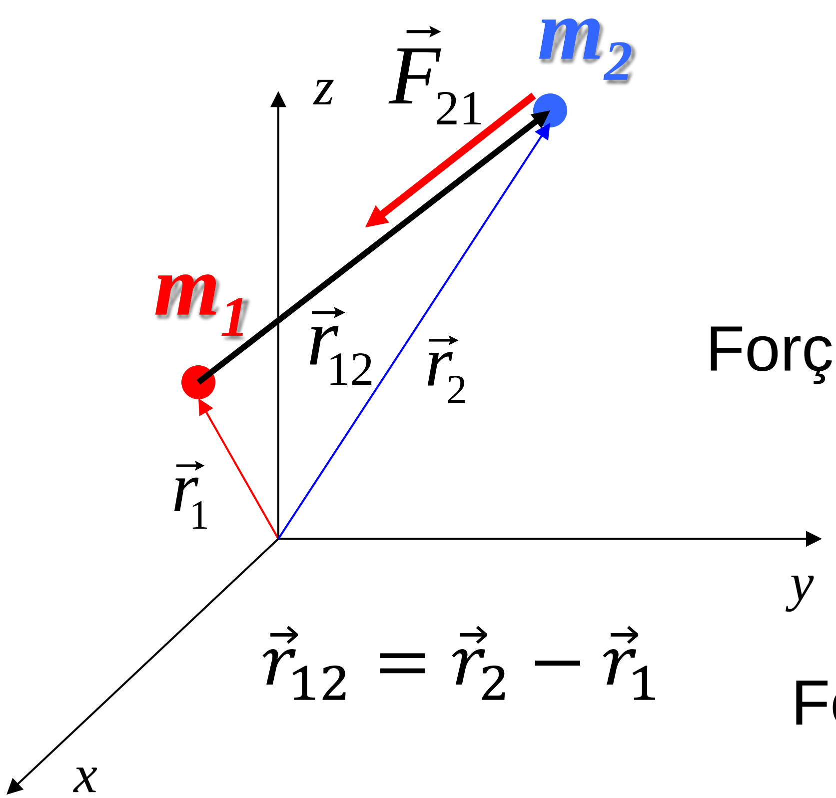
Força sobre m_2 devido à presença de m_1

Força sobre m_1 $\Rightarrow$ $\vec{F}_{12} = -\vec{F}_{21}$

Força gravitacional

Força Gravitacional

massas puntiformes



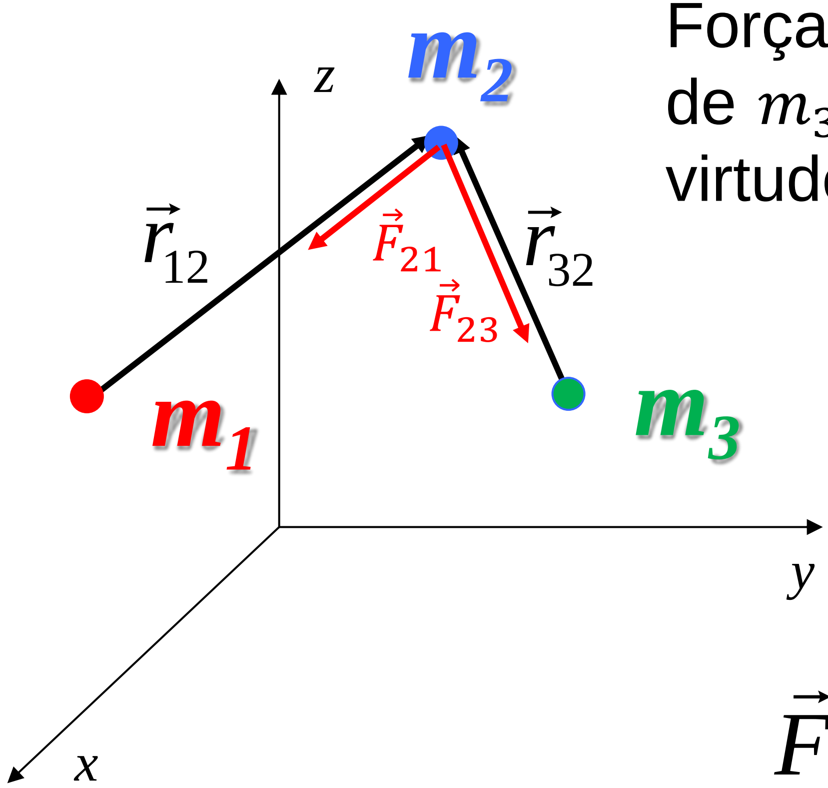
$$\vec{r}_{12} = \vec{r}_2 - \vec{r}_1$$

$$\vec{F}_{21} = -\frac{Gm_1m_2}{r_{12}^2}\hat{r}_{12}$$

Força sobre m_2 devido à presença de m_1

Força sobre m_1 $\Rightarrow$ $\vec{F}_{12} = -\vec{F}_{21}$

Princípio da Superposição



Força sobre m_2 em virtude da presença de m_3 não interfere na força sobre m_2 em virtude da presença de m_1

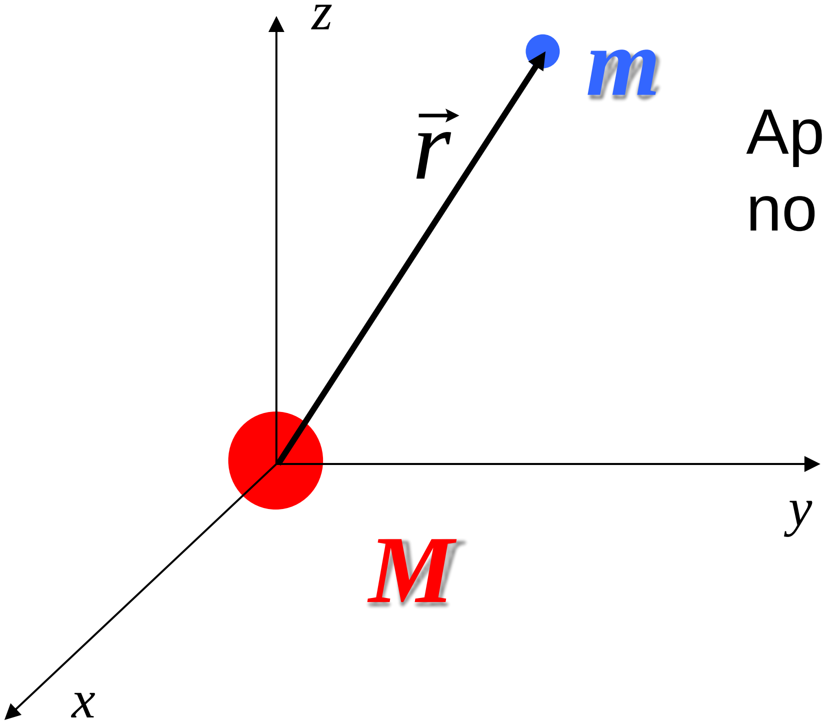
$$\vec{F} = \vec{F}_{21} + \vec{F}_{23}$$

$$\vec{F} = -\frac{Gm_1m_2}{r_{12}^2}\hat{r}_{12} - \frac{Gm_2m_3}{r_{32}^2}\hat{r}_{32}$$

Energia potencial gravitacional $\boldsymbol{U(r)}$

A força gravitacional é conservativa, e $\vec{\nabla} \times \vec{F} = 0 \ \rightarrow \vec{F} = -\vec{\nabla} U$

$$\Delta U = -W = -\int_{r_i}^{r_f} \vec{F} \cdot d\vec{r} = -G \frac{Mm}{r'}\bigg|_{r_i}^{r_f} = -G \frac{Mm}{r}$$

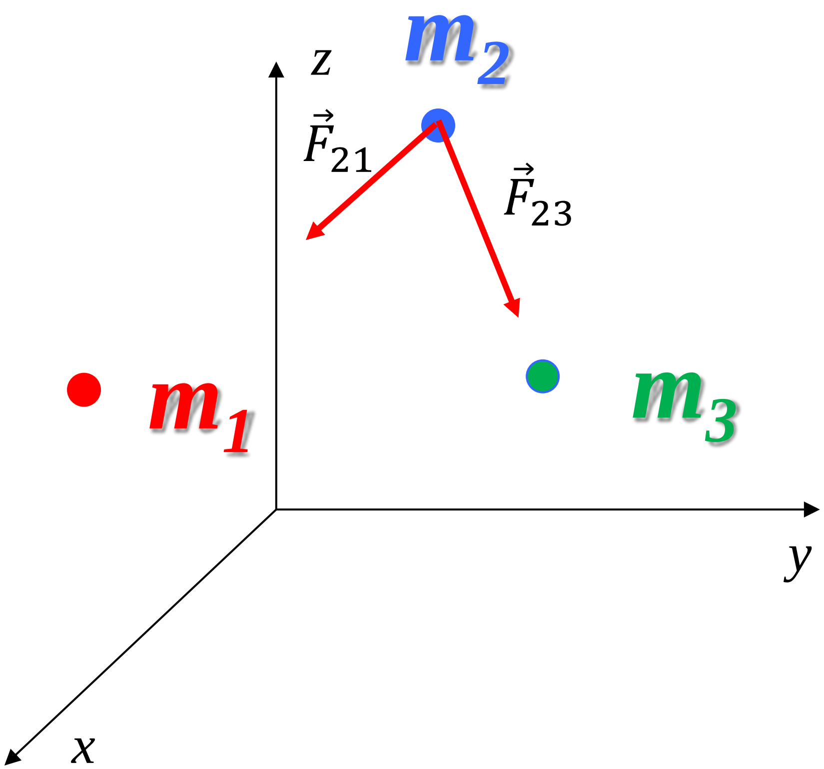


Após tomar o zero da energia potencial no infinito, $r_i \rightarrow \infty$ e $U(\infty) = 0$.

$$U(r) = -G \frac{Mm}{r}$$

Princípio da Superposição

Devido à relação entre a força gravitacional e a função energia potencial, temos que, como



$$\vec{F} = \vec{F}_{21} + \vec{F}_{23}$$

Teremos que

$$-\vec{\nabla}U = -\vec{\nabla}U_{21} - \vec{\nabla}U_{23}$$

$$U = U_{21} + U_{23}$$

Força gravitacional

Distribuição contínua de massa com densidade volumétrica ρ interagindo com massa puntiforme m.

$$\vec{F} = -Gm\int_V \frac{\rho(r')dV'}{r^2}\hat{r}$$

TM eq. (5.2); S eq. (6.7)

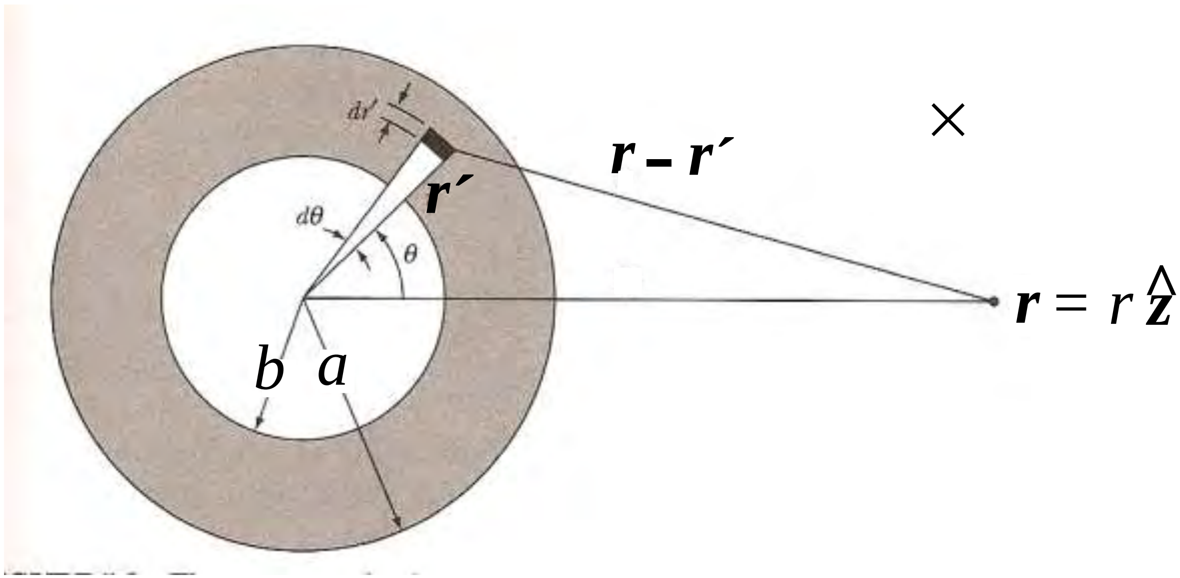
Potencial gravitacional gerado por uma massa puntiforme M e com o zero do potencial no infinito.

$$\Phi(r) = -\frac{GM}{r}$$ **TM eq. (5.6)**

Problemas

TM exemplo 5.1, pgs. 186-189

Calcular o potencial gravitacional dentro e fora de uma casca esférica homogênea de raio interno ***b*** e raio externo ***a***.



TM exemplo 5.1, pgs. 186–189

Potencial $\Phi(\boldsymbol{r}) = -G\int \frac{\rho(\boldsymbol{r}')\,\mathrm{d}^3\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|}$ escolhendo $\boldsymbol{r} = r\hat{\boldsymbol{z}}$:

$$\boldsymbol{r}-\boldsymbol{r}' = -r'\,\mathrm{sen}\,\theta\,\cos\varphi\,\hat{\boldsymbol{x}} - r'\,\mathrm{sen}\,\theta\,\mathrm{sen}\,\varphi\,\hat{\boldsymbol{y}} + (r - r'\cos\theta)\,\hat{\boldsymbol{z}}$$

$$|\boldsymbol{r}-\boldsymbol{r}'|^2 = r'^2\,\mathrm{sen}^2\,\theta + (r - r'\cos\theta)^2 = r'^2 + r^2 - 2rr'\cos\theta$$

$$\Phi(r) = -G\rho\int_0^{2\pi}\mathrm{d}\varphi\int_b^a \mathrm{d}r'\,r'^2\int_{-1}^{1}\mathrm{d}(\cos\theta)\,\frac{1}{\sqrt{r'^2+r^2-2rr'\cos\theta}}$$

$$\rho = \frac{M}{\frac{4}{3}\pi\,(a^3-b^3)},\quad \int_{-1}^{1}\frac{\mathrm{d}\xi}{\sqrt{r'^2+r^2-2rr'\xi}} = -\frac{1}{rr'}\left(\sqrt{r'^2+r^2-2rr'\xi}\right)_{-1}^{1}$$

$$= \frac{1}{rr'}\left(|r'+r| - |r'-r|\right)$$

Para $r' > r$: $\frac{1}{rr'}\left(|r'+r| - |r'-r|\right) = \frac{1}{rr'}\left(r'+r-r'+r\right) = \frac{2}{r'}$

Para $r' < r$: $\frac{1}{rr'}\left(|r'+r| - |r'-r|\right) = \frac{1}{rr'}\left(r'+r-r+r'\right) = \frac{2}{r}$

Portanto $\int_{-1}^{1}\frac{\mathrm{d}\xi}{\sqrt{r'^2+r^2-2rr'\xi}} = \frac{1}{rr'}\left(|r'+r| - |r'-r|\right) = \frac{2}{\max(r',r)}$

TM exemplo 5.1, pgs. 186–189

Potencial $\Phi(\boldsymbol{r}) = -G\int \frac{\rho(\boldsymbol{r}')\,\mathrm{d}^3\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|}$ escolhendo $\boldsymbol{r} = r\hat{\boldsymbol{z}}$:

$$\boldsymbol{r}-\boldsymbol{r}' = -r'\operatorname{sen}\theta\,\cos\varphi\,\hat{\boldsymbol{x}} - r'\operatorname{sen}\theta\,\operatorname{sen}\varphi\,\hat{\boldsymbol{y}} + (r - r'\cos\theta)\,\hat{\boldsymbol{z}}$$

$$|\boldsymbol{r}-\boldsymbol{r}'|^2 = r'^2\operatorname{sen}^2\theta + (r - r'\cos\theta)^2 = r'^2 + r^2 - 2rr'\cos\theta$$

$$\Phi(r) = -G\rho\int_0^{2\pi}\mathrm{d}\varphi\int_b^a \mathrm{d}r'\,r'^2\int_{-1}^{1}\mathrm{d}(\cos\theta)\,\frac{1}{\sqrt{r'^2+r^2-2rr'\cos\theta}}$$

$$\rho = \frac{M}{\frac{4}{3}\pi(a^3-b^3)},\quad \int_{-1}^{1}\frac{\mathrm{d}\xi}{\sqrt{r'^2+r^2-2rr'\xi}} = \frac{2}{\max(r',r)}$$

$$\Phi(r>a) = -\frac{3GM}{a^3-b^3}\,\frac{1}{r}\int_b^a \mathrm{d}r'\,r'^2 = -\frac{GM}{r}$$ **TM eq. (5.19)**

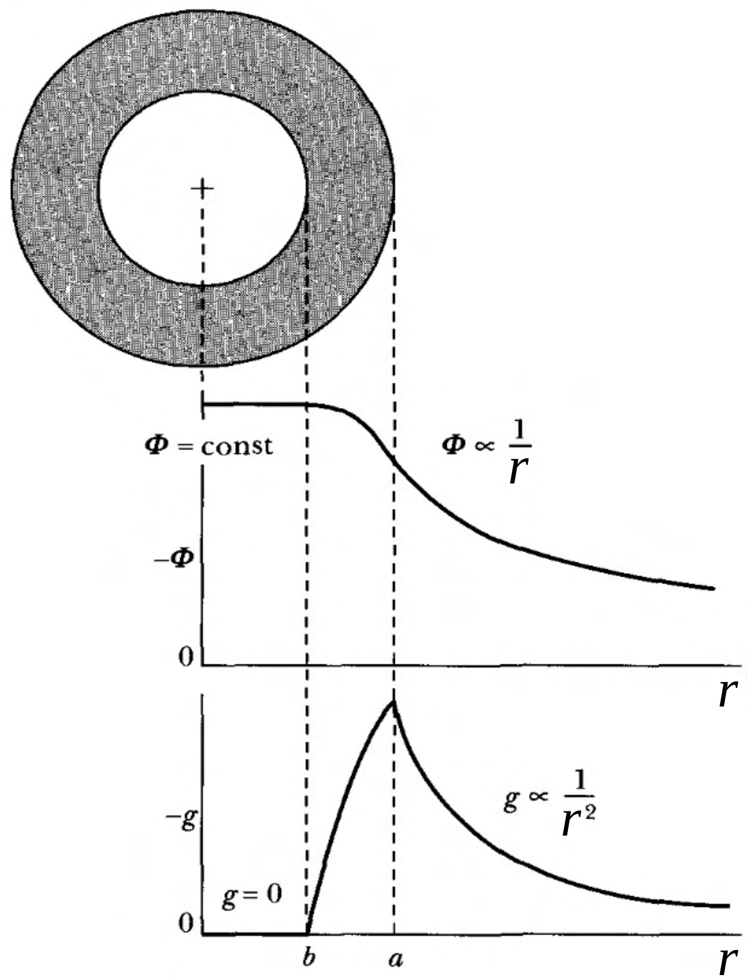
$$\Phi(b<r<a) = -\frac{3GM}{a^3-b^3}\left(\frac{1}{r}\int_b^r \mathrm{d}r'\,r'^2 + \int_r^a \mathrm{d}r'\,r'\right) = -\frac{GM}{r}\left(\frac{r^3-b^3}{a^3-b^3}\right)$$
$$-\frac{3GM}{2}\left(\frac{a^2-r^2}{a^3-b^3}\right) = \frac{GM}{2r}\left(\frac{2b^3-3a^2r+r^3}{a^3-b^3}\right)$$ **TM eq. (5.21)**

$$\Phi(r<b) = -\frac{3GM}{a^3-b^3}\int_b^a \mathrm{d}r'\,r' = -\frac{3GM}{2}\left(\frac{a^2-b^2}{a^3-b^3}\right)$$ **TM eq. (5.20)**

Problemas

TM exemplo 5.1, pgs. 186-189

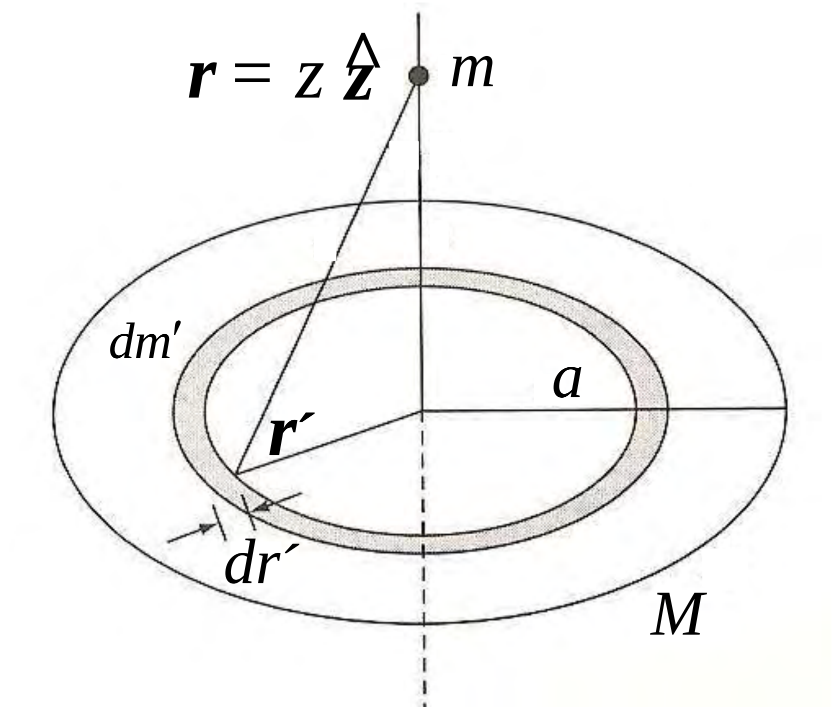
Campo gravitacional de uma casca esférica



Problemas

TM exemplo 5.4, pgs. 196-198

Calcular a força sobre uma massa puntiforme m localizada sobre o eixo do disco, a uma distância z do centro do mesmo. Considere um disco fino e homogêneo.



Potencial $\Phi(\boldsymbol{r}) = -G\int \frac{\sigma(\boldsymbol{r}')\,\mathrm{d}^2\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|}$ escolhendo $\boldsymbol{r} = z\hat{\boldsymbol{z}}$:

$$\boldsymbol{r}-\boldsymbol{r}' = -r'\cos\varphi\,\hat{\boldsymbol{x}} - r'\operatorname{sen}\varphi\,\hat{\boldsymbol{y}} + z\,\hat{\boldsymbol{z}}, \quad |\boldsymbol{r}-\boldsymbol{r}'|^2 = r'^2 + z^2$$

$$\Phi(z) = -G\sigma\int_0^{2\pi}\mathrm{d}\varphi\int_0^a \mathrm{d}r'\,\frac{r'}{\sqrt{r'^2+z^2}} = -2\pi G\sigma\left(\sqrt{r'^2+z^2}\right)_0^a$$

$$= -\frac{2GM}{a^2}\left(\sqrt{a^2+z^2}-|z|\right)$$ **TM eq. (5.43)** $\sigma = \frac{M}{\pi a^2}$

$$\boldsymbol{F}(z) = -m\nabla\Phi(z) = \frac{2GMm}{a^2}\left(\frac{z}{\sqrt{a^2+z^2}}-\frac{z}{|z|}\right)\hat{\boldsymbol{z}}$$ **TM eq. (5.45)**

Para um disco anular de raio externo a e interno b:

$$\Phi(z) = -\frac{2GM}{a^2-b^2}\left(\sqrt{a^2+z^2}-\sqrt{b^2+z^2}\right)$$

$$\boldsymbol{F}(z) = \frac{2GMm}{a^2-b^2}\left(\frac{z}{\sqrt{a^2+z^2}}-\frac{z}{\sqrt{b^2+z^2}}\right)\hat{\boldsymbol{z}}$$