F-315 (Mecânica Geral I)
Aula 26

Prof. Mário Noboru Tamashiro
Departamento de Física Aplicada, prédio A-5, sala 7

ramal 3521-5339
e-mail: mtamash@ifi.unicamp.br
http://www.ifi.unicamp.br/~mtamash/f315_mecgeral_i

Slides do prof. Antonio Vidiella Barranco:
http://www.ifi.unicamp.br/~vidiella/aulas.html

Teorema envolvendo a energia cinética $T = \frac{1}{2}\sum_\alpha m_\alpha \dot{\boldsymbol{r}}_\alpha^2$

$$\dot{\boldsymbol{r}}_\alpha = \frac{\partial \boldsymbol{r}_\alpha}{\partial t} + \sum_j \frac{\partial \boldsymbol{r}_\alpha}{\partial q_j}\dot{q}_j, \quad \dot{\boldsymbol{r}}_\alpha^2 = \left(\frac{\partial \boldsymbol{r}_\alpha}{\partial t}\right)^2 + 2\sum_j \frac{\partial \boldsymbol{r}_\alpha}{\partial t}\cdot\frac{\partial \boldsymbol{r}_\alpha}{\partial q_j}\dot{q}_j + \sum_{i,j}\left(\frac{\partial \boldsymbol{r}_\alpha}{\partial q_i}\dot{q}_i\right)\cdot\left(\frac{\partial \boldsymbol{r}_\alpha}{\partial q_j}\dot{q}_j\right)$$

$$T = \frac{1}{2}\sum_\alpha m_\alpha\left(\frac{\partial \boldsymbol{r}_\alpha}{\partial t}\right)^2 + \sum_j \dot{q}_j \underbrace{\sum_\alpha m_\alpha \frac{\partial \boldsymbol{r}_\alpha}{\partial t}\cdot\frac{\partial \boldsymbol{r}_\alpha}{\partial q_j}}_{b_j} + \sum_{i,j}\dot{q}_i\dot{q}_j \underbrace{\sum_\alpha \frac{1}{2} m_\alpha \left(\frac{\partial \boldsymbol{r}_\alpha}{\partial q_i}\right)\cdot\left(\frac{\partial \boldsymbol{r}_\alpha}{\partial q_j}\right)}_{a_{ij} = a_{ji}}$$

Sistemas / vínculos escleronômos: $\boldsymbol{r}_\alpha = \boldsymbol{r}_\alpha(\boldsymbol{q}) \;\rightarrow\; \partial \boldsymbol{r}_\alpha(\boldsymbol{q})/\partial t = 0$

T é uma *função homogênea quadrática* das velocidades generalizadas $\{\dot{q}_j\}$

TM eq. (7.121); S eq. (9.9)

$$T = \sum_{i,j} a_{ij}\dot{q}_i\dot{q}_j$$

$$\frac{\partial T}{\partial \dot{q}_j} = \sum_{i,k} a_{ik}\frac{\partial(\dot{q}_i\dot{q}_k)}{\partial \dot{q}_j} = 2\sum_i a_{ij}\dot{q}_i \;\rightarrow$$

TM eq. (7.122); S eq. (9.122)

$$\sum_j \frac{\partial T}{\partial \dot{q}_j}\dot{q}_j = 2\sum_j\Big(\sum_i a_{ij}\dot{q}_i\Big)\dot{q}_j = 2T$$

Conservação de energia: invariância temporal

$$\dot{\mathscr{L}} = \frac{\partial \mathscr{L}}{\partial t} + \sum_j \left(\frac{\partial \mathscr{L}}{\partial q_j} \dot{q}_j + \frac{\partial \mathscr{L}}{\partial \dot{q}_j} \ddot{q}_j \right) = \frac{\partial \mathscr{L}}{\partial t} + \sum_j \left[\dot{q}_j \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial \mathscr{L}}{\partial \dot{q}_j} \right) + \frac{\partial \mathscr{L}}{\partial \dot{q}_j} \ddot{q}_j \right]$$

$$= \frac{\partial \mathscr{L}}{\partial t} + \sum_j \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial \mathscr{L}}{\partial \dot{q}_j} \dot{q}_j \right). \quad \text{Se } \frac{\partial \mathscr{L}}{\partial t} = 0, \text{ então}$$

Função hamiltoniana $\mathscr{H}$ é uma constante, $\dot{\mathscr{H}} = 0$

TM eq. (7.128)

$$\mathscr{L} - \sum_j \frac{\partial \mathscr{L}}{\partial \dot{q}_j} \dot{q}_j \equiv -\mathscr{H}$$

Para $V = V(\boldsymbol{q})$, isto é, $\frac{\partial V}{\partial \dot{q}_j} = 0 \rightarrow \frac{\partial \mathscr{L}}{\partial \dot{q}_j} = \frac{\partial (T-V)}{\partial \dot{q}_j} = \frac{\partial T}{\partial \dot{q}_j}$:

$$-\mathscr{H} = T - V - \underbrace{\sum_j \frac{\partial T}{\partial \dot{q}_j} \dot{q}_j}_{2T} = -(T+V) = -E$$

$2T$ (TM seção 7.8): vínculos $\boldsymbol{q}(\boldsymbol{x})$ devem ser esclerônomos, $\partial \boldsymbol{q}(\boldsymbol{x})/\partial t = 0$

Para que $\mathscr{H} = E$: (a) vínculos $\boldsymbol{q}(\boldsymbol{x})$ devem ser escle-rônomos, $\frac{\partial \boldsymbol{q}(\boldsymbol{x})}{\partial t} = 0$; (b) $\frac{\partial V}{\partial \dot{q}_j} = 0$, não vale na presença de fricção/atrito e para a força de Lorentz $\boldsymbol{F} = q\dot{\boldsymbol{r}} \times \boldsymbol{B}$.

Condições para conservação da energia ($\dot{E} = 0$)

$$\mathscr{L} \equiv T - V\,, \quad E = T + V = 2T - \mathscr{L}\,, \quad \dot{E} = 2\dot{T} - \dot{\mathscr{L}}$$

$$\dot{E} = 2\dot{T} - \sum_j \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathscr{L}}{\partial \dot{q}_j}\dot{q}_j\right) - \frac{\partial \mathscr{L}}{\partial t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(2T - \sum_j \frac{\partial \mathscr{L}}{\partial \dot{q}_j}\dot{q}_j\right) - \frac{\partial \mathscr{L}}{\partial t}$$

Energia E se conserva, $\dot{E} = 0$, se:

$$2T = \sum_j \frac{\partial \mathscr{L}}{\partial \dot{q}_j}\dot{q}_j + \text{constante}\,, \qquad \frac{\partial \mathscr{L}}{\partial t} = 0$$

Conservação do momento linear: invariância translacional

Se $\frac{\partial \mathscr{L}}{\partial x_j} = 0 \;\rightarrow\; x_j$: coordenada cíclica ou ignorável

Momento (linear) conjugado $p_j \equiv \frac{\partial \mathscr{L}}{\partial \dot{x}_j}$ se conserva (é uma constante do movimento): $\dot{p}_j = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathscr{L}}{\partial \dot{x}_j}\right) = \frac{\partial \mathscr{L}}{\partial x_j} = 0$

Exemplo: massa m em campo gravitacional $\boldsymbol{g} = -g\hat{\boldsymbol{z}}$

$\mathscr{L} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz$, (x,y): coordenadas cíclicas

invariância translacional em $(x,y) \;\rightarrow\; \dot{p}_x = \dot{p}_y = 0$

z não é uma coordenada cíclica, p_z não se conserva:

$$\dot{p}_z = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathscr{L}}{\partial \dot{z}}\right) = \frac{\partial \mathscr{L}}{\partial z} = -mg \neq 0$$

Conservação do momento angular: invariância rotacional

Se q_j: coordenada angular cíclica → momento angular conjugado p_j se conserva

Exemplo: massa m em potencial de simetria cilíndrica

$$\mathscr{L} = \frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\varphi}^2 + \dot{z}^2) - V(\rho), \quad \rho \equiv \sqrt{x^2 + y^2}, \quad \operatorname{tg}\varphi \equiv y/x$$

(φ, z) são coordenadas cíclicas (rotações em φ e translações em z são invariantes)

$$L_z = p_\varphi \equiv \frac{\partial \mathscr{L}}{\partial \dot{\varphi}} = m\rho^2\dot{\varphi} \text{ se conserva } \rightarrow \dot{L}_z = 0$$

$$p_z \equiv \frac{\partial \mathscr{L}}{\partial \dot{z}} = m\dot{z} \text{ se conserva } \rightarrow \dot{p}_z = 0$$

Exemplo: massa m em potencial de simetria azimutal

$$\mathscr{L} = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \operatorname{sen}^2\theta\,\dot{\varphi}^2) - V(r,\theta),\ \varphi \text{ é cíclica}$$

$$r \equiv \sqrt{x^2+y^2+z^2},\quad \cos\theta \equiv z\Big/\sqrt{x^2+y^2+z^2},\quad \operatorname{tg}\varphi \equiv y/x$$

$$L_z = p_\varphi \equiv \frac{\partial\mathscr{L}}{\partial\dot{\varphi}} = mr^2\operatorname{sen}^2\theta\,\dot{\varphi} \text{ se conserva } \rightarrow\ \dot{L}_z = 0$$

Exemplo: massa m em potencial $V(r) = -k/r$

$$\mathscr{L} = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \operatorname{sen}^2\theta\,\dot{\varphi}^2) + k/r,\ \varphi \text{ cíclica} \rightarrow \dot{L}_z = 0$$

$$\boldsymbol{L}^2 \equiv p_\theta^2 + \frac{p_\varphi^2}{\operatorname{sen}^2\theta} = m^2r^4(\dot{\theta}^2 + \operatorname{sen}^2\theta\,\dot{\varphi}^2)\ \text{ se conserva}$$

para *qualquer* potencial central $V(r) = V(|\boldsymbol{r}|)\ \rightarrow\ \dot{\boldsymbol{L}}^2 = 0$

Vetor de Laplace–Runge–Lenz (G seção 3.9, pgs. 102–106): $\boldsymbol{\mathcal{A}} \equiv \boldsymbol{p}\times\boldsymbol{L} - \frac{mk}{r}\boldsymbol{r}$ se conserva $\rightarrow\ \dot{\boldsymbol{\mathcal{A}}} = \boldsymbol{0}$

Mecânica Lagrangiana

TM seção 7.9

Energia mecânica:

Homogeneidade do tempo $\Rightarrow$ $$\frac{\partial L}{\partial t} = 0$$

$$\frac{dL}{dt} = \sum_j \left(\frac{\partial L}{\partial q_j}\dot{q}_j + \frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j \right)$$

$$L - \sum_j \frac{\partial L}{\partial \dot{q}_j}\dot{q}_j = cte \equiv -H$$

TM eq. (7.128)

Função Hamiltoniana

Mecânica Lagrangiana

TM seção 7.9

Condições: $\dfrac{\partial x_i}{\partial t} = 0 \quad \Rightarrow \quad T = \sum_{j,k} a_{jk}\dot{q}_j \dot{q}_k$

$$\frac{\partial U}{\partial \dot{q}_j} = 0 \quad \Rightarrow \quad \frac{\partial L}{\partial \dot{q}_j} = \frac{\partial T}{\partial \dot{q}_j}$$

E obtemos

$$H = T + U = E$$

Energia mecânica conservada!

Mecânica Lagrangiana

TM seção 7.9

Leis de Conservação:

Homogeneidade do tempo	➡	Conservação da Energia Mecânica
Homogeneidade do espaço	➡	Conservação do Momento Linear
Isotropia do espaço	➡	Conservação do Momento Angular

Tomando a forma diferencial da função hamiltoniana $\mathscr{H}$ como uma transformada de Legendre de $-\mathscr{L}$:

$$\mathscr{H} \equiv -\mathscr{L} + \sum_j p_j \dot{q}_j, \quad p_j \equiv \frac{\partial \mathscr{L}}{\partial \dot{q}_j}, \quad \dot{p}_j = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathscr{L}}{\partial \dot{q}_j}\right) = \frac{\partial \mathscr{L}}{\partial q_j}$$

$$\mathrm{d}\mathscr{H} = -\frac{\partial \mathscr{L}}{\partial t}\,\mathrm{d}t - \sum_j \Big(\underbrace{\frac{\partial \mathscr{L}}{\partial q_j}}_{\dot{p}_j}\mathrm{d}q_j + \underbrace{\frac{\partial \mathscr{L}}{\partial \dot{q}_j}}_{p_j}\mathrm{d}\dot{q}_j\Big) + \sum_j \left(\dot{q}_j\,\mathrm{d}p_j + p_j\,\mathrm{d}\dot{q}_j\right)$$

Por outro lado, como a hamiltoniana $\mathscr{H} = \mathscr{H}(p_j, q_j, t)$

$\mathrm{d}\mathscr{H} = \frac{\partial \mathscr{H}}{\partial t}\,\mathrm{d}t + \sum_j \left(\frac{\partial \mathscr{H}}{\partial p_j}\mathrm{d}p_j + \frac{\partial \mathscr{H}}{\partial q_j}\mathrm{d}q_j\right)$. Comparando-as:

Equações de Hamilton: TM eqs. (7.160)–(7.163)

$$\dot{q}_j = \frac{\partial \mathscr{H}}{\partial p_j}, \quad -\dot{p}_j = \frac{\partial \mathscr{H}}{\partial q_j}, \quad -\frac{\partial \mathscr{L}}{\partial t} = \frac{\partial \mathscr{H}}{\partial t} = \dot{\mathscr{H}}$$

Equações de Hamilton

TM seção 7.10

Momentos generalizados

$$p_j \equiv \frac{\partial L}{\partial \dot{q}_j}$$ **TM eq. (7.151)**

Usando as equações de Lagrange, obtemos

$$\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = 0 \quad \Rightarrow \quad \dot{p}_j = \frac{\partial L}{\partial q_j}$$

TM eq. (7.152)

Equações de Hamilton

TM seção 7.10

A função Hamiltoniana pode ser escrita como

$$H(q_k, p_k, t) = \sum_j p_j \dot{q}_j - L(q_k, p_k, t)$$

TM eq. (7.155)

Equações de Hamilton

$$\boxed{\begin{aligned} \frac{\partial H}{\partial p_k} &= \dot{q}_k \\ \frac{\partial H}{\partial q_k} &= -\dot{p}_k \end{aligned}}$$

TM eqs. (7.160)

(7.161)