# F-315 (Mecânica Geral I)
# Aula 9

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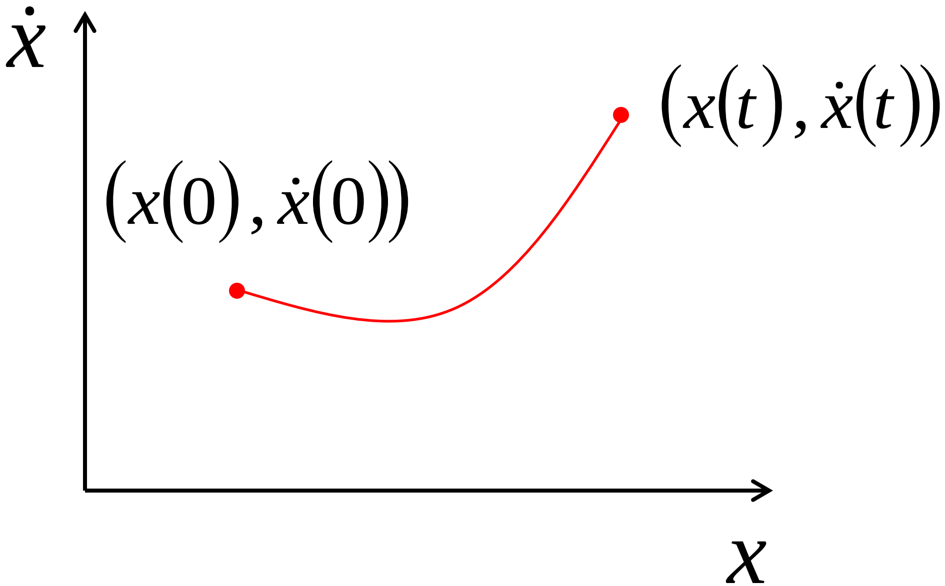
http://www.ifi.unicamp.br/~vidiella/aulas.html

# Retratos de Fase

Espaço de Fase: espaço bidimensional cujos pontos tem coordenadas $x(t)$ e $\dot{x}(t)$ Obs: 1D



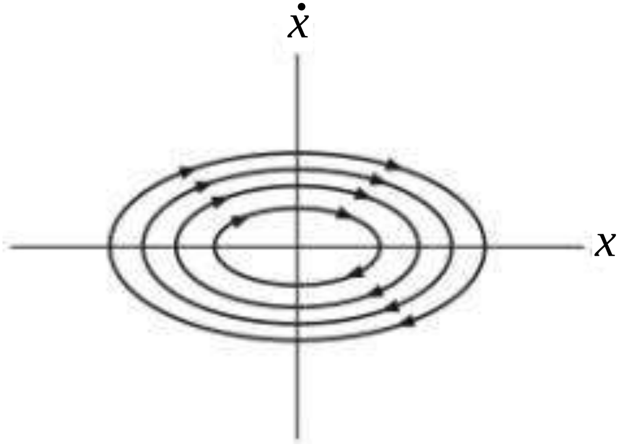

Conjunto de possíveis trajetórias:
Retrato de fase

# Oscilador harmônico simples

$$x(t) = x_m \cos(\omega_0 t + \theta)$$
$$\dot{x}(t) = -x_m \omega_0 \sin(\omega_0 t + \theta)$$

$$\frac{x^2}{x_m^2} + \frac{\dot{x}^2}{x_m^2 \omega_0^2} = 1$$

Família de elipses



# Classification of Harmonic Motion for Mass Spring Systems

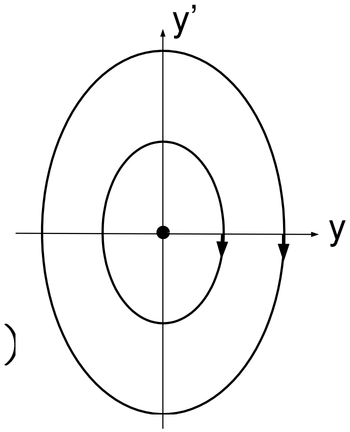
| | | | |
|---|---|---|---|
| $y'' + \frac{\mu}{m}y' + \frac{k}{m}y = 0$ | $\lambda_{1,2} = -\frac{\mu}{2m} \pm \frac{1}{2m}\sqrt{\mu^2 - 4km}$ | $m, k > 0$ | $\mu \geq 0$ |

**Undamped Case:** $\mu = 0$

$$\lambda = \pm i\omega_0$$

$$\omega_0 = \sqrt{k/m}$$

$$y(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t)$$




- oscillation
- phase portrait: center
- clockwise direction of rotation

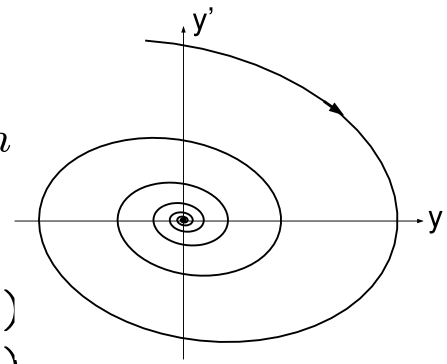
**Underdamped Case:** $0 < \mu^2 < 4km$

$$\lambda_{1,2} = -\alpha \pm i\omega$$

$$\alpha = \mu/2m$$

$$\omega = \sqrt{4km - \mu^2}/(2m = \sqrt{\omega_0^2 - \mu^2/4m^2}$$

$$y(t) = e^{-\alpha t}(c_1 \cos(\omega t) + c_2 \sin(\omega t))$$



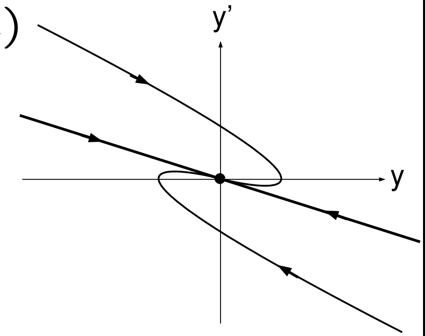

- damped oscillation
- phase portrait: spiral sink
- clockwise direction of rotation

**Critically Damped Case:** $\mu^2 = 4km$

$$\lambda_1 = \lambda_2 = -\mu/(2m)$$

$$y(t) = e^{\lambda_1 t}(c_1 + c_2 t)$$

- phase portrait: degenerate nodal sink

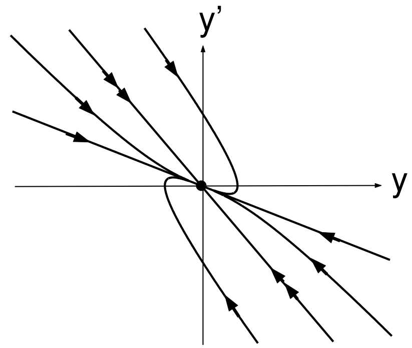



**Overdamped Case:** $\mu^2 > 4km$

$$\lambda_1 < \lambda_2 < 0$$

$$y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

- phase portrait: nodal sink
- both eigenlines: negative slopes



# Oscilador amortecido forçado

**TM seção 3.9, pgs. 131-132**

Forças impulsivas: função degrau

$$\frac{F(t)}{m} = \begin{cases} 0, & t < t_0 \\ a, & t > t_0 \end{cases} \qquad a \text{ cte}$$

Exemplo: Oscilador sub-amortecido submetido a uma força impulsiva do tipo função degrau partindo das condições iniciais $x(t_0) = x_0$; $\dot{x}(t_0) = 0$

# TM seção 3.9, pgs. 131–132

Força impulsiva: $F(t) = maH(t-t_0)$, $H(\tau) = \begin{cases} 0, & \tau < 0, \\ 1, & \tau > 0 \end{cases}$

$H(\tau)$: função degrau de Heaviside $= \int_{-\infty}^{\tau} \mathrm{d}t\, \delta(t)$ [Dirac]

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = \frac{1}{m}F(t),\ \gamma \equiv \frac{b}{2m},\ \omega_0 \equiv \sqrt{\frac{k}{m}},\ \omega_1 \equiv \sqrt{\omega_0^2 - \gamma^2}$$

$$x(t) = \frac{a}{\omega_0^2} + \mathrm{e}^{-\gamma(t-t_0)}\{A_1 \cos[\omega_1(t-t_0)] + A_2 \operatorname{sen}[\omega_1(t-t_0)]\}$$

$$\dot{x}(t) = -\gamma \mathrm{e}^{-\gamma(t-t_0)}\{A_1 \cos[\omega_1(t-t_0)] + A_2 \operatorname{sen}[\omega_1(t-t_0)]\} - \omega_1 \mathrm{e}^{-\gamma(t-t_0)}\{A_1 \operatorname{sen}[\omega_1(t-t_0)] - A_2 \cos[\omega_1(t-t_0)]\}$$

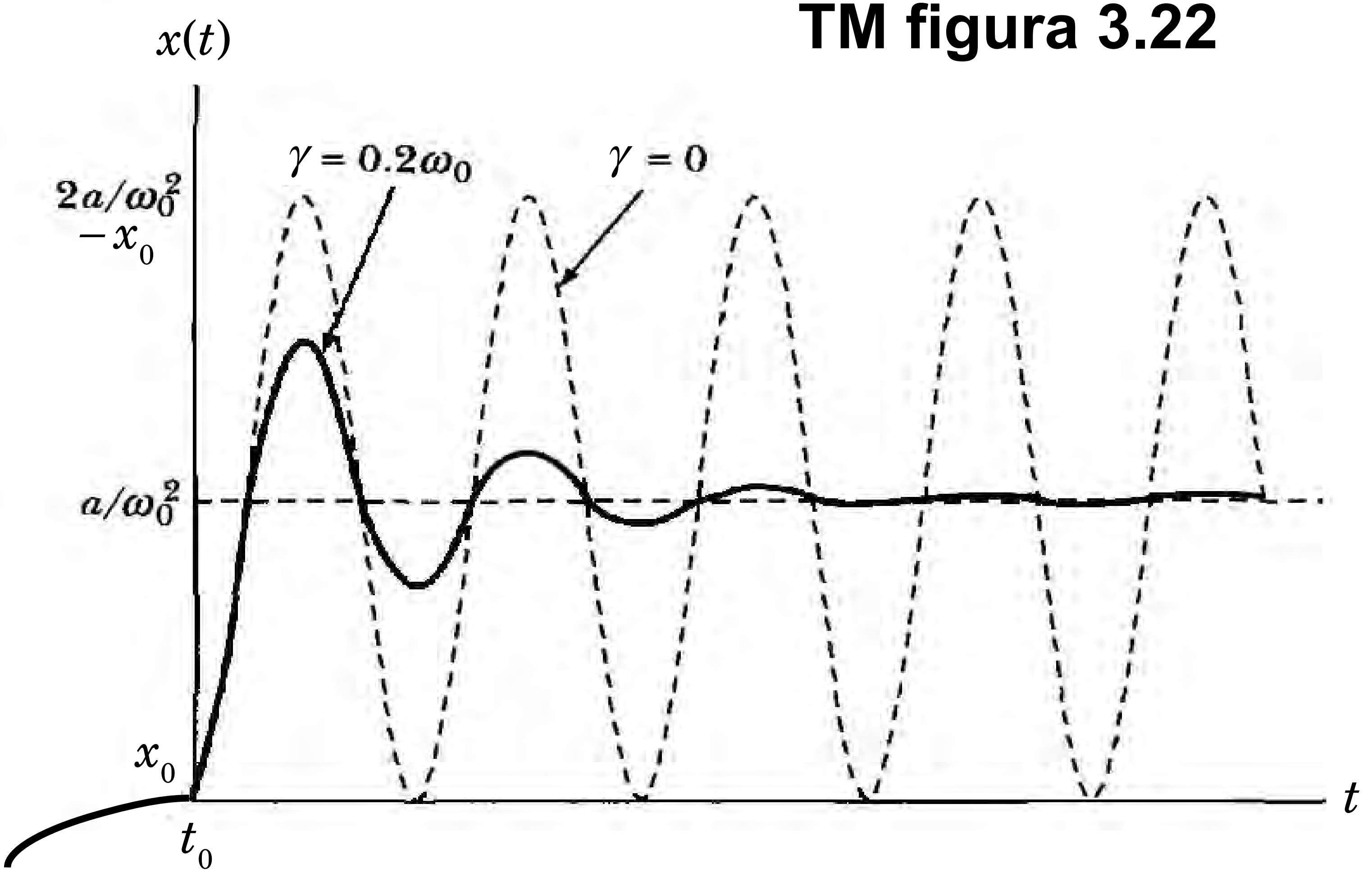
$$\left.\begin{array}{l} x(t_0) = x_0 = \frac{a}{\omega_0^2} + A_1 \\ \dot{x}(t_0) = 0 = -\gamma A_1 + \omega_1 A_2 \end{array}\right\} \rightarrow$$

**TM eq. (3.104)**

$$A_1 = x_0 - \frac{a}{\omega_0^2},\ A_2 = \frac{\gamma}{\omega_1}A_1$$

# Oscilador amortecido forçado

**TM figura 3.22**



# Oscilador amortecido forçado

## TM problema 3.35, pg. 142

Um oscilador sub-amortecido é submetido à força

$$\frac{F(t)}{m} = \begin{cases} 0, & t < 0 \\ a \sin \omega t, & 0 < t < \pi / \omega \\ 0, & t > \pi / \omega \end{cases} \qquad a \text{ cte}$$

Calcular a resposta do oscilador para $\omega = 2\omega_1$
com condições iniciais $x(0) = 0;\ \dot{x}(0) = 0$

# TM problema 3.35, pg. 142

Força impulsiva: $F(t) = ma\,H(t)H\left(\frac{\pi}{\omega} - t\right)\operatorname{sen}\omega t,\ \omega = 2\omega_1$

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = \frac{1}{m}F(t),\ \Delta^2 \equiv \delta^2 + 4\gamma^2\omega^2,\ \delta \equiv \omega_0^2 - \omega^2$$

Três regiões: $x_0(t<0) = 0;\ \ x_1\left(0 \le t \le \frac{\pi}{\omega}\right);\ \ x_2\left(t > \frac{\pi}{\omega}\right)$

$$x_1(t) = \mathrm{e}^{-\gamma t}\left(A_1\cos\omega_1 t + A_2\operatorname{sen}\omega_1 t\right) + \frac{a}{\Delta}\operatorname{sen}(\omega t - \varphi),$$

$$x_2(t) = \mathrm{e}^{-\gamma t}\left(B_1\cos\omega_1 t + B_2\operatorname{sen}\omega_1 t\right),\quad \operatorname{tg}\varphi = \frac{2\gamma\omega}{\delta}$$

Impondo as condições de contorno com $\frac{\omega_1\tau}{2} = \frac{\omega_1\pi}{\omega} = \frac{\pi}{2}$:

$$x_1(0) = \dot{x}_1(0) = 0;\ \ x_1\left(\frac{\pi}{\omega}\right) = x_2\left(\frac{\pi}{\omega}\right);\ \ \dot{x}_1\left(\frac{\pi}{\omega}\right) = \dot{x}_2\left(\frac{\pi}{\omega}\right)$$

$$A_1 = \frac{2a}{\Delta^2}\gamma\omega,\ \ A_2 = \frac{2a}{\Delta^2}(2\gamma^2 - \delta),\ \ \operatorname{sen}\varphi = \frac{2\gamma\omega}{\Delta},\ \ \cos\varphi = \frac{\delta}{\Delta}$$

$$B_1 = A_1 - \mathrm{e}^{\gamma\pi/\omega}A_2 = \frac{2a}{\Delta^2}\left[\gamma\omega - \mathrm{e}^{\gamma\pi/\omega}(2\gamma^2 - \delta)\right]$$

$$B_2 = \mathrm{e}^{\gamma\pi/\omega}A_1 + A_2 = \frac{2a}{\Delta^2}\left(\mathrm{e}^{\gamma\pi/\omega}\gamma\omega + 2\gamma^2 - \delta\right)$$