F-315 (Mecânica Geral I)
Aula 24

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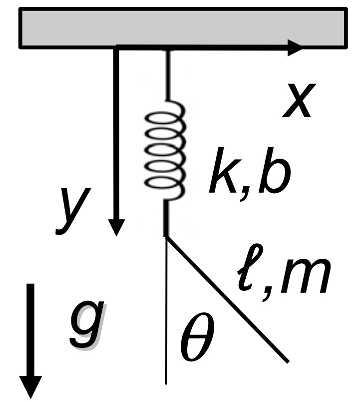
http://www.ifi.unicamp.br/~vidiella/aulas.html

Problema
Pêndulo composto por mola e haste



Considere uma haste homogênea de massa m e comprimento ℓ que oscila num plano vertical suspensa por uma de suas extremidades. O ponto de suspensão é a extremidade de uma mola (massa desprezível, constante de mola k e comprimento de repouso b) mantida na vertical.

a) Obtenha a lagrangiana para este pêndulo, usando como coordenadas generalizadas o ângulo θ em relação à direção vertical e o comprimento total y da mola (ver figura).

b) Determine o movimento de pequenas oscilações em torno das coordenadas de equilíbrio $\bar{y} = b + mg/k$ e $\bar{\theta} = 0$.

Problema: pêndulo composto por mola e haste

a) $\mathscr{L} = \frac{1}{2}m(\dot{X}^2 + \dot{Y}^2) + \frac{1}{2}I_{\text{haste}}^{\text{CM}}\dot{\theta}^2 - \frac{1}{2}k(y-b)^2 + mgY$

$I_{\text{haste}}^{\text{CM}} = \frac{m}{\ell}\int_{-\ell/2}^{\ell/2}\xi^2\,\mathrm{d}\xi = \frac{2}{3}\frac{m}{\ell}\left(\frac{\ell}{2}\right)^3 = \frac{1}{12}m\ell^2$

$X = \frac{1}{2}\ell\,\text{sen}\,\theta, \quad \dot{X} = \frac{1}{2}\ell\dot{\theta}\cos\theta$

$Y = y + \frac{1}{2}\ell\cos\theta, \quad \dot{Y} = \dot{y} - \frac{1}{2}\ell\dot{\theta}\,\text{sen}\,\theta, \quad \mathscr{L} = \frac{1}{6}m\ell^2\dot{\theta}^2 +$

$+ \frac{1}{2}m\dot{y}^2 - \frac{1}{2}m\ell\dot{y}\dot{\theta}\,\text{sen}\,\theta - \frac{1}{2}k(y-b)^2 + mg\left(y + \frac{1}{2}\ell\cos\theta\right)$

b) Expandindo $\mathscr{L}$ em torno de $\bar{y} \equiv b + \frac{mg}{k}$, $\bar{\theta} = 0$:

$\mathscr{L} \approx \frac{1}{6}m\ell^2\dot{\theta}^2 + \frac{1}{2}m\dot{y}^2 - \frac{1}{2}k(y-\bar{y})^2 + mg\left(b + \frac{mg}{2k} + \frac{1}{2}\ell\right)$

$-\frac{1}{4}mg\ell\theta^2 \rightarrow \delta\ddot{y} \approx -\frac{k}{m}\delta y \equiv -\omega^2\delta y, \quad \ddot{\theta} \approx -\frac{3g}{2\ell}\theta \equiv -\Omega^2\theta$

$y(t) = \bar{y} + [y(0) - \bar{y}]\cos\omega t + \frac{1}{\omega}\dot{y}(0)\,\text{sen}\,\omega t, \quad \omega \equiv \sqrt{k/m}$

$\theta(t) = \theta(0)\cos\Omega t + \frac{1}{\Omega}\dot{\theta}(0)\,\text{sen}\,\Omega t, \quad \Omega \equiv \sqrt{3g/2\ell}$

Mecânica Lagrangiana

TM seção 7.6

Equivalência entre as equações de Lagrange e as equações de Newton

Podemos obter a 2ª lei de Newton a partir das equações de Lagrange e vice-versa

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = 0 \quad \Leftrightarrow \quad m\ddot{x}_i = F_i$$ **TM eq. (7.98)**

Forças em coordenadas cartesianas $(x_1, x_2, x_3) = (x, y, z)$

$$V = V\{x_i\},\ T = T\{\dot{x}_i\} = \frac{1}{2} m \sum_j \dot{x}_j^2 \to \frac{\partial \mathscr{L}}{\partial x_i} = -\frac{\partial V}{\partial x_i} = F_i,\ \frac{\partial \mathscr{L}}{\partial \dot{x}_i} = \frac{\partial T}{\partial \dot{x}_i} = m\dot{x}_i \equiv p_i$$

$$\frac{\partial \mathscr{L}}{\partial x_i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathscr{L}}{\partial \dot{x}_i}\right) = F_i - \dot{p}_i = 0 \ \to \quad \dot{\boldsymbol{p}} \equiv \frac{\mathrm{d}}{\mathrm{d}t}\left(\nabla_{\dot{\boldsymbol{r}}} \mathscr{L}\right) = \boldsymbol{F} = -\nabla V = \nabla \mathscr{L}$$

Forças generalizadas em coordenadas esféricas $(q_1, q_2, q_3) = (r, \theta, \varphi)$, ***na ausência de vínculos***

TM eq. (7.112); S eq. (9.53)

$$Q_k \equiv -\frac{\partial V}{\partial q_k} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k}, \qquad V = V\{q_k\}$$

TM eq. (7.111); S eq. (9.50)

Momentos generalizados:

$$p_k \equiv \frac{\partial T}{\partial \dot{q}_k} \quad \to \quad \dot{p}_k = Q_k + \frac{\partial T}{\partial q_k}$$

TM eqs. (7.105) e (7.106); S eq. (9.33)

$$Q_k \equiv -\frac{\partial V}{\partial q_k} = -\sum_i \frac{\partial V}{\partial x_i}\frac{\partial x_i}{\partial q_k} = \sum_i F_i \frac{\partial x_i}{\partial q_k} = \boldsymbol{F}\cdot\frac{\partial \boldsymbol{r}}{\partial q_k} = \boldsymbol{F}\cdot\boldsymbol{\varepsilon}_k$$

$\boldsymbol{\varepsilon}_k \equiv \frac{\partial \boldsymbol{r}}{\partial q_k}$: vetores de base para coordenadas esféricas (Arfken 5ª ed., seção 2.10, pg. 150)

$$\boldsymbol{r} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}} = r\,\mathrm{sen}\,\theta\cos\varphi\,\hat{\boldsymbol{x}} + r\,\mathrm{sen}\,\theta\,\mathrm{sen}\,\varphi\,\hat{\boldsymbol{y}} + r\cos\theta\,\hat{\boldsymbol{z}}$$

$$\boldsymbol{\varepsilon}_r \equiv \frac{\partial \boldsymbol{r}}{\partial r} = \frac{\boldsymbol{r}}{r} = \hat{\boldsymbol{r}}$$

$$\boldsymbol{\varepsilon}_\theta \equiv \frac{\partial \boldsymbol{r}}{\partial \theta} = r\cos\theta\cos\varphi\,\hat{\boldsymbol{x}} + r\cos\theta\,\mathrm{sen}\,\varphi\,\hat{\boldsymbol{y}} - r\,\mathrm{sen}\,\theta\,\hat{\boldsymbol{z}} = r\,\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\varphi}}\times\boldsymbol{r}$$

$$\boldsymbol{\varepsilon}_\varphi \equiv \frac{\partial \boldsymbol{r}}{\partial \varphi} = -r\,\mathrm{sen}\,\theta\,\mathrm{sen}\,\varphi\,\hat{\boldsymbol{x}} + r\,\mathrm{sen}\,\theta\cos\varphi\,\hat{\boldsymbol{y}} = r\,\mathrm{sen}\,\theta\,\hat{\boldsymbol{\varphi}} = \hat{\boldsymbol{z}}\times\boldsymbol{r}$$

Arfken eq. (2.116)

$$\boldsymbol{\varepsilon}_r = \hat{\boldsymbol{r}}, \quad \boldsymbol{\varepsilon}_\theta = r\,\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\varphi}}\times\boldsymbol{r}, \quad \boldsymbol{\varepsilon}_\varphi = r\,\mathrm{sen}\,\theta\,\hat{\boldsymbol{\varphi}} = \hat{\boldsymbol{z}}\times\boldsymbol{r}$$

Decomposição das forças: $\boldsymbol{F} = F_x\hat{\boldsymbol{x}} + F_y\hat{\boldsymbol{y}} + F_z\hat{\boldsymbol{z}} = F_r\hat{\boldsymbol{r}} + F_\theta\hat{\boldsymbol{\theta}} + F_\varphi\hat{\boldsymbol{\varphi}}$

$Q_r = \boldsymbol{F}\cdot\boldsymbol{\varepsilon}_r = \boldsymbol{F}\cdot\hat{\boldsymbol{r}} = F_r$ (*força* radial)

$Q_\theta = \boldsymbol{F}\cdot\boldsymbol{\varepsilon}_\theta = \boldsymbol{F}\cdot(\hat{\boldsymbol{\varphi}}\times\boldsymbol{r}) = \hat{\boldsymbol{\varphi}}\cdot(\boldsymbol{r}\times\boldsymbol{F}) = \hat{\boldsymbol{\varphi}}\cdot\boldsymbol{\tau} = \tau_\varphi = rF_\theta$ (*torque* $\parallel \hat{\boldsymbol{\varphi}}$)

$Q_\varphi = \boldsymbol{F}\cdot\boldsymbol{\varepsilon}_\varphi = \boldsymbol{F}\cdot(\hat{\boldsymbol{z}}\times\boldsymbol{r}) = \hat{\boldsymbol{z}}\cdot(\boldsymbol{r}\times\boldsymbol{F}) = \hat{\boldsymbol{z}}\cdot\boldsymbol{\tau} = \tau_z = r\,\mathrm{sen}\,\theta\, F_\varphi$ (*torque* $\parallel \hat{\boldsymbol{z}}$)

$$\dot{\boldsymbol{r}} \equiv \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = \dot{r}\frac{\partial\boldsymbol{r}}{\partial r} + \dot{\theta}\frac{\partial\boldsymbol{r}}{\partial\theta} + \dot{\varphi}\frac{\partial\boldsymbol{r}}{\partial\varphi} = \dot{r}\,\boldsymbol{\varepsilon}_r + \dot{\theta}\boldsymbol{\varepsilon}_\theta + \dot{\varphi}\,\boldsymbol{\varepsilon}_\varphi = \dot{r}\,\hat{\boldsymbol{r}} + r\dot{\theta}\,\hat{\boldsymbol{\theta}} + r\,\mathrm{sen}\,\theta\,\dot{\varphi}\,\hat{\boldsymbol{\varphi}}$$

$$T = \frac{1}{2}m\dot{\boldsymbol{r}}^2 = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\,\mathrm{sen}^2\,\theta\,\dot{\varphi}^2\right), \qquad p_k \equiv \frac{\partial T}{\partial\dot{q}_k} = \boldsymbol{p}\cdot\boldsymbol{\varepsilon}_k$$

$$p_r = m\dot{r}, \quad p_\theta = mr^2\dot{\theta} = m|\boldsymbol{\varepsilon}_\theta|^2\dot{\theta}, \quad p_\varphi = mr^2\,\mathrm{sen}^2\,\theta\,\dot{\varphi} = m|\boldsymbol{\varepsilon}_\varphi|^2\dot{\varphi}$$

$$\boldsymbol{p} = m\dot{\boldsymbol{r}} = m\left(\dot{r}\,\hat{\boldsymbol{r}} + r\dot{\theta}\,\hat{\boldsymbol{\theta}} + r\,\mathrm{sen}\,\theta\,\dot{\varphi}\,\hat{\boldsymbol{\varphi}}\right) = p_r\hat{\boldsymbol{r}} + \frac{1}{|\boldsymbol{\varepsilon}_\theta|}p_\theta\,\hat{\boldsymbol{\theta}} + \frac{1}{|\boldsymbol{\varepsilon}_\varphi|}p_\varphi\,\hat{\boldsymbol{\varphi}}$$

Forças de reação Q_i^* com vínculos ***holonômicos*** $f_k = f_k\{q_i;t\} = 0$:

TM eqs. (7.64), (7.65) e (7.66)

$$\delta f_k = \sum_i \frac{\partial f_k}{\partial q_i}\mathrm{d}q_i = 0, \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathscr{L}}{\partial\dot{q}_i}\right) - \frac{\partial\mathscr{L}}{\partial q_i} = \sum_k \lambda_k(t)\frac{\partial f_k}{\partial q_i} \equiv Q_i^*$$

Mecânica Lagrangiana

TM seção 7.5

Equações com multiplicadores de Lagrange

Vínculos da forma $\sum_j \frac{\partial f_k}{\partial q_j} dq_j = 0 \quad j = 1,2,\ldots s;\, k = 1,2,\ldots m$

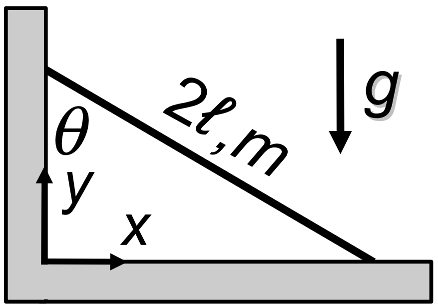
Podem ser incorporados nas equações de Lagrange

$$\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} + \sum_k \lambda_k(t)\frac{\partial f_k}{\partial q_j} = 0$$ **TM eq. (7.65)**

E as forças de vínculo generalizadas são dadas por

$$Q_j = \sum_k \lambda_k \frac{\partial f_k}{\partial q_j}$$ **TM eq. (7.66)**

Symon problema 9.12, pg. 435
Kibble & Berkshire problema 10.12, pg. 250
Forças de vínculo em haste apoiada



Uma haste uniforme de comprimento 2ℓ e massa m é apoiada entre duas superfícies lisas, uma horizontal (piso) e outra vertical (parede). Ao ser largada, ela desliza sem atrito, mantendo-se sempre num mesmo plano vertical.

a) Obtenha a lagrangiana do sistema, usando como coordenadas generalizadas a posição do CM da haste e o ângulo θ que ela forma com a parede (ver figura).

b) Obtenha as três equações de Lagrange do sistema com os vínculos e multiplicadores de Lagrange apropriados.

c) Usando $\theta(0) = \dot{\theta}(0) = 0$ como condições iniciais, obtenha as forças de reação da parede e do piso em função de θ.

a) $\mathscr{L} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I^{\text{CM}}_{\text{haste}}\dot{\theta}^2 - mgy, \quad (x,y) \equiv (X,Y)_{\text{haste}}$

$$I^{\text{CM}}_{\text{haste}} = \frac{1}{12}m\,(2\ell)^2 = \frac{1}{3}m\ell^2$$

b) $f_1(x,\theta) = x - \ell\,\text{sen}\,\theta = 0, \quad f_2(y,\theta) = y - \ell\cos\theta = 0$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathscr{L}}{\partial\dot{x}}\right) = m\ddot{x} = \frac{\partial\mathscr{L}}{\partial x} + \lambda_1\frac{\partial f_1}{\partial x} = \lambda_1$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathscr{L}}{\partial\dot{y}}\right) = m\ddot{y} = \frac{\partial\mathscr{L}}{\partial y} + \lambda_2\frac{\partial f_2}{\partial y} = \lambda_2 - mg$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathscr{L}}{\partial\dot{\theta}}\right) = I^{\text{CM}}_{\text{haste}}\ddot{\theta} = \frac{\partial\mathscr{L}}{\partial\theta} + \lambda_1\frac{\partial f_1}{\partial\theta} + \lambda_2\frac{\partial f_2}{\partial\theta}$$

$$= -\lambda_1\,\ell\cos\theta + \lambda_2\,\ell\,\text{sen}\,\theta$$

c) $x = \ell\,\text{sen}\,\theta, \quad y = \ell\cos\theta, \quad \dot{x} = \ell\dot{\theta}\cos\theta, \quad \dot{y} = -\ell\dot{\theta}\,\text{sen}\,\theta$

$$\ddot{x} = \ell\ddot{\theta}\cos\theta - \ell\dot{\theta}^2\,\text{sen}\,\theta, \quad \ddot{y} = -\ell\ddot{\theta}\,\text{sen}\,\theta - \ell\dot{\theta}^2\cos\theta$$

$$\lambda_1 = m\ddot{x} = m\ell\ddot{\theta}\cos\theta - m\ell\dot{\theta}^2\,\text{sen}\,\theta$$

$$\lambda_2 = m(g+\ddot{y}) = mg - m\ell\ddot{\theta}\,\mathrm{sen}\,\theta - m\ell\dot{\theta}^2\cos\theta$$

$$I^{\mathrm{CM}}_{\mathrm{haste}}\ddot{\theta} = -\lambda_1\,\ell\cos\theta + \lambda_2\,\ell\,\mathrm{sen}\,\theta \;\rightarrow\; \ddot{\theta} = \frac{3g}{4\ell}\,\mathrm{sen}\,\theta$$

$$E = \tfrac{1}{2}m(\dot{x}^2+\dot{y}^2) + \tfrac{1}{2}I^{\mathrm{CM}}_{\mathrm{haste}}\dot{\theta}^2 + mgy = mg\ell$$

$$= \tfrac{1}{2}m\ell^2\dot{\theta}^2 + \tfrac{1}{6}m\ell^2\dot{\theta}^2 + mg\ell\cos\theta \;\rightarrow\; \dot{\theta}^2 = \frac{3g}{2\ell}(1-\cos\theta)$$

Substituindo em λ_1 e λ_2, obtemos as forças de reação:

$Q_x = \lambda_1 = \tfrac{3}{2}mg\,\mathrm{sen}\,\theta\left(\tfrac{3}{2}\cos\theta - 1\right)$: reação da parede

$Q_y = \lambda_2 = \tfrac{1}{4}mg(1-3\cos\theta)^2$: reação do piso

$Q_\theta = -\lambda_1\,\ell\cos\theta + \lambda_2\,\ell\,\mathrm{sen}\,\theta = \tfrac{1}{4}mg\ell\,\mathrm{sen}\,\theta = \tau_z$: torques

$$\boldsymbol{\tau} = \boldsymbol{r}_x \times Q_x\hat{\boldsymbol{x}} + \boldsymbol{r}_y \times Q_y\hat{\boldsymbol{y}} = \tau_z\hat{\boldsymbol{z}}, \quad \boldsymbol{r}_x \equiv (-x, y), \quad \boldsymbol{r}_y \equiv (x, -y)$$

A reação da parede se anula para $\cos\theta_x = \frac{2}{3}$, enquanto a reação do piso se anula para $\cos\theta_y = \frac{1}{3}$.