F-315 (Mecânica Geral I)
Aula 22

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http://www.ifi.unicamp.br/~vidiella/aulas.html

Cálculo Variacional

TM seção 6.5

Funções com várias variáveis

$$f = f\{y_i(x), y_i'(x); x\} \quad i = 1,2\ldots n$$

Analogamente,

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \sum_i \left[\frac{\partial f}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial f}{\partial y_i'} \right) \right] \eta_i(x) dx$$

Resulta no conjunto de equações de Euler

$$\frac{\partial f}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial f}{\partial y_i'} \right) = 0$$ **TM eq. (6.57)**

Cálculo Variacional

TM seção 6.6

Equações de Euler com vínculos

TM eq. (6.69)

$$\frac{\partial f}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial f}{\partial y_i'}\right) + \sum_j \lambda_j(x)\frac{\partial g_j}{\partial y_i} = 0$$

Onde os vínculos $g_j\{y_i; x\} = 0$ e $i = 1,2,\ldots n$, $j = 1,2\ldots m$

As funções λ_j são os multiplicadores de Lagrange

$f = f\{y_i, y_i'; x\}$: múltiplas variáveis dependentes $\{y_i(x)\}$
$i = 1, \cdots, n$, com vínculos $g_k = g_k\{y_i; x\} = 0$, $k = 1, \cdots, m$

Exemplo: $g_1 = \sum_{i=1}^{3} x_i^2 - r^2 = 0$ ($\boldsymbol{r}$ sobre a esfera de raio r)

$$y_i(\alpha, x) = y_i(0, x) + \alpha\eta_i(x), \quad \frac{\partial y_i}{\partial \alpha} = \eta_i(x), \quad \frac{\partial y_i'}{\partial \alpha} = \frac{\mathrm{d}\eta_i(x)}{\mathrm{d}x} = \eta_i'(x)$$

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \sum_{i=1}^{n} \left(\frac{\partial f}{\partial y_i} \frac{\partial y_i}{\partial \alpha} + \frac{\partial f}{\partial y_i'} \frac{\partial y_i'}{\partial \alpha} \right) \mathrm{d}x = \int_{x_1}^{x_2} \sum_{i=1}^{n} \left[\frac{\partial f}{\partial y_i} \eta_i(x) + \frac{\partial f}{\partial y_i'} \eta_i'(x) \right] \mathrm{d}x$$

$= \int_{x_1}^{x_2} \sum_{i=1}^{n} \left[\frac{\partial f}{\partial y_i} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y_i'} \right) \right] \eta_i(x)\, \mathrm{d}x = 0$. Agora os $\{\eta_i(x)\}$ não são mais independentes, mas estão vinculados através da condição sobre $\mathrm{d}g_k = \sum_{i=1}^{n} \frac{\partial g_k}{\partial y_i} \frac{\partial y_i}{\partial \alpha} \mathrm{d}\alpha = \sum_{i=1}^{n} \frac{\partial g_k}{\partial y_i} \eta_i(x)\, \mathrm{d}\alpha = 0$.

Exemplo: um vínculo ($m = 1$) com $n = 2$, $g(y_1, y_2; x)$

$$\frac{\mathrm{d}g}{\mathrm{d}\alpha} = \frac{\partial g}{\partial y_1}\eta_1(x) + \frac{\partial g}{\partial y_2}\eta_2(x) = 0 \;\rightarrow\; \frac{\eta_2(x)}{\eta_1(x)} = -\frac{\partial g/\partial y_1}{\partial g/\partial y_2}$$

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left\{ \left[\frac{\partial f}{\partial y_1} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y_1'}\right)\right]\eta_1(x) + \left[\frac{\partial f}{\partial y_2} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y_2'}\right)\right]\eta_2(x) \right\} \mathrm{d}x$$

$$= \int_{x_1}^{x_2} \left\{ \left[\frac{\partial f}{\partial y_1} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y_1'}\right)\right] - \left[\frac{\partial f}{\partial y_2} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y_2'}\right)\right] \frac{\partial g/\partial y_1}{\partial g/\partial y_2} \right\} \eta_1(x)\,\mathrm{d}x = 0.$$

Agora o integrando entre colchetes se anula $\forall \eta_1(x) \rightarrow$

$$\lambda(x) \equiv -\left[\frac{\partial f}{\partial y_1} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y_1'}\right)\right]\left(\frac{\partial g}{\partial y_1}\right)^{-1} = -\left[\frac{\partial f}{\partial y_2} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y_2'}\right)\right]\left(\frac{\partial g}{\partial y_2}\right)^{-1}$$

é o chamado ***multiplicador de Lagrange.***

$$\frac{\partial f}{\partial y_1} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y_1'}\right) + \lambda(x)\frac{\partial g}{\partial y_1} = \frac{\partial f}{\partial y_2} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y_2'}\right) + \lambda(x)\frac{\partial g}{\partial y_2} = 0$$

Para m vínculos ***holonômicos*** $g_k = g_k\{y_i;x\} = 0$, isto é equivalente a extremizar $\tilde{J} \equiv \int_{x_1}^{x_2} \left[f + \sum_{k=1}^{m} \lambda_k(x) g_k\right] \mathrm{d}x$ como se $\{y_i(x), \lambda_k(x)\}$ fossem variáveis independentes entre si.

Equações de Euler com vínculos:

TM eq. (6.69)

$$\frac{\partial f}{\partial y_i} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y_i'}\right) + \sum_{k=1}^{m} \lambda_k(x) \frac{\partial g_k}{\partial y_i} = 0$$

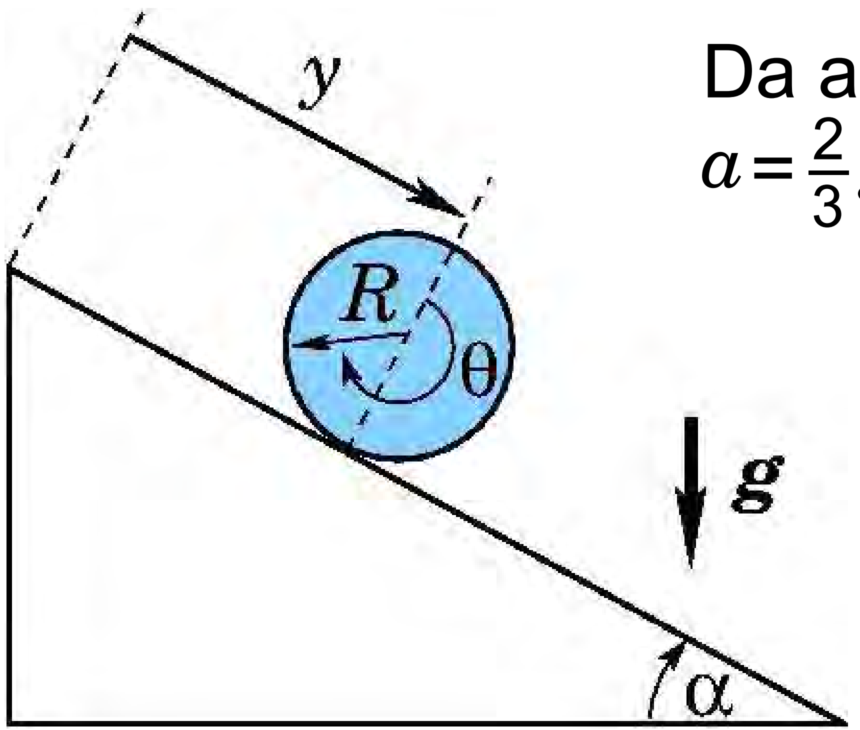
O caso geral ***não-holonômico*** $g_k = g_k\{y_i, y_i'; x\} = 0$ não é trivial: ver FLANNERY, *Am. J. Phys.* **73,** 265–272 (2005).

No entanto, se o vínculo está na forma integral $K\{y_i\} = = \int_{x_1}^{x_2} g\{y_i, y_i'; x\}\, \mathrm{d}x$, então extremiza-se $\int_{x_1}^{x_2} (f + \lambda g)\, \mathrm{d}x$.

Problema

TM exemplo 6.5, pgs. 221-222

Disco de raio R rolando sem escorregar em um plano inclinado. Obter a equação de vínculo $g(y,\theta)$.



Da aula 16, aceleração:
$a = \frac{2}{3} g \operatorname{sen} \alpha$

$$y = R\theta$$

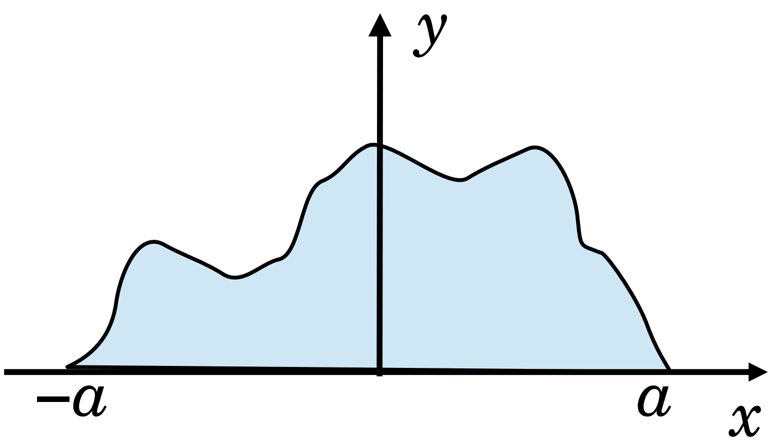
$$g(y,\theta) = y - R\theta = 0$$

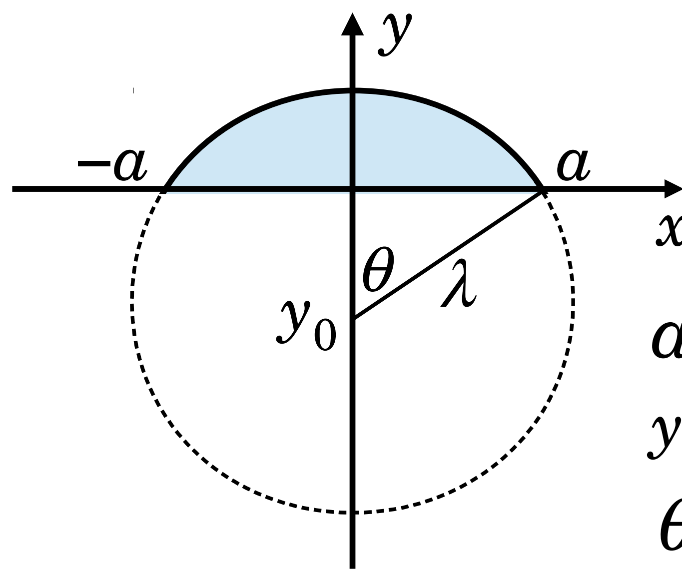
$$\frac{\partial g}{\partial y} = 1 \;,\; \frac{\partial g}{\partial \theta} = -R$$

Problema

TM exemplo 6.6, pgs. 222-224

Problema isoperimétrico: Encontrar a curva $y(x)$, de comprimento ℓ, limitada inferiormente pelo eixo x, que passa pelos pontos $(-a,0)$ e $(a,0)$ de modo que a área circunscrita seja máxima.





$$a = \lambda \operatorname{sen}\theta$$

$$y_0 = \lambda \cos\theta$$

$$\theta = \ell / 2\lambda$$

TM exemplo 6.6 (problema de Dido), pgs. 222–224

Problema de maximizar a área $A = \int_{-a}^{a} y\,\mathrm{d}x$ com comprimento fixo $\ell = \int_{-a}^{a} \mathrm{d}s = \int_{-a}^{a} \sqrt{1+y'^2}\,\mathrm{d}x$. Condições de contorno: $y(x=\pm a)=0$.

Maximizar $\tilde{J} = \int_{-a}^{a} \left(y+\lambda\sqrt{1+y'^2}\right)\mathrm{d}x \equiv \int_{-a}^{a} \tilde{f}\,\mathrm{d}x$. A função $\tilde{f}(y,y';x)$ satisfaz a equação de Euler: $\frac{\partial\tilde{f}}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial\tilde{f}}{\partial y'}\right) = 0$

$$\frac{\partial\tilde{f}}{\partial y} = 1 = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial\tilde{f}}{\partial y'}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\lambda y'}{(1+y'^2)^{1/2}}\right] \rightarrow \frac{\lambda y'}{(1+y'^2)^{1/2}} = x + \text{constante} = x - x_0 \rightarrow$$

$$\lambda^2 y'^2 = (x-x_0)^2(1+y'^2) \rightarrow y'(x) = \frac{\pm(x-x_0)}{[\lambda^2-(x-x_0)^2]^{1/2}} \rightarrow y(x) = y_0 \mp [\lambda^2-(x-x_0)^2]^{1/2}$$

Caso geral, circunferência de raio λ: TM eq. (6.85)

$$(x-x_0)^2 + (y-y_0)^2 = \lambda^2, \quad x_0 = 0, \quad y_0 = \lambda\cos\frac{\ell}{2\lambda}, \quad a = \lambda\,\mathrm{sen}\,\frac{\ell}{2\lambda}$$

Equação para a provém de $\ell = \int_{-a}^{a}\mathrm{d}s = \lambda\int_{-a/\lambda}^{a/\lambda}\frac{\mathrm{d}u}{\sqrt{1-u^2}} = 2\lambda\,\mathrm{arcsen}\,\frac{a}{\lambda}$.

G problema 2.9 (curva catenária), pg. 65

Problema da curva catenária: minimizar potencial gravitacional

$V = \rho g \int_{-a}^{a} y\,\mathrm{d}s = \rho g \int_{-a}^{a} y\sqrt{1+y'^2}\,\mathrm{d}x$ com comprimento fixo $\ell = \int_{-a}^{a} \mathrm{d}s$

$= \int_{-a}^{a} \sqrt{1+y'^2}\,\mathrm{d}x$. Condições de contorno: $b \equiv y(x=a) - y(x=-a)$.

Minimizar $\tilde{J} = \int_{-a}^{a} (y+\lambda)\sqrt{1+y'^2}\,\mathrm{d}x \equiv \int_{-a}^{a} \tilde{f}\,\mathrm{d}x$. A função $\tilde{f}(y,y';x)$

satisfaz a equação de Euler: $\frac{\partial \tilde{f}}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial \tilde{f}}{\partial y'}\right) = 0$

$$\frac{\partial \tilde{f}}{\partial y} = (1+y'^2)^{1/2} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial \tilde{f}}{\partial y'}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{(y+\lambda)y'}{(1+y'^2)^{1/2}}\right] = \frac{y'^2+(y+\lambda)y''}{(1+y'^2)^{1/2}} - \frac{(y+\lambda)y'^2y''}{(1+y'^2)^{3/2}}$$

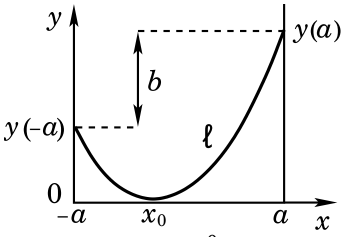
$$\rightarrow\ (y+\lambda)y'' = 1+y'^2 \ \rightarrow\ \frac{2y'y''}{1+y'^2} = \frac{2y'}{y+\lambda} \ \rightarrow\ \frac{\mathrm{d}}{\mathrm{d}x}\ln(1+y'^2) = \frac{\mathrm{d}}{\mathrm{d}x}\ln(y+\lambda)^2$$

$$\rightarrow\ \ln(1+y'^2) = C + \ln(y+\lambda)^2 \ \rightarrow\ 1+y'^2 = \left(\frac{y+\lambda}{\lambda c}\right)^2 \rightarrow\ y' = \frac{\mathrm{d}y}{\mathrm{d}x} =$$

$$= \pm\frac{1}{\lambda c}\sqrt{(y+\lambda)^2-(\lambda c)^2} \ \rightarrow\ \Delta x \equiv x - x_0 = \pm\lambda c \int_{y_0/\lambda}^{y/\lambda} \frac{\mathrm{d}u}{\sqrt{(1+u)^2-c^2}}$$

G problema 2.9 (curva catenária), pg. 65

Escolhendo x_0 tal que $y_0 \equiv y(x_0) = 0$ e $y'(x_0) = 0 \ \rightarrow\ c = 1$, ou seja, x_0 representa um *mínimo* da função $y(x)$, conforme mostra a figura ao lado:



$$\Delta x = \lambda c \int_{y_0/\lambda}^{y/\lambda} \frac{\mathrm{d}u}{\sqrt{(1+u)^2-c^2}} = \lambda c \ln\left[1+u+\sqrt{(1+u)^2-c^2}\,\right]_{y_0/\lambda}^{y/\lambda} \overset{y_0=0}{\underset{c=1}{=}} \lambda \operatorname{arccosh}\left(\frac{y+\lambda}{\lambda}\right)$$

$$b \equiv y(a) - y(-a) = \lambda \cosh \frac{\Delta x}{\lambda}\Big|_{-a}^{a} = -2\lambda \operatorname{senh}\frac{a}{\lambda} \operatorname{senh}\frac{x_0}{\lambda},\ y'(x) = \operatorname{senh}\frac{\Delta x}{\lambda}$$

$$\ell = \int_{-a}^{a} \mathrm{d}s = \int_{-a}^{a} \cosh\frac{\Delta x}{\lambda}\,\mathrm{d}x = \lambda \operatorname{senh}\frac{\Delta x}{\lambda}\Big|_{-a}^{a} = 2\lambda \operatorname{senh}\frac{a}{\lambda} \cosh\frac{x_0}{\lambda}$$

Curva catenária com mínimo em $(x_0, y_0 \equiv 0)$

$$y(x) = \lambda\left[\cosh\left(\frac{x-x_0}{\lambda}\right) - 1\right],\quad \sqrt{\ell^2 - b^2} = 2\lambda \operatorname{senh}\frac{a}{\lambda},\quad \operatorname{tgh}\frac{x_0}{\lambda} = -\frac{b}{\ell}$$

TM seção 6.4 (equação de Euler II), pgs. 216–217

Derivada *total*: $\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\partial f}{\partial y'}\frac{\mathrm{d}y'}{\mathrm{d}x} + \frac{\partial f}{\partial x} = y'\frac{\partial f}{\partial y} + y''\frac{\partial f}{\partial y'} + \frac{\partial f}{\partial x}$

Temos também $\frac{\mathrm{d}}{\mathrm{d}x}\left(y'\frac{\partial f}{\partial y'}\right) = y''\frac{\partial f}{\partial y'} + y'\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right)$, ou seja:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(y'\frac{\partial f}{\partial y'}\right) = \frac{\mathrm{d}f}{\mathrm{d}x} - \frac{\partial f}{\partial x} - y'\left[\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right)\right]$$

= 0 (eq. de Euler)

***Equação de Euler (II)*:**

TM eq. (6.39)

$$\frac{\partial f}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}x}\left(f - y'\frac{\partial f}{\partial y'}\right) = 0$$

Assim, se $f = f(y, y'; x)$ não depende *explicitamente* de x:

TM eq. (6.40); Taylor eq. (6.43)

$$\frac{\partial f}{\partial x} = 0 \quad \rightarrow \quad f - y'\frac{\partial f}{\partial y'} = \text{constante}$$

Problemas isoperimétrico e da curva catenária revisitados

Nestes dois casos: $\frac{\partial \tilde{f}}{\partial x} = 0 \;\rightarrow\; \tilde{f} - y' \frac{\partial \tilde{f}}{\partial y'} = \text{constante.}$

Problema isoperimétrico: $\tilde{f}(y, y'; x) = y + \lambda\sqrt{1 + y'^2}$

$$\tilde{f} - y' \frac{\partial \tilde{f}}{\partial y'} = \tilde{f} - \frac{\lambda y'^2}{(1+y'^2)^{1/2}} = \tilde{f} - \frac{(\tilde{f}-y)y'^2}{1+y'^2} = \frac{\tilde{f}+yy'^2}{1+y'^2} = y + \frac{\lambda}{(1+y'^2)^{1/2}} = \text{constante}$$

$$(y - y_0)(1 + y'^2)^{1/2} = -\lambda \;\rightarrow\; y' = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\pm[\lambda^2 - (y-y_0)^2]^{1/2}}{y - y_0} \;\rightarrow\; \pm(x - x_0) =$$

$$= \int_{\substack{y(x_0) \\ = y_0 + \lambda}}^{y} \frac{(y'-y_0)\,\mathrm{d}y'}{[\lambda^2 - (y'-y_0)^2]^{1/2}} = -[\lambda^2 - (y - y_0)^2]^{1/2} \;\rightarrow$$

TM eq. (6.85)

$$(x - x_0)^2 + (y - y_0)^2 = \lambda^2$$

Problema da curva catenária: $\tilde{f}(y, y'; x) = (y + \lambda)\sqrt{1 + y'^2}$

$$\tilde{f} - y' \frac{\partial \tilde{f}}{\partial y'} = \tilde{f} - \frac{(y+\lambda)y'^2}{(1+y'^2)^{1/2}} = \tilde{f} - \frac{\tilde{f}y'^2}{1+y'^2} = \frac{\tilde{f}}{1+y'^2} = \frac{y+\lambda}{(1+y'^2)^{1/2}} = \text{constante} = \lambda c$$

$$\rightarrow 1 + y'^2 = \frac{1}{(\lambda c)^2}(y + \lambda)^2 \rightarrow$$

Curva catenária com mínimo em $(x_0, y_0 \equiv 0)$

$$x(y) = x_0 + \lambda c \int_{y_0/\lambda}^{y/\lambda} \frac{\mathrm{d}u}{\sqrt{(1+u)^2 - c^2}} \overset{\substack{y_0=0 \\ c=1}}{=} x_0 + \lambda \,\text{arccosh}\left(\frac{y+\lambda}{\lambda}\right)$$