F-315 (Mecânica Geral I)
Aula 25

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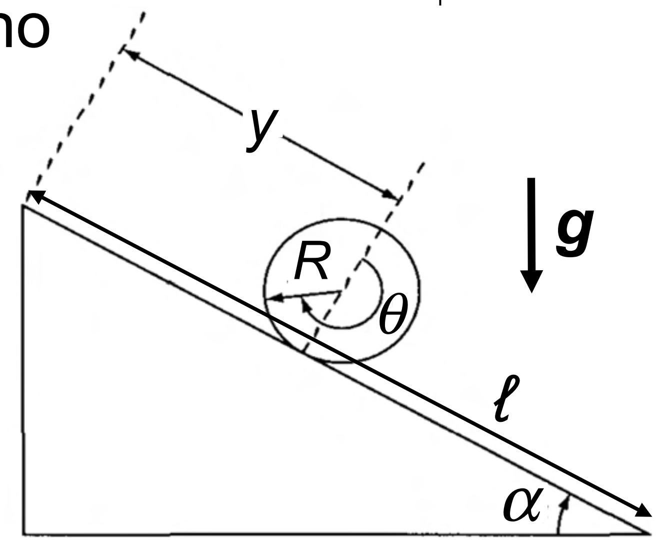
http://www.ifi.unicamp.br/~vidiella/aulas.html

Mecânica Lagrangiana

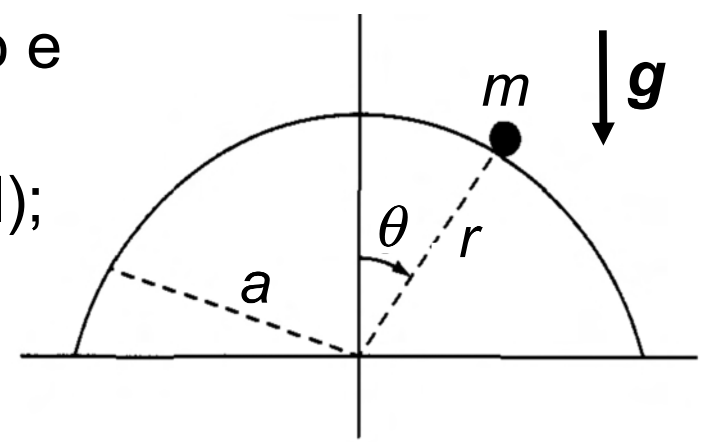
TM exemplo 7.9, pgs. 250-252

Considere um disco rolando em um plano inclinado, sem deslizar: a) encontrar as equações de movimento; b) calcular as forças generalizadas.



TM exemplo 7.10, pgs. 252-254

Uma partícula de massa *m* parte do repouso do topo de um hemisfério fixo e liso de raio *a*, deslizando sobre ele. Calcular: a) a força de vínculo (normal); b) o ângulo em que a partícula perde contato com o hemisfério.



TM exemplo 7.9

$$\mathscr{L} = \tfrac{1}{2}M\dot{y}^2 + \tfrac{1}{2}I\dot{\theta}^2 - Mg(\ell - y)\,\mathrm{sen}\,\alpha,\;\; f(y,\theta) = y - R\theta = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathscr{L}}{\partial\dot{y}}\right) = M\ddot{y} = \frac{\partial\mathscr{L}}{\partial y} + \lambda\frac{\partial f}{\partial y} = Mg\,\mathrm{sen}\,\alpha + \lambda$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathscr{L}}{\partial\dot{\theta}}\right) = I\ddot{\theta} = \tfrac{1}{2}MR^2\ddot{\theta} = \frac{\partial\mathscr{L}}{\partial\theta} + \lambda\frac{\partial f}{\partial\theta} = -\lambda R$$

$$\to\quad \ddot{y} = \tfrac{2}{3}g\,\mathrm{sen}\,\alpha,\;\; \ddot{\theta} = \tfrac{2}{3R}g\,\mathrm{sen}\,\alpha,\;\; Q_y = \lambda = -\tfrac{1}{3}Mg\,\mathrm{sen}\,\alpha$$

$$Q_\theta = -\lambda R = \tfrac{1}{3}MgR\,\mathrm{sen}\,\alpha$$

TM exemplo 7.10

$$\mathscr{L} = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - mgr\cos\theta,\;\; f(r,\theta) = r - a = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathscr{L}}{\partial\dot{r}}\right) = m\ddot{r} = \frac{\partial\mathscr{L}}{\partial r} + \lambda\frac{\partial f}{\partial r} = mr\dot{\theta}^2 - mg\cos\theta + \lambda$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathscr{L}}{\partial\dot{\theta}}\right) = m(r^2\ddot{\theta} + 2r\dot{r}\dot{\theta}) = \frac{\partial\mathscr{L}}{\partial\theta} + \lambda\frac{\partial f}{\partial\theta} = mgr\,\mathrm{sen}\,\theta$$

TM exemplo 7.10, pgs. 252–254

Usando o vínculo $r = a \rightarrow \dot{r} = 0 \rightarrow \ddot{r} = 0$ enquanto a partícula encontra-se em contato com o hemisfério:

$$\ddot{\theta} = \frac{g}{a}\,\mathrm{sen}\,\theta, \qquad \lambda = mg\cos\theta - ma\dot{\theta}^2$$

Usando a conservação de $E = T + V$ e a condição inicial $\dot{\theta} = 0$ para $\theta = 0$:

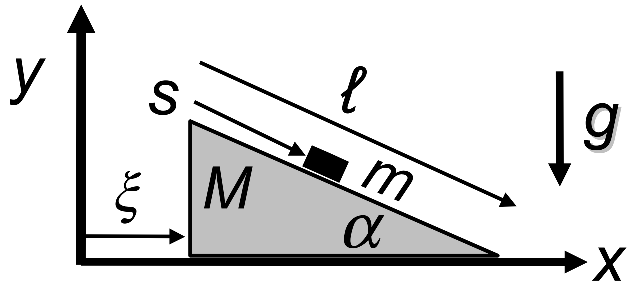
$$E = \frac{1}{2}ma^2\dot{\theta}^2 + mga\cos\theta = mga \;\rightarrow\; \dot{\theta}^2 = \frac{2g}{a}(1-\cos\theta)$$

Substituindo em $\lambda = mg\cos\theta - ma\dot{\theta}^2$:

$$\lambda = mg\cos\theta - 2mg(1-\cos\theta) = mg\,(3\cos\theta - 2)$$

associada à força normal radial $F_r = Q_r = \lambda\frac{\partial f}{\partial r}$, que se anula para $\cos\theta = \frac{2}{3}$.

TM problema 7.6, pg. 280 (modificado)
Kibble & Berkshire problema 3.14, pg. 69



Considere um sistema composto de uma cunha de inclinação α e massa *M* colocada sobre um plano horizontal, sobre a qual há um pequeno bloco de massa *m*, conforme mostra a figura. Não há atrito entre as superfícies.

a) Obtenha a lagrangiana do sistema $\mathscr{L}(\xi, s, \dot{\xi}, \dot{s})$, bem como a aceleração da cunha $\ddot{\xi}$ e do bloco em relação à cunha $\ddot{s}$.

b) Reescreva as equações de Lagrange do sistema com os vínculos e multiplicadores de Lagrange apropriados.

c) Obtenha as forças de vínculo generalizadas $\{Q_j\}$ e o módulo da força de reação normal $|\boldsymbol{N}|$ da cunha sobre o bloco.

TM problema 7.6 (modificado), pg. 280

a) $\mathscr{L} = \frac{1}{2}M\dot{\xi}^2 + \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy, \quad x = \xi + s\cos\alpha$

$$\dot{x} = \dot{\xi} + \dot{s}\cos\alpha, \quad y = (\ell - s)\,\mathrm{sen}\,\alpha, \quad \dot{y} = -\dot{s}\,\mathrm{sen}\,\alpha$$

$$\mathscr{L} = \frac{1}{2}M\dot{\xi}^2 + \frac{1}{2}m(\dot{\xi}^2 + \dot{s}^2 + 2\dot{\xi}\dot{s}\cos\alpha) - mg(\ell - s)\,\mathrm{sen}\,\alpha$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathscr{L}}{\partial\dot{\xi}}\right) = (M + m)\ddot{\xi} + m\ddot{s}\cos\alpha = \frac{\partial\mathscr{L}}{\partial\xi} = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathscr{L}}{\partial\dot{s}}\right) = m(\ddot{s} + \ddot{\xi}\cos\alpha) = \frac{\partial\mathscr{L}}{\partial s} = mg\,\mathrm{sen}\,\alpha$$

$$\rightarrow \quad \ddot{\xi} = -\frac{mg\,\mathrm{sen}\,\alpha\,\cos\alpha}{M + m\,\mathrm{sen}^2\,\alpha}, \qquad \ddot{s} = \frac{(M+m)g\,\mathrm{sen}\,\alpha}{M + m\,\mathrm{sen}^2\,\alpha}$$

b) $\mathscr{L} = \frac{1}{2}M\dot{\xi}^2 + \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy$

$$f_1(x, \xi, s) = x - \xi - s\cos\alpha = 0$$

$$f_2(y, s) = y - (\ell - s)\,\mathrm{sen}\,\alpha = 0$$

TM problema 7.6 (modificado), pg. 280

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathscr{L}}{\partial\dot{x}}\right) = m\ddot{x} = \frac{\partial\mathscr{L}}{\partial x} + \lambda_1\frac{\partial f_1}{\partial x} = \lambda_1$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathscr{L}}{\partial\dot{y}}\right) = m\ddot{y} = \frac{\partial\mathscr{L}}{\partial y} + \lambda_2\frac{\partial f_2}{\partial y} = \lambda_2 - mg$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathscr{L}}{\partial\dot{\xi}}\right) = M\ddot{\xi} = \frac{\partial\mathscr{L}}{\partial\xi} + \lambda_1\frac{\partial f_1}{\partial\xi} = -\lambda_1$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathscr{L}}{\partial\dot{s}}\right) = 0 = \frac{\partial\mathscr{L}}{\partial s} + \lambda_1\frac{\partial f_1}{\partial s} + \lambda_2\frac{\partial f_2}{\partial s} = -\lambda_1\cos\alpha + \lambda_2\operatorname{sen}\alpha$$

Resolvendo, obtemos $\ddot{\xi}$ e $\ddot{s}$ do item a, além dos $\{\lambda_k\}$:

$$\lambda_1 = m\ddot{x} = m(\ddot{\xi} + \ddot{s}\cos\alpha) = \frac{Mmg\operatorname{sen}\alpha\cos\alpha}{M+m\operatorname{sen}^2\alpha}$$

$$\lambda_2 = mg + m\ddot{y} = mg - m\ddot{s}\operatorname{sen}\alpha = \frac{Mmg\cos^2\alpha}{M+m\operatorname{sen}^2\alpha}$$

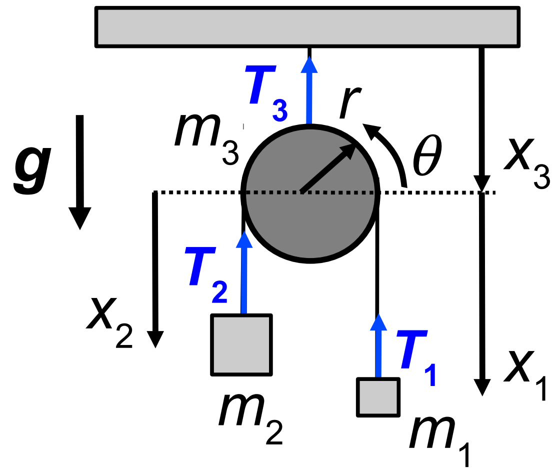
c) $N_x = Q_x = \lambda_1, \quad N_y = Q_y = \lambda_2, \quad Q_\xi = -\lambda_1, \quad Q_s = 0$

$$\boldsymbol{N} = N_x\hat{\boldsymbol{x}} + N_y\hat{\boldsymbol{y}}, \quad |\boldsymbol{N}| = \sqrt{N_x^2 + N_y^2} = \frac{Mmg\cos\alpha}{M+m\operatorname{sen}^2\alpha}$$

Máquina de Atwood

TM exemplo 2.9 (modificado), pgs. 71-72

Taylor problema 7.17, pg. 284



Considere a máquina de Atwood representada na figura acima, composta de dois blocos de massas $m_1=m$ e $m_2=\alpha m$ conectados por um fio inextensível de comprimento ℓ e massa desprezível, que passa através de uma polia ideal circular de raio r, massa $m_3=2\beta m$, cuja posição x_3 é fixa.

a) Obtenha a lagrangiana do sistema $\mathscr{L}(\{x_i\},\{\dot{x}_i\},\theta,\dot{\theta})$ e as equações dos vínculos $f_k(\{x_i\},\theta)$.

b) Obtenha as equações de Euler-Lagrange e as forças generalizadas de vínculo $\{Q_k\}$ pelo método de multiplicadores de Lagrange, relacionando-as com as reações $\{T_k\}$.

TM exemplo 2.9 (modificado), pgs. 71–72

a) $\mathscr{L} = \frac{1}{2}m\left[(\dot{x}_1+\dot{x}_3)^2 + \alpha(\dot{x}_2+\dot{x}_3)^2 + 2\beta\dot{x}_3^2\right] + \frac{1}{2}\beta m r^2\dot{\theta}^2$
$\quad + mgx_1 + \alpha mgx_2 + Mgx_3, \quad M \equiv (1+\alpha+2\beta)m$

$f_1(x_1,x_2) = x_1 + x_2 + \pi r - \ell = 0 \;\rightarrow\; \dot{x}_1 = -\dot{x}_2$

$f_2(x_1,\theta) = x_1 + r\theta - x_1^0 = 0 \;\rightarrow\; \dot{x}_1 = -r\dot{\theta}$

$f_3(x_3) = x_3 - x_3^0 = 0 \;\rightarrow\; \dot{x}_3 = 0$

b) $p_1 = \frac{\partial\mathscr{L}}{\partial\dot{x}_1} = m(\dot{x}_1+\dot{x}_3),\; \dot{p}_1 = \frac{\partial\mathscr{L}}{\partial x_1} + \sum_{k=1}^{3}\lambda_k\frac{\partial f_k}{\partial x_1} = mg + \lambda_1 + \lambda_2$

$p_2 = \frac{\partial\mathscr{L}}{\partial\dot{x}_2} = \alpha m(\dot{x}_2+\dot{x}_3),\; \dot{p}_2 = \frac{\partial\mathscr{L}}{\partial x_2} + \lambda_1\frac{\partial f_1}{\partial x_2} = \alpha mg + \lambda_1$

$p_3 = \frac{\partial\mathscr{L}}{\partial\dot{x}_3} = m(\dot{x}_1+\alpha\dot{x}_2) + M\dot{x}_3,\; \dot{p}_3 = \frac{\partial\mathscr{L}}{\partial x_3} + \lambda_3\frac{\partial f_3}{\partial x_3} = Mg + \lambda_3$

$p_\theta = \frac{\partial\mathscr{L}}{\partial\dot{\theta}} = \beta m r^2\dot{\theta},\; \dot{p}_\theta = \frac{\partial\mathscr{L}}{\partial\theta} + \lambda_2\frac{\partial f_2}{\partial\theta} = r\lambda_2$

TM exemplo 2.9 (modificado), pgs. 71–72

Resolvendo para $\{\ddot{x}_k, \ddot{\theta}, \lambda_k\}$:

$$\ddot{x}_1 = -\ddot{x}_2 = \frac{1-\alpha}{1+\alpha+\beta} g, \ \ddot{\theta} = -\frac{\ddot{x}_1}{r} = \frac{\alpha-1}{1+\alpha+\beta}\frac{g}{r}, \ \lambda_1 = -\frac{(2+\beta)\alpha}{1+\alpha+\beta} mg$$

$$\lambda_2 = \frac{(\alpha-1)\beta}{1+\alpha+\beta} mg, \ \lambda_3 = -\frac{4\alpha+3\beta+3\alpha\beta+2\beta^2}{1+\alpha+\beta} mg$$

Forças generalizadas $\{Q_k\}$:

$$Q_1 = \lambda_1 \frac{\partial f_1}{\partial x_1} + \lambda_2 \frac{\partial f_2}{\partial x_1} = \lambda_1 + \lambda_2 = -\frac{2\alpha+\beta}{1+\alpha+\beta} mg = T_1$$

$$Q_2 = \lambda_1 \frac{\partial f_1}{\partial x_2} = \lambda_1 = -\frac{(2+\beta)\alpha}{1+\alpha+\beta} mg = T_2$$

$$Q_3 = \lambda_3 \frac{\partial f_3}{\partial x_3} = \lambda_3 = -\frac{4\alpha+3\beta+3\alpha\beta+2\beta^2}{1+\alpha+\beta} mg = T_3$$

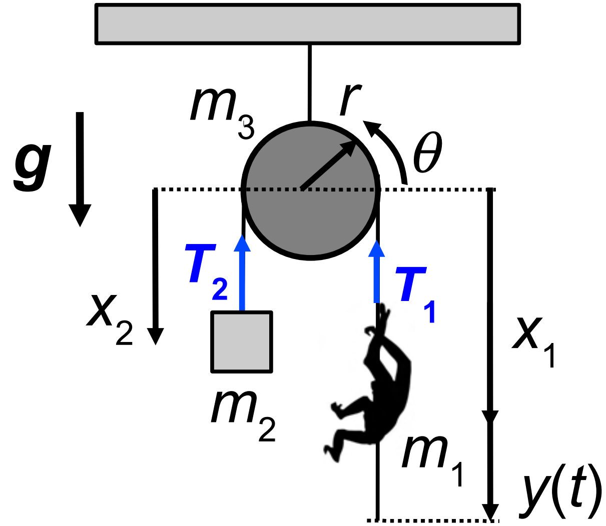
$$Q_\theta = \lambda_2 \frac{\partial f_2}{\partial \theta} = r\lambda_2 = \frac{(\alpha-1)\beta}{1+\alpha+\beta} mgr = -r(T_2 - T_1) = \tau_z$$

Assim, para $\{x_1, x_2, x_3\}$, $\boldsymbol{Q} = \boldsymbol{T}$, enquanto $Q_\theta = \tau_z$.

Máquina de Atwood

TM exemplo 2.9 (modificado), pgs. 71-72

Taylor problema 7.17, pg. 284



Considere a mesma máquina de Atwood do problema anterior, substituindo a massa $m_1=m$ por um macaco que sobe, relativamente ao fio, a uma velocidade ***conhecida*** $-\dot{y}(t)$.

a) Obtenha a lagrangiana do sistema $\mathscr{L}(x_1,\dot{x}_1;y)$ já eliminando os vínculos e a equação de Euler-Lagrange para $x_1(t)$. Resolva-a com as condições iniciais $y(t=0)=y_0$ e $\dot{y}(t=0)=\dot{y}_0$.

b) Obtenha as tensões no fio T_1 *e* T_2 através das forças generalizadas de vínculo $\{Q_k\}$ pelo método de multiplicadores de Lagrange. Note que a coordenada x_3 é irrelevante.

TM exemplo 2.9 (modificado c/macaco), pgs. 71–72

a) $\mathscr{L} = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m(\alpha+\beta)(\dot{x}_1+\dot{y})^2 + mgx_1$
$+ \alpha mg(\ell - y - \pi r - x_1)$

$\dot{p}_1 = m\ddot{x}_1 + m(\alpha+\beta)(\ddot{x}_1+\ddot{y}) = (1-\alpha)mg \;\rightarrow$

$x_1(t) = x_1^0 + \dot{x}_1^0 t + \left(\frac{1-\alpha}{1+\alpha+\beta}\right)\frac{gt^2}{2} - \frac{\beta}{1+\alpha+\beta}[y(t) - y_0 - \dot{y}_0 t]$

b) $\mathscr{L} = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\alpha\dot{x}_2^2 + \frac{1}{2}\beta mr^2\dot{\theta}^2 + mgx_1 + \alpha mgx_2$

$f_1 = x_1 + x_2 + \pi r + y(t) - \ell, \;\; f_2 = x_2 - r\theta - x_2^0$

$\dot{p}_1 = m\ddot{x}_1 = mg + \lambda_1, \;\; \dot{p}_2 = \alpha m\ddot{x}_2 = \alpha mg + \lambda_1 + \lambda_2$

$\dot{p}_\theta = \beta mr^2\ddot{\theta} = -r\lambda_2 \;\rightarrow\; \ddot{x}_1 = \frac{1-\alpha}{1+\alpha+\beta}g - \frac{\beta}{1+\alpha+\beta}\ddot{y}$

$T_1 = Q_1 = \lambda_1 = m\ddot{x}_1 - mg = -\frac{2\alpha+\beta}{1+\alpha+\beta}mg - \frac{\beta}{1+\alpha+\beta}m\ddot{y}$

$T_2 = Q_2 = \lambda_1 + \lambda_2 = \alpha m(\ddot{x}_2 - g) = -\frac{(2+\beta)\alpha}{1+\alpha+\beta}mg - \frac{(1+\alpha)\alpha}{1+\alpha+\beta}m\ddot{y}$