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CLASSICAL DYNAMICS

OF PARTICLES AND SYSTEMS

FIFTH EDITION

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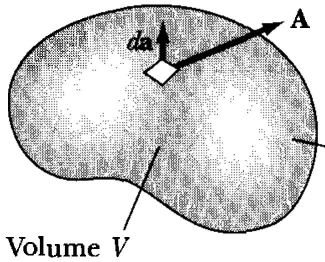

FIGURE 1-23 The differential da is an element of area on a surface S that surrounds a closed volume V.

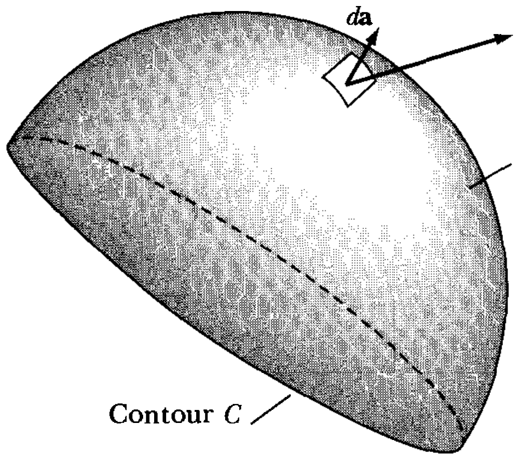

FIGURE 1-24 A contour path C defines an open surface S. A line integral around the path C and a surface integral over the surface S is required for Stokes's theorem.

is hoped, a simpler line integral (one dimensional). Both Gauss's and Stokes's theorems have wide application in vector calculus. In addition to mechanics, they are also useful in electromagnetic applications and in potential theory.

PROBLEMS

1-1. Find the transformation matrix that rotates the axis x_3 of a rectangular coordinate system 45° toward x_1 around the x_2-axis.

1-2. Prove Equations 1.10 and 1.11 from trigonometric considerations.

1-3. Find the transformation matrix that rotates a rectangular coordinate system through an angle of 120° about an axis making equal angles with the original three coordinate axes.

1-4. Show
(a) $(\mathbf{AB})^t = \mathbf{B}^t\mathbf{A}^t$ (b) $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$

1-5. Show by direct expansion that $|\boldsymbol{\lambda}|^2 = 1$. For simplicity, take $\boldsymbol{\lambda}$ to be a two-dimensional orthogonal transformation matrix.

1-6. Show that Equation 1.15 can be obtained by using the requirement that the transformation leaves unchanged the length of a line segment.

1-7. Consider a unit cube with one corner at the origin and three adjacent sides lying along the three axes of a rectangular coordinate system. Find the vectors describing the diagonals of the cube. What is the angle between any pair of diagonals?

1-8. Let **A** be a vector from the origin to a point P fixed in space. Let **r** be a vector from the origin to a variable point $Q(x_1, x_2, x_3)$. Show that

$$\mathbf{A} \cdot \mathbf{r} = A^2$$

is the equation of a plane perpendicular to **A** and passing through the point P.

1-9. For the two vectors

$$\mathbf{A} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}, \quad \mathbf{B} = -2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$$

find

(a) $\mathbf{A} - \mathbf{B}$ and $|\mathbf{A} - \mathbf{B}|$ (b) component of **B** along **A** (c) angle between **A** and **B**
(d) $\mathbf{A} \times \mathbf{B}$ (e) $(\mathbf{A} - \mathbf{B}) \times (\mathbf{A} + \mathbf{B})$

1-10. A particle moves in a plane elliptical orbit described by the position vector

$$\mathbf{r} = 2b \sin \omega t \, \mathbf{i} + b \cos \omega t \, \mathbf{j}$$

(a) Find **v**, **a**, and the particle speed.
(b) What is the angle between **v** and **a** at time $t = \pi/2\omega$?

1-11. Show that the *triple scalar product* $(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}$ can be written as

$$(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

Show also that the product is unaffected by an interchange of the scalar and vector product operations or by a change in the order of **A**, **B**, **C**, as long as they are in cyclic order; that is,

$$(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} = \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = (\mathbf{C} \times \mathbf{A}) \cdot \mathbf{B}, \qquad \text{etc.}$$

We may therefore use the notation **ABC** to denote the triple scalar product. Finally, give a geometric interpretation of **ABC** by computing the volume of the parallelepiped defined by the three vectors **A**, **B**, **C**.

1-12. Let **a**, **b**, **c** be three constant vectors drawn from the origin to the points A, B, C. What is the distance from the origin to the plane defined by the points A, B, C? What is the area of the triangle ABC?

1-13. **X** is an unknown vector satisfying the following relations involving the known vectors **A** and **B** and the scalar ϕ,

$$\mathbf{A} \times \mathbf{X} = \mathbf{B}, \quad \mathbf{A} \cdot \mathbf{X} = \phi.$$

Express **X** in terms of **A**, **B**, ϕ, and the magnitude of **A**.

1-14. Consider the following matrices:

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & 0 & 1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & -1 & 2 \\ 1 & 1 & 3 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 2 & 1 \\ 4 & 3 \\ 1 & 0 \end{pmatrix}$$

Find the following
(a) $|\mathbf{AB}|$ (b) $\mathbf{AC}$ (c) $\mathbf{ABC}$ (d) $\mathbf{AB} - \mathbf{B}^t\mathbf{A}^t$

1-15. Find the values of α needed to make the following transformation orthogonal.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & -\alpha \\ 0 & \alpha & \alpha \end{pmatrix}$$

1-16. What surface is represented by $\mathbf{r} \cdot \mathbf{a} =$ const. that is described if $\mathbf{a}$ is a vector of constant magnitude and direction from the origin and $\mathbf{r}$ is the position vector to the point $P(x_1, x_2, x_3)$ on the surface?

1-17. Obtain the cosine law of plane trigonometry by interpreting the product $(\mathbf{A} - \mathbf{B}) \cdot (\mathbf{A} - \mathbf{B})$ and the expansion of the product.

1-18. Obtain the sine law of plane trigonometry by interpreting the product $\mathbf{A} \times \mathbf{B}$ and the alternate representation $(\mathbf{A} - \mathbf{B}) \times \mathbf{B}$.

1-19. Derive the following expressions by using vector algebra:
(a) $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$
(b) $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

1-20. Show that

(a) $\sum_{i,j} \varepsilon_{ijk}\,\delta_{ij} = 0$ (b) $\sum_{j,k} \varepsilon_{ijk}\,\varepsilon_{ljk} = 2\delta_{il}$ (c) $\sum_{i,j,k} \varepsilon_{ijk}\,\varepsilon_{ijk} = 6$

1-21. Show (see also Problem 1-11) that

$$\mathbf{ABC} = \sum_{i,j,k} \varepsilon_{ijk} A_i B_j C_k$$

1-22. Evaluate the sum $\sum_k \varepsilon_{ijk}\varepsilon_{lmk}$ (which contains 3 terms) by considering the result for all possible combinations of i, j, l, m; that is,
(a) $i = j$ (b) $i = l$ (c) $i = m$ (d) $j = l$ (e) $j = m$ (f) $l = m$
(g) $i \neq l$ or m (h) $j \neq l$ or m

Show that

$$\sum_k \varepsilon_{ijk}\varepsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

and then use this result to prove

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}$$

1-23. Use the ε_{ijk} notation and derive the identity

$$(\mathbf{A} \times \mathbf{B}) \times (\mathbf{C} \times \mathbf{D}) = (\mathbf{ABD})\mathbf{C} - (\mathbf{ABC})\mathbf{D}$$

1-24. Let **A** be an arbitrary vector, and let **e** be a unit vector in some fixed direction. Show that

$$\mathbf{A} = \mathbf{e}(\mathbf{A} \cdot \mathbf{e}) + \mathbf{e} \times (\mathbf{A} \times \mathbf{e})$$

What is the geometrical significance of each of the two terms of the expansion?

1-25. Find the components of the acceleration vector **a** in spherical coordinates.

1-26. A particle moves with $v =$ const. along the curve $r = k(1 + \cos\theta)$ (a *cardioid*). Find $\ddot{\mathbf{r}} \cdot \mathbf{e}_r = \mathbf{a} \cdot \mathbf{e}_r, |\mathbf{a}|$, and $\dot{\theta}$.

1-27. If **r** and $\dot{\mathbf{r}} = \mathbf{v}$ are both explicit functions of time, show that

$$\frac{d}{dt}[\mathbf{r} \times (\mathbf{v} \times \mathbf{r})] = r^2\mathbf{a} + (\mathbf{r} \cdot \mathbf{v})\mathbf{v} - (v^2 + \mathbf{r} \cdot \mathbf{a})\mathbf{r}$$

1-28. Show that

$$\nabla(\ln|\mathbf{r}|) = \frac{\mathbf{r}}{r^2}$$

1-29. Find the angle between the surfaces defined by $r^2 = 9$ and $x + y + z^2 = 1$ at the point $(2, -2, 1)$.

1-30. Show that $\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi$.

1-31. Show that

(a) $\nabla r^n = nr^{(n-2)}\mathbf{r}$ **(b)** $\nabla f(r) = \frac{\mathbf{r}}{r}\frac{df}{dr}$ **(c)** $\nabla^2(\ln r) = \frac{1}{r^2}$

1-32. Show that

$$\int (2a\mathbf{r} \cdot \dot{\mathbf{r}} + 2b\dot{\mathbf{r}} \cdot \ddot{\mathbf{r}})\, dt = a\mathbf{r}^2 + b\dot{r}^2 + \text{const.}$$

where **r** is the vector from the origin to the point (x_1, x_2, x_3). The quantities r and $\dot{r}$ are the magnitudes of the vectors **r** and $\dot{\mathbf{r}}$, respectively, and a and b are constants.

1-33. Show that

$$\int \left(\frac{\dot{\mathbf{r}}}{r} - \frac{\mathbf{r}\dot{r}}{r^2}\right) dt = \frac{\mathbf{r}}{r} + \mathbf{C}$$

where **C** is a constant vector.

1-34. Evaluate the integral

$$\int \mathbf{A} \times \ddot{\mathbf{A}} dt$$

1-35. Show that the volume common to the intersecting cylinders defined by $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$ is $V = 16a^3/3$.

1-36. Find the value of the integral $\int_S \mathbf{A} \cdot d\mathbf{a}$, where $\mathbf{A} = x\mathbf{i} - y\mathbf{j} + z\mathbf{k}$ and S is the closed surface defined by the cylinder $c^2 = x^2 + y^2$. The top and bottom of the cylinder are at $z = d$ and 0, respectively.

1-37. Find the value of the integral $\int_S \mathbf{A} \cdot d\mathbf{a}$, where $\mathbf{A} = (x^2 + y^2 + z^2)(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$ and the surface S is defined by the sphere $R^2 = x^2 + y^2 + z^2$. Do the integral directly and also by using Gauss's theorem.

1-38. Find the value of the integral $\int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{a}$ if the vector $\mathbf{A} = y\mathbf{i} + z\mathbf{j} + x\mathbf{k}$ and S is the surface defined by the paraboloid $z = 1 - x^2 - y^2$, where $z \geq 0$.

1-39. A plane passes through the three points $(x, y, z) = (1, 0, 0), (0, 2, 0), (0, 0, 3)$. **(a)** Find a unit vector perpendicular to the plane. **(b)** Find the distance from the point (1, 1, 1) to the closest point of the plane and the coordinates of the closest point.

1-40. The height of a hill in meters is given by $z = 2xy - 3x^2 - 4y^2 - 18x + 28y + 12$, where x is the distance east and y is the distance north of the origin. **(a)** Where is the top of the hill and how high is it? **(b)** How steep is the hill at $x = y = 1$, that is, what is the angle between a vector perpendicular to the hill and the z axis? **(c)** In which compass direction is the slope at $x = y = 1$ steepest?

1-41. For what values of a are the vectors $\mathbf{A} = 2a\mathbf{i} - 2\mathbf{j} + a\mathbf{k}$ and $\mathbf{B} = a\mathbf{i} + 2a\mathbf{j} + 2\mathbf{k}$ perpendicular?

Newtonian mechanics is therefore subject to fundamental limitations when small distances or high velocities are encountered. Difficulties with Newtonian mechanics may also occur when massive objects or enormous distances are involved. A practical limitation also occurs when the number of bodies constituting the system is large. In Chapter 8, we see that we cannot obtain a general solution in closed form for the motion of a system of more than two interacting bodies even for the relatively simple case of gravitational interaction. To calculate the motion in a three-body system, we must resort to a numerical approximation procedure. Although such a method is in principle capable of any desired accuracy, the labor involved is considerable. The motion in even more complex systems (for example, the system composed of all the major objects in the solar system) can likewise be computed, but the procedure rapidly becomes too unwieldy to be of much use for any larger system. To calculate the motion of the individual molecules in, say, a cubic centimeter of gas containing $\approx 10^{19}$ molecules is clearly out of the question. A successful method of calculating the *average* properties of such systems was developed in the latter part of the nineteenth century by Boltzmann, Maxwell, Gibbs, Liouville, and others. These procedures allowed the dynamics of systems to be calculated from probability theory, and a *statistical mechanics* was evolved. Some comments regarding the formulation of statistical concepts in mechanics are found in Section 7.13.

PROBLEMS

2-1. Suppose that the force acting on a particle is factorable into one of the following forms:
(a) $F(x_i, t) = f(x_i)g(t)$ **(b)** $F(\dot{x}_i, t) = f(\dot{x}_i)g(t)$ **(c)** $F(x_i, \dot{x}_i) = f(x_i)g(\dot{x}_i)$
For which cases are the equations of motion integrable?

2-2. A particle of mass m is constrained to move on the surface of a sphere of radius R by an applied force $\mathbf{F}(\theta, \phi)$. Write the equation of motion.

2-3. If a projectile is fired from the origin of the coordinate system with an initial velocity v_0 and in a direction making an angle α with the horizontal, calculate the time required for the projectile to cross a line passing through the origin and making an angle $\beta < \alpha$ with the horizontal.

2-4. A clown is juggling four balls simultaneously. Students use a video tape to determine that it takes the clown 0.9 s to cycle each ball through his hands (including catching, transferring, and throwing) and to be ready to catch the next ball. What is the minimum vertical speed the clown must throw up each ball?

2-5. A jet fighter pilot knows he is able to withstand an acceleration of $9g$ before blacking out. The pilot points his plane vertically down while traveling at Mach 3 speed and intends to pull up in a circular maneuver before crashing into the ground. **(a)** Where does the maximum acceleration occur in the maneuver? **(b)** What is the minimum radius the pilot can take?

2-6. In the blizzard of '88, a rancher was forced to drop hay bales from an airplane to feed her cattle. The plane flew horizontally at 160 km/hr and dropped the bales from a height of 80 m above the flat range. **(a)** She wanted the bales of hay to land 30 m behind the cattle so as to not hit them. Where should she push the bales out of the airplane? **(b)** To not hit the cattle, what is the largest time error she could make while pushing the bales out of the airplane? Ignore air resistance.

2-7. Include air resistance for the bales of hay in the previous problem. A bale of hay has a mass of about 30 kg and an average area of about 0.2 m^2. Let the resistance be proportional to the square of the speed and let $c_W = 0.8$. Plot the trajectories with a computer if the hay bales land 30 m behind the cattle for both including air resistance and not. If the bales of hay were released at the same time in the two cases, what is the distance between landing positions of the bales?

2-8. A projectile is fired with a velocity v_0 such that it passes through two points both a distance h above the horizontal. Show that if the gun is adjusted for maximum range, the separation of the points is

$$d = \frac{v_0}{g}\sqrt{v_0^2 - 4gh}$$

2-9. Consider a projectile fired vertically in a constant gravitational field. For the same initial velocities, compare the times required for the projectile to reach its maximum height **(a)** for zero resisting force, **(b)** for a resisting force proportional to the instantaneous velocity of the projectile.

2-10. Repeat Example 2.4 by performing a calculation using a computer to solve Equation 2.22. Use the following values: $m = 1$ kg, $v_0 = 10$ m/s, $x_0 = 0$, and $k = 0.1\ s^{-1}$. Make plots of v versus t, x versus t, and v versus x. Compare with the results of Example 2.4 to see if your results are reasonable.

2-11. Consider a particle of mass m whose motion starts from rest in a constant gravitational field. If a resisting force proportional to the square of the velocity (i.e., kmv^2) is encountered, show that the distance s the particle falls in accelerating from v_0 to v_1 is given by

$$s(v_0 \rightarrow v_1) = \frac{1}{2k}\ln\left[\frac{g - kv_0^2}{g - kv_1^2}\right]$$

2-12. A particle is projected vertically upward in a constant gravitational field with an initial speed v_0. Show that if there is a retarding force proportional to the square of the instantaneous speed, the speed of the particle when it returns to the initial position is

$$\frac{v_0 v_t}{\sqrt{v_0^2 + v_t^2}}$$

where v_t is the terminal speed.

2-13. A particle moves in a medium under the influence of a retarding force equal to $mk(v^3 + a^2 v)$, where k and a are constants. Show that for any value of the initial speed the particle will never move a distance greater than $\pi/2ka$ and that the particle comes to rest only for $t \to \infty$.

2-14. A projectile is fired with initial speed v_0 at an elevation angle of α up a hill of slope $\beta (\alpha > \beta)$.
(a) How far up the hill will the projectile land?
(b) At what angle α will the range be a maximum?
(c) What is the maximum range?

2-15. A particle of mass m slides down an inclined plane under the influence of gravity. If the motion is resisted by a force $f = kmv^2$, show that the time required to move a distance d after starting from rest is

$$t = \frac{\cosh^{-1}(e^{kd})}{\sqrt{kg \sin \theta}}$$

where θ is the angle of inclination of the plane.

2-16. A particle is projected with an initial velocity v_0 up a slope that makes an angle α with the horizontal. Assume frictionless motion and find the time required for the particle to return to its starting position. Find the time for $v_0 = 2.4$ m/s and $\alpha = 26°$.

2-17. A strong softball player smacks the ball at a height of 0.7 m above home plate. The ball leaves the player's bat at an elevation angle of 35° and travels toward a fence 2 m high and 60 m away in center field. What must the initial speed of the softball be to clear the center field fence? Ignore air resistance.

2-18. Include air resistance proportional to the square of the ball's speed in the previous problem. Let the drag coefficient be $c_W = 0.5$, the softball radius be 5 cm and the mass be 200 g. **(a)** Find the initial speed of the softball needed now to clear the fence. **(b)** For this speed, find the initial elevation angle that allows the ball to most easily clear the fence. By how much does the ball now vertically clear the fence?

2-19. If a projectile moves such that its distance from the point of projection is always increasing, find the maximum angle above the horizontal with which the particle could have been projected. (Assume no air resistance.)

2-20. A gun fires a projectile of mass 10 kg of the type to which the curves of Figure 2-3 apply. The muzzle velocity is 140 m/s. Through what angle must the barrel be elevated to hit a target on the same horizontal plane as the gun and 1000 m away? Compare the results with those for the case of no retardation.

2-21. Show directly that the time rate of change of the angular momentum about the origin for a projectile fired from the origin (constant g) is equal to the moment of force (or torque) about the origin.

2-22. The motion of a charged particle in an electromagnetic field can be obtained from the **Lorentz equation*** for the force on a particle in such a field. If the electric field vector is **E** and the magnetic field vector is **B**, the force on a particle of mass m that

*See, for example, Heald and Marion, *Classical Electromagnetic Radiation* (95, Section 1.7).

carries a charge q and has a velocity $\mathbf{v}$ is given by

$$\mathbf{F} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B}$$

where we assume that $v \ll c$(speed of light).

(a) If there is no electric field and if the particle enters the magnetic field in a direction perpendicular to the lines of magnetic flux, show that the trajectory is a circle with radius

$$r = \frac{mv}{qB} = \frac{v}{\omega_c}$$

where $\omega_c \equiv qB/m$ is the *cyclotron frequency.*

(b) Choose the z-axis to lie in the direction of $\mathbf{B}$ and let the plane containing $\mathbf{E}$ and $\mathbf{B}$ be the yz–plane. Thus

$$\mathbf{B} = B\mathbf{k}, \quad \mathbf{E} = E_y\mathbf{j} + E_z\mathbf{k}$$

Show that the z component of the motion is given by

$$z(t) = z_0 + \dot{z}_0 t + \frac{qE_z}{2m}t^2$$

where

$$z(0) \equiv z_0 \quad \text{and} \quad \dot{z}(0) \equiv \dot{z}_0$$

(c) Continue the calculation and obtain expressions for $\dot{x}(t)$ and $\dot{y}(t)$. Show that the time averages of these velocity components are

$$\langle \dot{x} \rangle = \frac{E_y}{B}, \quad \langle \dot{y} \rangle = 0$$

(Show that the motion is periodic and then average over one complete period.)

(d) Integrate the velocity equations found in (c) and show (with the initial conditions $x(0) = -A/\omega_c$, $\dot{x}(0) = E_y/B$, $y(0) = 0$, $\dot{y}(0) = A$) *that*

$$x(t) = \frac{-A}{\omega_c} \cos \omega_c t + \frac{E_y}{B} t, \quad y(t) = \frac{A}{\omega_c} \sin \omega_c t$$

These are the parametric equations of a trochoid. Sketch the projection of the trajectory on the xy–plane for the cases (i) $A > |E_y/B|$, (ii) $A < |E_y/B|$, and (iii) $A = |E_y/B|$. (The last case yields a cycloid.)

2-23. A particle of mass $m = 1$ kg is subjected to a one-dimensional force $F(t) = kte^{-\alpha t}$, where $k = 1$ N/s and $\alpha = 0.5$ s^{-1}. If the particle is initially at rest, calculate and plot with the aid of a computer the position, speed, and acceleration of the particle as a function of time.

2-24. A skier weighing 90 kg starts from rest down a hill inclined at 17°. He skis 100 m down the hill and then coasts for 70 m along level snow until he stops. Find the coefficient of kinetic friction between the skis and the snow. What velocity does the skier have at the bottom of the hill?

2-25. A block of mass $m = 1.62$ kg slides down a frictionless incline (Figure 2-A). The block is released a height $h = 3.91$ m above the bottom of the loop.

(a) What is the force of the inclined track on the block at the bottom (point A)?
(b) What is the force of the track on the block at point B?
(c) At what speed does the block leave the track?
(d) How far away from point A does the block land on level ground?
(e) Sketch the potential energy $U(x)$ of the block. Indicate the total energy on the sketch.

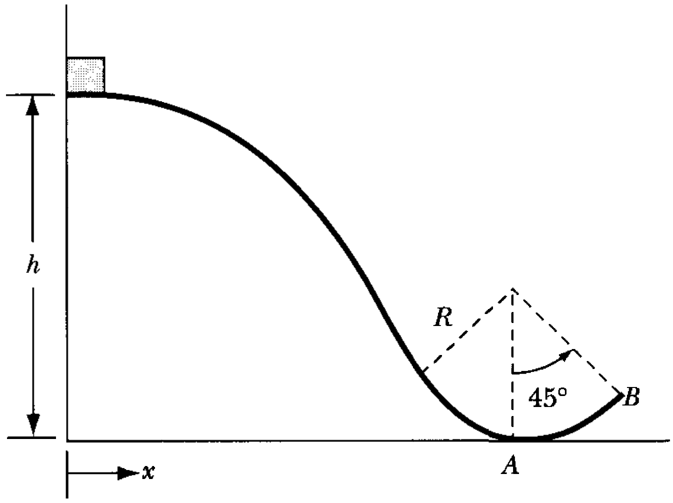

FIGURE 2-A Problem 2-25.

2-26. A child slides a block of mass 2 kg along a slick kitchen floor. If the initial speed is 4 m/s and the block hits a spring with spring constant 6 N/m, what is the maximum compression of the spring? What is the result if the block slides across 2 m of a rough floor that has $\mu_k = 0.2$?

2-27. A rope having a total mass of 0.4 kg and total length 4 m has 0.6 m of the rope hanging vertically down off a work bench. How much work must be done to place all the rope on the bench?

2-28. A superball of mass M and a marble of mass m are dropped from a height h with the marble just on top of the superball. A superball has a coefficient of restitution of nearly 1 (i.e., its collision is essentially elastic). Ignore the sizes of the superball and marble. The superball collides with the floor, rebounds, and smacks the marble, which moves back up. How high does the marble go if all the motion is vertical? How high does the superball go?

2-29. An automobile driver traveling down an 8% grade slams on his brakes and skids 30 m before hitting a parked car. A lawyer hires an expert who measures the coefficient of kinetic friction between the tires and road to be $\mu_k = 0.45$. Is the lawyer correct to accuse the driver of exceeding the 25-MPH speed limit? Explain.

2-30. A student drops a water-filled balloon from the roof of the tallest building in town trying to hit her roommate on the ground (who is too quick). The first student ducks back but hears the water splash 4.021 s after dropping the balloon. If the speed of sound is 331 m/s, find the height of the building, neglecting air resistance.

2-31. In Example 2.10, the initial velocity of the incoming charged particle had no component along the x-axis. Show that, even if it had an x component, the subsequent motion of the particle would be the same—that only the radius of the helix would be altered.

2-32. Two blocks of unequal mass are connected by a string over a smooth pulley (Figure 2-B). If the coefficient of kinetic friction is μ_k, what angle θ of the incline allows the masses to move at a constant speed?

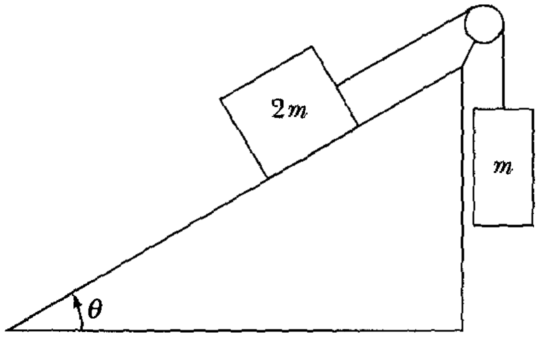

FIGURE 2-B Problem 2-32.

2-33. Perform a computer calculation for an object moving vertically in air under gravity and experiencing a retarding force proportional to the square of the object's speed (see Equation 2.21). Use variables m for mass and r for the object's radius. All the objects are dropped from rest from the top of a 100-m-tall building. Use a value of $c_W = 0.5$ and make computer plots of height y, speed v, and acceleration a versus t for the following conditions and answer the questions:

(**a**) A baseball of $m = 0.145$ kg and $r = 0.0366$ m.

(**b**) A ping-pong ball of $m = 0.0024$ kg and $r = 0.019$ m.

(**c**) A raindrop of $r = 0.003$ m.

(**d**) Do all the objects reach their terminal speeds? Discuss the values of the terminal velocities and explain their differences.

(**e**) Why can a baseball be thrown farther than a ping-pong ball even though the baseball is so much more massive?

(**f**) Discuss the terminal speeds of big and small raindrops. What are the terminal speeds of raindrops having radii 0.002 m and 0.004 m?

2-34. A particle is released from rest ($y = 0$) and falls under the influence of gravity and air resistance. Find the relationship between v and the distance of falling y when the air resistance is equal to (**a**) αv and (**b**) βv^2.

2-35. Perform the numerical calculations of Example 2.7 for the values given in Figure 2-8. Plot both Figures 2-8 and 2-9. Do not duplicate the solution in Appendix H; compose your own solution.

2-36. A gun is located on a bluff of height h overlooking a river valley. If the muzzle velocity is v_0, find the expression for the range as a function of the elevation angle of the gun. Solve numerically for the maximum range out into the valley for a given h and v_0.

2-37. A particle of mass m has speed $v = \alpha/x$, where x is its displacement. Find the force $F(x)$ responsible.

2-38. The speed of a particle of mass m varies with the distance x as $v(x) = \alpha x^{-n}$. Assume $v(x = 0) = 0$ at $t = 0$. **(a)** Find the force $F(x)$ responsible. **(b)** Determine $x(t)$ and **(c)** $F(t)$.

2-39. A boat with initial speed v_0 is launched on a lake. The boat is slowed by the water by a force $F = -\alpha e^{\beta v}$. **(a)** Find an expression for the speed $v(t)$. **(b)** Find the time and **(c)** distance for the boat to stop.

2-40. A particle moves in a two-dimensional orbit defined by

$$x(t) = A(2\alpha t - \sin \alpha t)$$
$$y(t) = A(1 - \cos \alpha t)$$

(a) Find the tangential acceleration a_t and normal acceleration a_n as a function of time where the tangential and normal components are taken with respect to the velocity.
(b) Determine at what times in the orbit a_n has a maximum.

2-41. A train moves along the tracks at a constant speed u. A woman on the train throws a ball of mass m straight ahead with a speed v with respect to herself. **(a)** What is the kinetic energy gain of the ball as measured by a person on the train? **(b)** by a person standing by the railroad track? **(c)** How much work is done by the woman throwing he ball and **(d)** by the train?

2-42. A solid cube of uniform density and sides of b is in equilibrium on top of a cylinder of radius R (Figure 2-C). The planes of four sides of the cube are parallel to the axis of the cylinder. The contact between cube and sphere is perfectly rough. Under what conditions is the equilibrium stable or not stable?

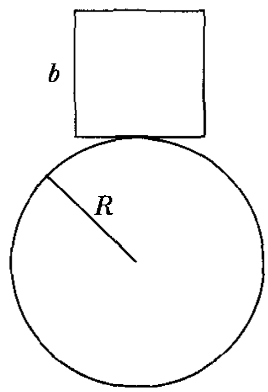

FIGURE 2-C Problem 2-42.

2-43. A particle is under the influence of a force $F = -kx + kx^3/\alpha^2$, where k and α are constants and k is positive. Determine $U(x)$ and discuss the motion. What happens when $E = (1/4)k\alpha^2$?

2-44. Solve Example 2.12 by using forces rather than energy. How can you determine whether the system equilibrium is stable or unstable?

2-45. Describe how to determine whether an equilibrium is stable or unstable when $(d^2U/dx^2)_0 = 0$.

2-46. Write the criteria for determining whether an equilibrium is stable or unstable when all derivatives up through order n, $(d^nU/dx^n)_0 = 0$.

2-47. Consider a particle moving in the region $x > 0$ under the influence of the potential

$$U(x) = U_0\left(\frac{a}{x} + \frac{x}{a}\right)$$

where $U_0 = 1$ J and $a = 2$ m. Plot the potential, find the equilibrium points, and determine whether they are maxima or minima.

2-48. Two gravitationally bound stars with equal masses m, separated by a distance d, revolve about their center of mass in circular orbits. Show that the period τ is proportional to $d^{3/2}$ (Kepler's Third Law) and find the proportionality constant.

2-49. Two gravitationally bound stars with unequal masses m_1 and m_2, separated by a distance d, revolve about their center of mass in circular orbits. Show that the period τ is proportional to $d^{3/2}$ (Kepler's Third Law) and find the proportionality constant.

2-50. According to special relativity, a particle of rest mass m_0 accelerated in one dimension by a force F obeys the equation of motion $dp/dt = F$. Here $p = m_0v/(1 - v^2/c^2)^{1/2}$ is the relativistic momentum, which reduces to m_0v for $v^2/c^2 \ll 1$. **(a)** For the case of constant F and initial conditions $x(0) = 0 = v(0)$, find $x(t)$ and $v(t)$. **(b)** Sketch your result for $v(t)$. **(c)** Suppose that $F/m_0 = 10\text{ m/s}^2$ ($\approx g$ on Earth). How much time is required for the particle to reach half the speed of light and of 99% the speed of light?

2-51. Let us make the (unrealistic) assumption that a boat of mass m gliding with initial velocity v_0 in water is slowed by a viscous retarding force of magnitude bv^2, where b is a constant. **(a)** Find and sketch $v(t)$. How long does it take the boat to reach a speed of $v_0/1000$? **(b)** Find $x(t)$. How far does the boat travel in this time? Let $m = 200$ kg, $v_0 = 2$ m/s, and $b = 0.2\text{ Nm}^{-2}\text{s}^2$.

2-52. A particle of mass m moving in one dimension has potential energy $U(x) = U_0[2(x/a)^2 - (x/a)^4]$, where U_0 and a are positive constants. **(a)** Find the force $F(x)$, which acts on the particle. **(b)** Sketch $U(x)$. Find the positions of stable and unstable equilibrium. **(c)** What is the angular frequency ω of oscillations about the point of stable equilibrium? **(d)** What is the minimum speed the particle must have at the origin to escape to infinity? **(e)** At $t = 0$ the particle is at the origin and its velocity is positive and equal in magnitude to the escape speed of part **(d)**. Find $x(t)$ and sketch the result.

2-53. Which of the following forces are conservative? If conservative, find the potential energy $U(\mathbf{r})$. **(a)** $F_x = ayz + bx + c$, $F_y = axz + bz$, $F_z = axy + by$. **(b)** $F_x = -ze^{-x}$, $F_y = \ln z$, $F_z = e^{-x} + y/z$. **(c)** $\mathbf{F} = \mathbf{e}_r a/r$ (a, b, c are constants).

2-54. A potato of mass 0.5 kg moves under Earth's gravity with an air resistive force of $-kmv$. **(a)** Find the terminal velocity if the potato is released from rest and $k = 0.01\text{ s}^{-1}$. **(b)** Find the maximum height of the potato if it has the same value of k, but it is initially shot directly upward with a student-made potato gun with an initial velocity of 120 m/s.

2-55. A pumpkin of mass 5 kg shot out of a student-made cannon under air pressure at an elevation angle of 45° fell at a distance of 142 m from the cannon. The students used light beams and photocells to measure the initial velocity of 54 m/s. If the air resistive force was $F = -kmv$, what was the value of k?

PROBLEMS

3-1. A simple harmonic oscillator consists of a 100-g mass attached to a spring whose force constant is 10^4 dyne/cm. The mass is displaced 3 cm and released from rest. Calculate **(a)** the natural frequency ν_0 and the period τ_0, **(b)** the total energy, and **(c)** the maximum speed.

3-2. Allow the motion in the preceding problem to take place in a resisting medium. After oscillating for 10 s, the maximum amplitude decreases to half the initial value. Calculate **(a)** the damping parameter β, **(b)** the frequency ν_1 (compare with the undamped frequency ν_0), and **(c)** the decrement of the motion.

3-3. The oscillator of Problem 3-1 is set into motion by giving it an initial velocity of 1 cm/s at its equilibrium position. Calculate **(a)** the maximum displacement and **(b)** the maximum potential energy.

3-4. Consider a simple harmonic oscillator. Calculate the *time* averages of the kinetic and potential energies over one cycle, and show that these quantities are equal. Why is this a reasonable result? Next calculate the *space* averages of the kinetic and potential energies. Discuss the results.

3-5. Obtain an expression for the fraction of a complete period that a simple harmonic oscillator spends within a small interval Δx at a position x. Sketch curves of this function versus x for several different amplitudes. Discuss the physical significance of the results. Comment on the areas under the various curves.

3-6. Two masses $m_1 = 100$ g and $m_2 = 200$ g slide freely in a horizontal frictionless track and are connected by a spring whose force constant is $k = 0.5$ N/m. Find the frequency of oscillatory motion for this system.

3-7. A body of uniform cross-sectional area $A = 1$ cm^2 and of mass density $\rho = 0.8$ g/cm^3 floats in a liquid of density $\rho_0 = 1$ g/cm^3 and at equilibrium displaces a volume $V = 0.8$ cm^3. Show that the period of small oscillations about the equilibrium position is given by

$$\tau = 2\pi\sqrt{V/gA}$$

where g is the gravitational field strength. Determine the value of τ.

3-8. A pendulum is suspended from the cusp of a cycloid* cut in a rigid support (Figure 3-A). The path described by the pendulum bob is cycloidal and is given by

$$x = a(\phi - \sin\phi), \qquad y = a(\cos\phi - 1)$$

where the length of the pendulum is $l = 4a$, and where ϕ is the angle of rotation of the circle generating the cycloid. Show that the oscillations are exactly isochronous with a frequency $\omega_0 = \sqrt{g/l}$, independent of the amplitude.

*The reader unfamiliar with the properties of cycloids should consult a text on analytic geometry.

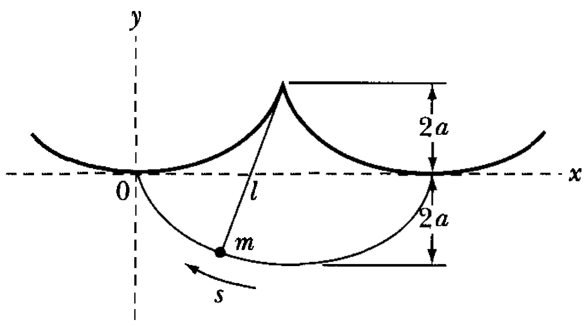

FIGURE 3-A Problem 3-8.

3-9. A particle of mass m is at rest at the end of a spring (force constant $= k$) hanging from a fixed support. At $t = 0$, a constant downward force F is applied to the mass and acts for a time t_0. Show that, after the force is removed, the displacement of the mass from its equilibrium position ($x = x_0$, where x is down) is

$$x - x_0 = \frac{F}{k}[\cos\omega_0(t - t_0) - \cos\omega_0 t]$$

where $\omega_0^2 = k/m$.

3-10. If the amplitude of a damped oscillator decreases to $1/e$ of its initial value after n periods, show that the frequency of the oscillator must be approximately $[1 - (8\pi^2 n^2)^{-1}]$ times the frequency of the corresponding undamped oscillator.

3-11. Derive the expressions for the energy and energy-loss curves shown in Figure 3-8 for the damped oscillator. For a lightly damped oscillator, calculate the *average rate* at which the damped oscillator loses energy (i.e., compute a time average over one cycle).

3-12. A simple pendulum consists of a mass m suspended from a fixed point by a weightless, extensionless rod of length l. Obtain the equation of motion and, in the approximation that $\sin\theta \cong \theta$, show that the natural frequency is $\omega_0 = \sqrt{g/l}$, where g is the gravitational field strength. Discuss the motion in the event that the motion takes place in a viscous medium with retarding force $2m\sqrt{gl}\,\dot{\theta}$.

3-13. Show that Equation 3.43 is indeed the solution for critical damping by assuming a solution of the form $x(t) = y(t)\exp(-\beta t)$ and determining the function $y(t)$.

3-14. Express the displacement $x(t)$ and the velocity $\dot{x}(t)$ for the overdamped oscillator in terms of hyperbolic functions.

3-15. Reproduce Figures 3-10b and c for the same values given in Example 3.2, but instead let $\beta = 0.1\ \text{s}^{-1}$ and $\delta = \pi$ rad. How many times does the system cross the $x = 0$ line before the amplitude finally falls below 10^{-2} of its maximum value? Which plot, b or c, is more useful for determining this number? Explain.

3-16. Discuss the motion of a particle described by Equation 3.34 in the event that $b < 0$ (i.e., the damping resistance is *negative*).

3-17. For a damped, driven oscillator, show that the average kinetic energy is the same at a frequency of a given number of octaves* above the kinetic energy resonance as at a frequency of the same number of octaves below resonance.

3-18. Show that, if a driven oscillator is only lightly damped and driven near resonance, the Q of the system is approximately

$$Q \cong 2\pi \times \left(\frac{\text{Total energy}}{\text{Energy loss during one period}}\right)$$

3-19. For a lightly damped oscillator, show that $Q \cong \omega_0/\Delta\omega$ (Equation 3.65).

3-20. Plot a *velocity* resonance curve for a driven, damped oscillator with $Q = 6$, and show that the full width of the curve between the points corresponding to $\dot{x}_{\text{max}}/\sqrt{2}$ is approximately equal to $\omega_0/6$.

3-21. Use a computer to produce a phase space diagram similar to Figure 3-11 for the case of critical damping. Show analytically that the equation of the line that the phase paths approach asymptotically is $\dot{x} = -\beta x$. Show the phase paths for at least three initial positions above and below the line.

3-22. Let the initial position and speed of an overdamped, nondriven oscillator be x_0 and v_0, respectively.

(a) Show that the values of the amplitudes A_1 and A_2 in Equation 3.44 have the values $A_1 = \dfrac{\beta_2 x_0 + v_0}{\beta_2 - \beta_1}$ and $A_2 = -\dfrac{\beta_1 x_0 + v_0}{\beta_2 - \beta_1}$ where $\beta_1 = \beta - \omega_2$ and $\beta_2 = \beta + \omega_2$.

(b) Show that when $A_1 = 0$, the phase paths of Figure 3-11 must be along the dashed curve given by $\dot{x} = -\beta_2 x$, otherwise the asymptotic paths are along the other dashed curve given by $\dot{x} = -\beta_1 x$. *Hint:* Note that $\beta_2 > \beta_1$ and find the asymptotic paths when $t \to \infty$.

3-23. To better understand underdamped motion, use a computer to plot $x(t)$ of Equation 3.40 (with $A = 1$ m) and its two components $[e^{-\beta t}$ and $\cos(\omega_1 t - \delta)]$ and comparisons (with $\beta = 0$) on the same plot as in Figure 3-6. Let $\omega_0 = 1$ rad/s and make separate plots for $\beta^2/\omega_0^2 = 0.1, 0.5$, and 0.9 and for δ (in radians) $= 0, \pi/2$, and π. Have only one value of δ and β on each plot (i.e., nine plots). Discuss the results.

3-24. For $\beta = 0.2\ \text{s}^{-1}$, produce computer plots like those shown in Figure 3-15 for a sinusoidal driven, damped oscillator where $x_p(t)$, $x_c(t)$, and the sum $x(t)$ are shown. Let $k = 1\ \text{kg/s}^2$ and $m = 1$ kg. Do this for values of ω/ω_1 of 1/9, 1/3, 1.1, 3, and 6. For the $x_c(t)$ solution (Equation 3.40), let the phase angle $\delta = 0$ and the amplitude $A = -1$ m. For the $x_p(t)$ solution (Equation 3.60), let $A = 1\ \text{m/s}^2$ but calculate δ. What do you observe about the relative amplitudes of the two solutions as ω increases? Why does this occur? For $\omega/\omega_1 = 6$, let $A = 20\ \text{m/s}^2$ for $x_p(t)$ and produce the plot again.

3-25. For values of $\beta = 1\ \text{s}^{-1}$, $k = 1\ \text{kg/s}^2$, and $m = 1$ kg, produce computer plots like those shown in Figure 3-15 for a sinusoidal driven, damped oscillator where $x_p(t)$, $x_c(t)$,

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*An octave is a frequency interval in which the highest frequency is just twice the lowest frequency.

and the sum $x(t)$ are shown. Do this for values of ω/ω_0 of 1/9, 1/3, 1.1, 3, and 6. For the critically damped $x_c(t)$ solution of Equation 3.43, let $A = -1$ m and $B = 1$ m/s. For the $x_p(t)$ solution of Equation 3.60, let $A = 1$ m/s^2 and calculate δ. What do you observe about the relative amplitudes of the two solutions as ω increases? Why does this occur? For $\omega/\omega_0 = 6$, let $A = 20$ m/s^2 for $x_p(t)$ and produce the plot again.

3-26. Figure 3-B illustrates a mass m_1 driven by a sinusoidal force whose frequency is ω. The mass m_1 is attached to a rigid support by a spring of force constant k and slides on a second mass m_2. The frictional force between m_1 and m_2 is represented by the damping parameter b_1, and the frictional force between m_2 and the support is represented by b_2. Construct the electrical analog of this system and calculate the impedance.

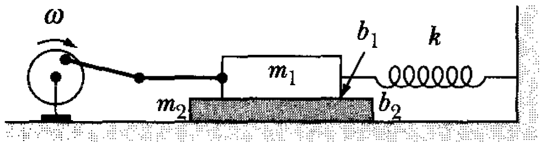

FIGURE 3-B Problem 3-26.

3-27. Show that the Fourier series of Equation 3.89 can be expressed as

$$F(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} c_n \cos(n\omega t - \phi_n)$$

Relate the coefficients c_n to the a_n and b_n of Equation 3.90.

3-28. Obtain the Fourier expansion of the function

$$F(t) = \begin{cases} -1, & -\pi/\omega < t < 0 \\ +1, & 0 < t < \pi/\omega \end{cases}$$

in the interval $-\pi/\omega < t < \pi/\omega$. Take $\omega = 1$ rad/s. In the periodical interval, calculate and plot the sums of the first two terms, the first three terms, and the first four terms to demonstrate the convergence of the series.

3-29. Obtain the Fourier series representing the function

$$F(t) = \begin{cases} 0, & -2\pi/\omega < t < 0 \\ \sin\omega t, & 0 < t < 2\pi/\omega \end{cases}$$

3-30. Obtain the Fourier representation of the output of a full-wave rectifier. Plot the first three terms of the expansion and compare with the exact function.

3-31. A damped linear oscillator, originally at rest in its equilibrium position, is subjected to a forcing function given by

$$\frac{F(t)}{m} = \begin{cases} 0, & t < 0 \\ a \times (t/\tau), & 0 < t < \tau \\ a, & t > \tau \end{cases}$$

Find the response function. Allow $\tau \to 0$ and show that the solution becomes that for a step function.

3-32. Obtain the response of a linear oscillator to a step function and to an impulse function (in the limit $\tau \rightarrow 0$) for overdamping. Sketch the response functions.

3-33. Calculate the maximum values of the amplitudes of the response functions shown in Figures 3-22 and 3-24. Obtain numerical values for $\beta = 0.2\omega_0$ when $a = 2$ m/s^2, $\omega_0 = 1$ rad/s, and $t_0 = 0$.

3-34. Consider an undamped linear oscillator with a natural frequency $\omega_0 = 0.5$ rad/s and the step function $a = 1$ m/s^2. Calculate and sketch the response function for an impulse forcing function acting for a time $\tau = 2\pi/\omega_0$. Give a physical interpretation of the results.

3-35. Obtain the response of a linear oscillator to the forcing function

$$\frac{F(t)}{m} = \begin{cases} 0, & t < 0 \\ a \sin \omega t, & 0 < t < \pi/\omega \\ 0, & t > \pi/\omega \end{cases}$$

3-36. Derive an expression for the displacement of a linear oscillator analogous to Equation 3.110 but for the initial conditions $x(t_0) = x_0$ and $\dot{x}(t_0) = \dot{x}_0$.

3-37. Derive the Green's method solution for the response caused by an arbitrary forcing function. Consider the function to consist of a series of step functions—that is, start from Equation 3.105 rather than from Equation 3.110.

3-38. Use Green's method to obtain the response of a damped oscillator to a forcing function of the form

$$F(t) = \begin{cases} 0 & t < 0 \\ F_0 e^{-\gamma t} \sin \omega t & t > 0 \end{cases}$$

3-39. Consider the periodic function

$$F(t) = \begin{cases} \sin \omega t, & 0 < t < \pi/\omega \\ 0, & \pi/\omega < t < 2\pi/\omega \end{cases}$$

which represents the positive portions of a sine function. (Such a function represents, for example, the output of a half-wave rectifying circuit.) Find the Fourier representation and plot the sum of the first four terms.

3-40. An automobile with a mass of 1000 kg, including passengers, settles 1.0 cm closer to the road for every additional 100 kg of passengers. It is driven with a constant horizontal component of speed 20 km/h over a washboard road with sinusoidal bumps. The amplitude and wavelength of the sine curve are 5.0 cm and 20 cm, respectively. The distance between the front and back wheels is 2.4 m. Find the amplitude of oscillation of the automobile, assuming it moves vertically as an undamped driven harmonic oscillator. Neglect the mass of the wheels and springs and assume that the wheels are always in contact with the road.

3-41. **(a)** Use the general solutions $x(t)$ to the differential equation $d^2x/dt^2 + 2\beta dx/dt + \omega_0^2 x = 0$ for underdamped, critically damped, and overdamped motion and choose the constants of integration to satisfy the initial conditions $x = x_0$ and $v = v_0 = 0$ at $t = 0$. **(b)** Use a computer to plot the results for $x(t)/x_0$ as a function of $\omega_0 t$ in the three cases $\beta = (1/2)\omega_0$, $\beta = \omega_0$, and $\beta = 2\omega_0$. Show all three curves on a single plot.

3-42. An undamped driven harmonic oscillator satisfies the equation of motion $m(d^2x/dt^2 + \omega_0{}^2 x) = F(t)$. The driving force $F(t) = F_0 \sin(\omega t)$ is switched on at $t = 0$. (**a**) Find $x(t)$ for $t > 0$ for the initial conditions $x = 0$ and $v = 0$ at $t = 0$. (**b**) Find $x(t)$ for $\omega = \omega_0$ by taking the limit $\omega \rightarrow \omega_0$ in your result for part (**a**). Sketch your result for $x(t)$.
Hint: In part (**a**) look for a particular solution of the differential equation of the form $x = A \sin(\omega t)$ and determine A. Add the solution of the homogeneous equation to this to obtain the general solution of the inhomogeneous equation.

3-43. A point mass m slides without friction on a horizontal table at one end of a massless spring of natural length a and spring constant k as shown in Figure 3-C. The spring is attached to the table so it can rotate freely without friction. The net force on the mass is the central force $F(r) = -k(r - a)$. (**a**) Find and sketch both the potential energy $U(r)$ and the effective potential $U_{\text{eff}}(r)$. (**b**) What angular velocity ω_0 is required for a circular orbit with radius r_0? (**c**) Derive the frequency of small oscillations ω about the circular orbit with radius r_0. Express your answers for (**b**) and (**c**) in terms of k, m, r_0, and a.

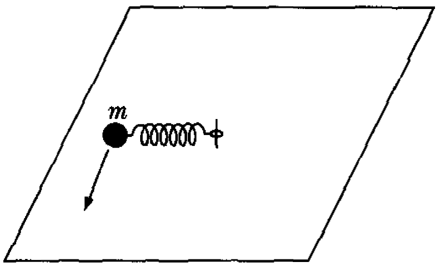

FIGURE 3-C Problem 3-43.

3-44. Consider a damped harmonic oscillator. After four cycles the amplitude of the oscillator has dropped to $1/e$ of its initial value. Find the ratio of the frequency of the damped oscillator to its natural frequency.

3-45. A grandfather clock has a pendulum length of 0.7 m and mass bob of 0.4 kg. A mass of 2 kg falls 0.8 m in seven days to keep the amplitude (from equilibrium) of the pendulum oscillation steady at 0.03 rad. What is the Q of the system?

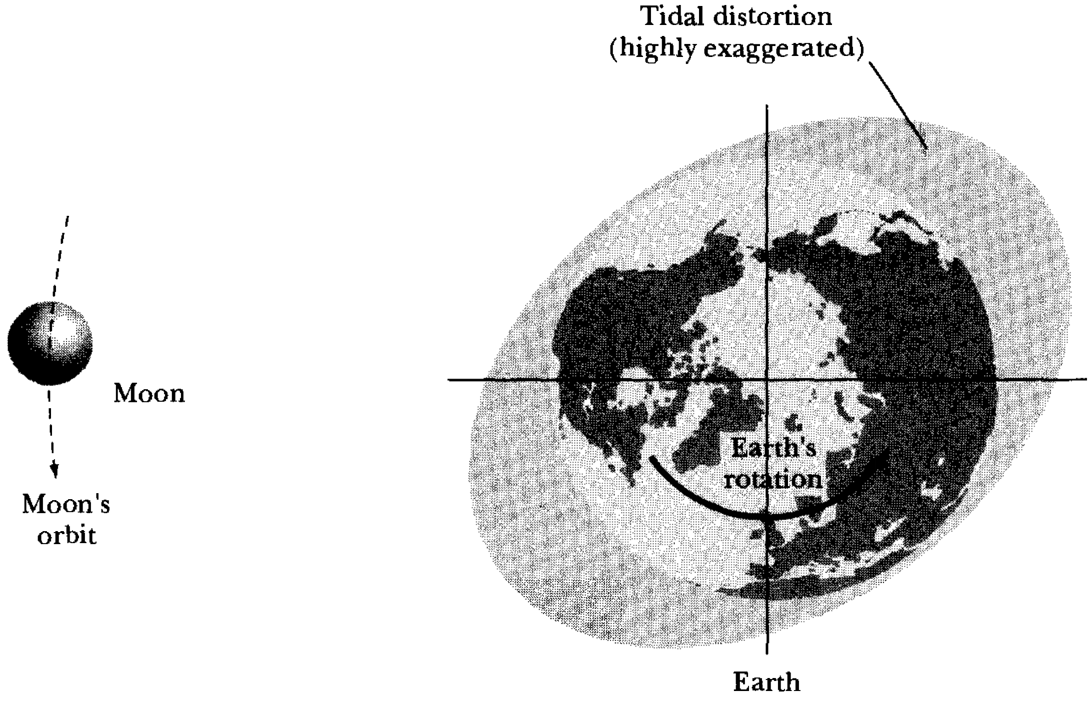

FIGURE 5-13 Some effects cause the high tides to not be exactly along the Earth-Moon axis.

Tidal friction between water and Earth leads to a significant amount of energy loss on Earth. Earth is not rigid, and it is also distorted by tidal forces.

In addition to the effects just discussed, remember that as Earth rotates, the Moon is also orbiting Earth. This leads to the result that there are not quite exactly two high tides per day, because they occur once every 12 h and 26 min (Problem 5-19). The plane of the moon's orbit about Earth is also not perpendicular to Earth's rotation axis. This causes one high tide each day to be slightly higher than the other. The tidal friction between water and land mentioned previously also results in Earth "dragging" the ocean with it as Earth rotates. This causes the high tides to be not quite along the Earth-Moon axis, but rather several degrees apart as shown in Figure 5-13.

PROBLEMS

5-1. Sketch the equipotential surfaces and the lines of force for two point masses separated by a certain distance. Next, consider one of the masses to have a fictitious negative mass $-M$. Sketch the equipotential surfaces and lines of force for this case. To what kind of physical situation does this set of equipotentials and field lines apply? (Note that the lines of force have *direction*; indicate this with appropriate arrows.)

5-2. If the field vector is independent of the radial distance within a sphere, find the function describing the density $\rho = \rho(r)$ of the sphere.

5-3. Assuming that air resistance is unimportant, calculate the minimum velocity a particle must have at the surface of Earth to escape from Earth's gravitational field. Obtain a numerical value for the result. (This velocity is called the *escape velocity.*)

5-4. A particle at rest is attracted toward a center of force according to the relation $F = -mk^2/x^3$. Show that the time required for the particle to reach the force center from a distance d is d^2/k.

5-5. A particle falls to Earth starting from rest at a great height (many times Earth's radius). Neglect air resistance and show that the particle requires approximately $\frac{9}{11}$ of the total time of fall to traverse the first half of the distance.

5-6. Compute directly the gravitational force on a unit mass at a point exterior to a homogeneous sphere of matter.

5-7. Calculate the gravitational potential due to a thin rod of length l and mass M at a distance R from the center of the rod and in a direction perpendicular to the rod.

5-8. Calculate the gravitational field vector due to a homogeneous cylinder at exterior points on the axis of the cylinder. Perform the calculation **(a)** by computing the force directly and **(b)** by computing the potential first.

5-9. Calculate the potential due to a thin circular ring of radius a and mass M for points lying in the plane of the ring and exterior to it. The result can be expressed as an elliptic integral.* Assume that the distance from the center of the ring to the field point is large compared with the radius of the ring. Expand the expression for the potential and find the first correction term.

5-10. Find the potential at off-axis points due to a thin circular ring of radius a and mass M. Let R be the distance from the center of the ring to the field point, and let θ be the angle between the line connecting the center of the ring with the field point and the axis of the ring. Assume $R \gg a$ so that terms of order $(a/R)^3$ and higher may be neglected.

5-11. Consider a massive body of arbitrary shape and a spherical surface that is exterior to and does not contain the body. Show that the average value of the potential due to the body taken over the spherical surface is equal to the value of the potential at the center of the sphere.

5-12. In the previous problem, let the massive body be inside the spherical surface. Now show that the average value of the potential over the surface of the sphere is equal to the value of the potential that would exist on the surface of the sphere if all the mass of the body were concentrated at the center of the sphere.

5-13. A planet of density ρ_1 (spherical core, radius R_1) with a thick spherical cloud of dust (density ρ_2, radius R_2) is discovered. What is the force on a particle of mass m placed within the dust cloud?

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*See Appendix B for a list of some elliptic integrals.

5-14. Show that the gravitational self-energy (energy of assembly piecewise from infinity) of a uniform sphere of mass M and radius R is

$$U = -\frac{3}{5}\frac{GM^2}{R}$$

5-15. A particle is dropped into a hole drilled straight through the center of Earth. Neglecting rotational effects, show that the particle's motion is simple harmonic if you assume Earth has uniform density. Show that the period of the oscillation is about 84 min.

5-16. A uniformly solid sphere of mass M and radius R is fixed a distance h above a thin infinite sheet of mass density ρ_s (mass/area). With what force does the sphere attract the sheet?

5-17. Newton's model of the tidal height, using the two water wells dug to the center of Earth, used the fact that the pressure at the bottom of the two wells should be the same. Assume water is incompressible and find the tidal height difference h, Equation 5.55, due to the Moon using this model. (*Hint*: $\int_0^{y_{max}} \rho g_y\, dy = \int_0^{x_{max}} \rho g_x\, dx$; $h = x_{max} - y_{max}$, where $x_{max} + y_{max} = 2R_{earth}$, and R_{earth} is Earth's median radius.)

5-18. Show that the ratio of maximum tidal heights due to the Moon and Sun is given by

$$\frac{M_m}{M_s}\left(\frac{R_{Es}}{D}\right)^3$$

and that this value is 2.2. R_{Es} is the distance between the Sun and Earth, and M_s is the Sun's mass.

5-19. The orbital revolution of the Moon about Earth takes about 27.3 days and is in the same direction as Earth's rotation (24 h). Use this information to show that high tides occur everywhere on Earth every 12 h and 26 min.

5-20. A thin disk of mass M and radius R lies in the (x, y) plane with the z-axis passing through the center of the disk. Calculate the gravitational potential $\Phi(z)$ and the gravitational field $\mathbf{g}(z) = -\nabla\Phi(z) = -\hat{\mathbf{k}}\, d\Phi(z)/dz$ on the z-axis.

5-21. A point mass m is located a distance D from the nearest end of a thin rod of mass M and length L along the axis of the rod. Find the gravitational force exerted on the point mass by the rod.

PROBLEMS

6-1. Consider the line connecting $(x_1, y_1) = (0, 0)$ and $(x_2, y_2) = (1, 1)$. Show explicitly that the function $y(x) = x$ produces a minimum path length by using the varied function $y(\alpha, x) = x + \alpha \sin \pi(1 - x)$. Use the first few terms in the expansion of the resulting elliptic integral to show the equivalent of Equation 6.4.

6-2. Show that the shortest distance between two points on a plane is a straight line.

6-3. Show that the shortest distance between two points in (three-dimensional) space is a straight line.

6-4. Show that the geodesic on the surface of a right circular cylinder is a segment of a helix.

6-5. Consider the surface generated by revolving a line connecting two fixed points (x_1, y_1) and (x_2, y_2) about an axis coplanar with the two points. Find the equation of the line connecting the points such that the surface area generated by the revolution (i.e., the area of the surface of revolution) is a minimum. Obtain the solution by using Equation 6.39.

6-6. Reexamine the problem of the brachistochrone (Example 6.2) and show that the time required for a particle to move (frictionlessly) to the *minimum* point of the cycloid is $\pi\sqrt{a/g}$, *independent* of the starting point.

6-7. Consider light passing from one medium with index of refraction n_1 into another medium with index of refraction n_2 (Figure 6-A). Use Fermat's principle to minimize time, and derive the law of refraction: $n_1 \sin \theta_1 = n_2 \sin \theta_2$.

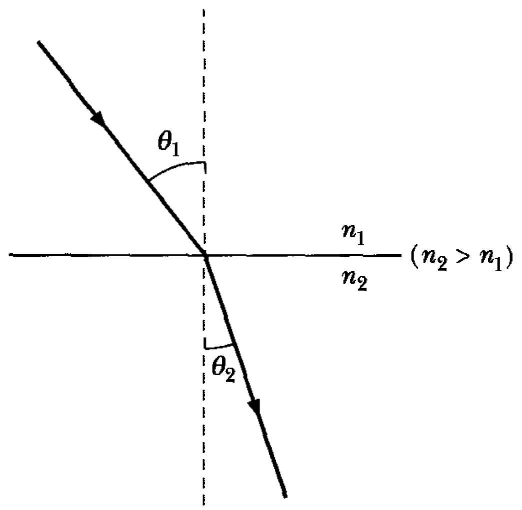

FIGURE 6-A Problem 6-7.

6-8. Find the dimensions of the parallelepiped of maximum volume circumscribed by **(a)** a sphere of radius R; **(b)** an ellipsoid with semiaxes a, b, c.

6-9. Find an expression involving the function $\phi(x_1, x_2, x_3)$ that has a minimum average value of the square of its gradient within a certain volume V of space.

6-10. Find the ratio of the radius R to the height H of a right-circular cylinder of fixed volume V that minimizes the surface area A.

6-11. A disk of radius R rolls without slipping inside the parabola $y = ax^2$. Find the equation of constraint. Express the condition that allows the disk to roll so that it contacts the parabola at one and only one point, independent of its position.

6-12. Repeat Example 6.4, finding the shortest path between any two points on the surface of a sphere, but use the method of the Euler equations with an auxiliary condition imposed.

6-13. Repeat Example 6.6 but do not use the constraint that the $y = 0$ line is the bottom part of the area. Show that the plane curve of a given length, which encloses a maximum area, is a circle.

6-14. Find the shortest path between the (x, y, z) points $(0, -1, 0)$ and $(0, 1, 0)$ on the conical surface $z = 1 - \sqrt{x^2 + y^2}$. What is the length of the path? Note: this is the shortest mountain path around a volcano.

6-15. **(a)** Find the curve $y(x)$ that passes through the endpoints $(0, 0)$ and $(1, 1)$ and minimizes the functional $I[y] = \int_0^1 [(dy/dx)^2 - y^2]\,dx$. **(b)** What is the minimum value of the integral? **(c)** Evaluate $I[y]$ for a straight line $y = x$ between the points $(0, 0)$ and $(1, 1)$.

6-16. **(a)** What curve on the surface $z = x^{3/2}$ joining the points $(x, y, z) = (0, 0, 0)$ and $(1, 1, 1)$ has the shortest arc length? **(b)** Use a computer to produce a plot showing the surface and the shortest curve on a single plot.

6-17. The corners of a rectangle lie on the ellipse $(x/a)^2 + (y/b)^2 = 1$. **(a)** Where should the corners be located in order to maximize the area of the rectangle? **(b)** What fraction of the area of the ellipse is covered by the rectangle with maximum area?

6-18. A particle of mass m is constrained to move under gravity with no friction on the surface $xy = z$. What is the trajectory of the particle if it starts from rest at $(x, y, z) = (1, -1, -1)$ with the z-axis vertical?

If the particles have a gravitational interaction, then $n = -2$, and

$$\langle T \rangle = -\frac{1}{2}\langle U \rangle, \quad n = -2$$

This relation is useful in calculating, for example, the energetics in planetary motion.

PROBLEMS

7-1. A disk rolls without slipping across a horizontal plane. The plane of the disk remains vertical, but it is free to rotate about a vertical axis. What generalized coordinates may be used to describe the motion? Write a differential equation describing the rolling constraint. Is this equation integrable? Justify your answer by a physical argument. Is the constraint holonomic?

7-2. Work out Example 7.6 showing all the steps, in particular those leading to Equations 7.36 and 7.41. Explain why the sign of the acceleration a cannot affect the frequency ω. Give an argument why the signs of a^2 and g^2 in the solution of ω^2 in Equation 7.42 are the same.

7-3. A sphere of radius ρ is constrained to roll without slipping on the lower half of the inner surface of a hollow cylinder of inside radius R. Determine the Lagrangian function, the equation of constraint, and Lagrange's equations of motion. Find the frequency of small oscillations.

7-4. A particle moves in a plane under the influence of a force $f = -Ar^{\alpha-1}$ directed toward the origin; A and α (> 0) are constants. Choose appropriate generalized coordinates, and let the potential energy be zero at the origin. Find the Lagrangian equations of motion. Is the angular momentum about the origin conserved? Is the total energy conserved?

7-5. Consider a vertical plane in a constant gravitational field. Let the origin of a coordinate system be located at some point in this plane. A particle of mass m moves in the vertical plane under the influence of gravity and under the influence of an additional force $f = -Ar^{\alpha-1}$ directed toward the origin (r is the distance from the origin; A and α [$\neq 0$ or 1] are constants). Choose appropriate generalized coordinates, and find the Lagrangian equations of motion. Is the angular momentum about the origin conserved? Explain.

7-6. A hoop of mass m and radius R rolls without slipping down an inclined plane of mass M, which makes an angle α with the horizontal. Find the Lagrange equations and the integrals of the motion if the plane can slide without friction along a horizontal surface.

7-7. A double pendulum consists of two simple pendula, with one pendulum suspended from the bob of the other. If the two pendula have equal lengths and have bobs of equal mass and if both pendula are confined to move in the same plane, find Lagrange's equations of motion for the system. Do not assume small angles.

7-8. Consider a region of space divided by a plane. The potential energy of a particle in region 1 is U_1 and in region 2 it is U_2. If a particle of mass m and with speed v_1 in region 1 passes from region 1 to region 2 such that its path in region 1 makes an angle θ_1 with the normal to the plane of separation and an angle θ_2 with the normal when in region 2, show that

$$\frac{\sin \theta_1}{\sin \theta_2} = \left(1 + \frac{U_1 - U_2}{T_1}\right)^{1/2}$$

where $T_1 = \frac{1}{2}mv_1^2$. What is the optical analog of this problem?

7-9. A disk of mass M and radius R rolls without slipping down a plane inclined from the horizontal by an angle α. The disk has a short weightless axle of negligible radius. From this axis is suspended a simple pendulum of length $l < R$ and whose bob has a mass m. Consider that the motion of the pendulum takes place in the plane of the disk, and find Lagrange's equations for the system.

7-10. Two blocks, each of mass M, are connected by an extensionless, uniform string of length l. One block is placed on a smooth horizontal surface, and the other block hangs over the side, the string passing over a frictionless pulley. Describe the motion of the system **(a)** when the mass of the string is negligible and **(b)** when the string has a mass m.

7-11. A particle of mass m is constrained to move on a circle of radius R. The circle rotates in space about one point on the circle, which is fixed. The rotation takes place in the plane of the circle and with constant angular speed ω. In the absence of a gravitational force, show that the particle's motion about one end of a diameter passing through the pivot point and the center of the circle is the same as that of a plane pendulum in a uniform gravitational field. Explain why this is a reasonable result.

7-12. A particle of mass m rests on a smooth plane. The plane is raised to an inclination angle θ at a constant rate α ($\theta = 0$ at $t = 0$), causing the particle to move down the plane. Determine the motion of the particle.

7-13. A simple pendulum of length b and bob with mass m is attached to a massless support moving horizontally with constant acceleration a. Determine **(a)** the equations of motion and **(b)** the period for small oscillations.

7-14. A simple pendulum of length b and bob with mass m is attached to a massless support moving vertically upward with constant acceleration a. Determine **(a)** the equations of motion and **(b)** the period for small oscillations.

7-15. A pendulum consists of a mass m suspended by a massless spring with unextended length b and spring constant k. Find Lagrange's equations of motion.

7-16. The point of support of a simple pendulum of mass m and length b is driven horizontally by $x = a \sin \omega t$. Find the pendulum's equation of motion.

7-17. A particle of mass m can slide freely along a wire AB whose perpendicular distance to the origin O is h (see Figure 7-A, page 282). The line OC rotates about the origin

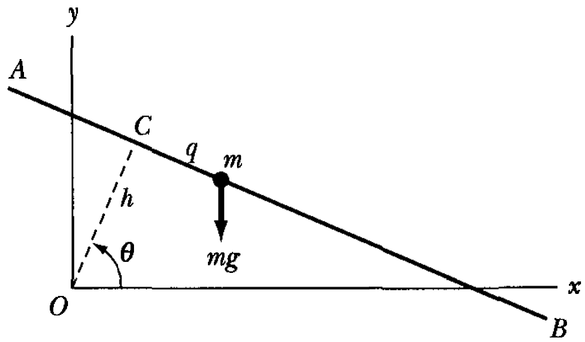

FIGURE 7-A Problem 7-17.

at a constant angular velocity $\dot{\theta} = \omega$. The position of the particle can be described in terms of the angle θ and the distance q to the point C. If the particle is subject to a gravitational force, and if the initial conditions are

$$\theta(0) = 0, \quad q(0) = 0, \quad \dot{q}(0) = 0$$

show that the time dependence of the coordinate q is

$$q(t) = \frac{g}{2\omega^2} (\cosh \omega t - \cos \omega t)$$

Sketch this result. Compute the Hamiltonian for the system, and compare with the total energy. Is the total energy conserved?

7-18. A pendulum is constructed by attaching a mass m to an extensionless string of length l. The upper end of the string is connected to the uppermost point on a vertical disk of radius R ($R < l/\pi$) as in Figure 7-B. Obtain the pendulum's equation of motion, and find the frequency of small oscillations. Find the line about which the angular motion extends equally in either direction (i.e., $\theta_1 = \theta_2$).

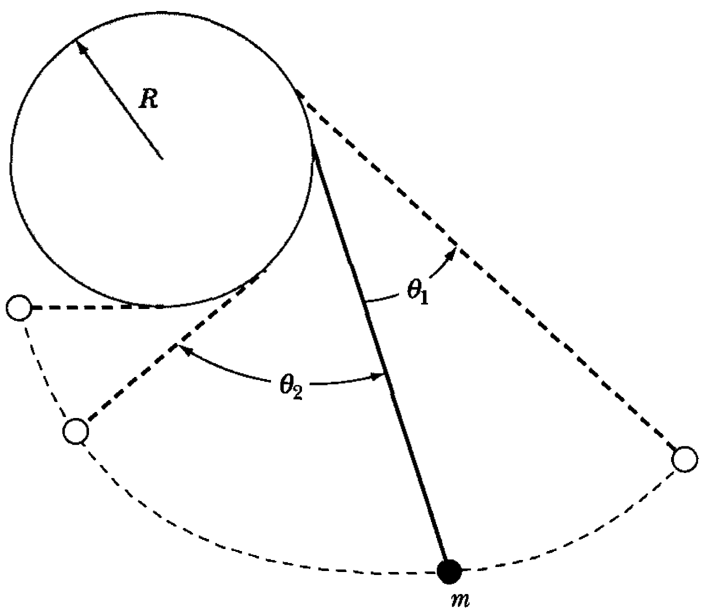

FIGURE 7-B Problem 7-18.

7-19. Two masses m_1 and m_2 ($m_1 \neq m_2$) are connected by a rigid rod of length d and of negligible mass. An extensionless string of length l_1 is attached to m_1 and connected to a fixed point of support P. Similarly, a string of length l_2 ($l_1 \neq l_2$) connects m_2 and P. Obtain the equation describing the motion in the plane of m_1, m_2, and P, and find the frequency of small oscillations around the equilibrium position.

7-20. A circular hoop is suspended in a horizontal plane by three strings, each of length l, which are attached symmetrically to the hoop and are connected to fixed points lying in a plane above the hoop. At equilibrium, each string is vertical. Show that the frequency of small rotational oscillations about the vertical through the center of the hoop is the same as that for a simple pendulum of length l.

7-21. A particle is constrained to move (without friction) on a circular wire rotating with constant angular speed ω about a vertical diameter. Find the equilibrium position of the particle, and calculate the frequency of small oscillations around this position. Find and interpret physically a critical angular velocity $\omega = \omega_c$ that divides the particle's motion into two distinct types. Construct phase diagrams for the two cases $\omega < \omega_c$ and $\omega > \omega_c$.

7-22. A particle of mass m moves in one dimension under the influence of a force

$$F(x, t) = \frac{k}{x^2} e^{-(t/\tau)}$$

where k and τ are positive constants. Compute the Lagrangian and Hamiltonian functions. Compare the Hamiltonian and the total energy, and discuss the conservation of energy for the system.

7-23. Consider a particle of mass m moving freely in a conservative force field whose potential function is U. Find the Hamiltonian function, and show that the canonical equations of motion reduce to Newton's equations. (Use rectangular coordinates.)

7-24. Consider a simple plane pendulum consisting of a mass m attached to a string of length l. After the pendulum is set into motion, the length of the string is shortened at a constant rate

$$\frac{dl}{dt} = -\alpha = \text{constant}$$

The suspension point remains fixed. Compute the Lagrangian and Hamiltonian functions. Compare the Hamiltonian and the total energy, and discuss the conservation of energy for the system.

7-25. A particle of mass m moves under the influence of gravity along the helix $z = k\theta$, $r =$ constant, where k is a constant and z is vertical. Obtain the Hamiltonian equations of motion.

7-26. Determine the Hamiltonian and Hamilton's equations of motion for **(a)** a simple pendulum and **(b)** a simple Atwood machine (single pulley).

7-27. A massless spring of length b and spring constant k connects two particles of masses m_1 and m_2. The system rests on a smooth table and may oscillate and rotate.

(a) Determine Lagrange's equations of motion.
(b) What are the generalized momenta associated with any cyclic coordinates?
(c) Determine Hamilton's equations of motion.

7-28. A particle of mass m is attracted to a force center with the force of magnitude k/r^2. Use plane polar coordinates and find Hamilton's equations of motion.

7-29. Consider the pendulum described in Problem 7-15. The pendulum's point of support rises vertically with constant acceleration a.
(a) Use the Lagrangian method to find the equations of motion.
(b) Determine the Hamiltonian and Hamilton's equations of motion.
(c) What is the period of small oscillations?

7-30. Consider any two continuous functions of the generalized coordinates and momenta $g(q_k, p_k)$ and $h(q_k, p_k)$. The **Poisson brackets** are defined by

$$[g, h] \equiv \sum_k \left(\frac{\partial g}{\partial q_k}\frac{\partial h}{\partial p_k} - \frac{\partial g}{\partial p_k}\frac{\partial h}{\partial q_k} \right)$$

Verify the following properties of the Poisson brackets:

(a) $\dfrac{dg}{dt} = [g, H] + \dfrac{\partial g}{dt}$ **(b)** $\dot{q}_j = [q_i, H],\ \dot{p}_j = [p_j, H]$

(c) $[p_l, p_j] = 0,\ [q_l, q_j] = 0$ **(d)** $[q_l, p_j] = \delta_{lj}$

where H is the Hamiltonian. If the Poisson bracket of two quantities vanishes, the quantities are said to *commute*. If the Poisson bracket of two quantities equals unity, the quantities are said to be *canonically conjugate*. **(e)** Show that any quantity that does not depend explicitly on the time and that commutes with the Hamiltonian is a constant of the motion of the system. Poisson-bracket formalism is of considerable importance in quantum mechanics.

7-31. A spherical pendulum consists of a bob of mass m attached to a weightless, extensionless rod of length l. The end of the rod opposite the bob pivots freely (in all directions) about some fixed point. Set up the Hamiltonian function in spherical coordinates. (If $p_\phi = 0$, the result is the same as that for the plane pendulum.) Combine the term that depends on p_ϕ with the ordinary potential energy term to define as *effective* potential $V(\theta, p_\phi)$. Sketch V as a function of θ for several values of p_ϕ, including $p_\phi = 0$. Discuss the features of the motion, pointing out the differences between $p_\phi = 0$ and $p_\phi \neq 0$. Discuss the limiting case of the conical pendulum (θ = constant) with reference to the V-θ diagram.

7-32. A particle moves in a spherically symmetric force field with potential energy given by $U(r) = -k/r$. Calculate the Hamiltonian function in spherical coordinates, and obtain the canonical equations of motion. Sketch the path that a representative point for the system would follow on a surface H = constant in phase space. Begin by showing that the motion must lie in a plane so that the phase space is four dimensional (r, θ, p_r, p_θ, but only the first three are nontrivial). Calculate the projection of the phase path on the r-p_r plane, then take into account the variation with θ.

7-33. Determine the Hamiltonian and Hamilton's equations of motion for the double Atwood machine of Example 7.8.

7-34. A particle of mass m slides down a smooth circular wedge of mass M as shown in Figure 7-C. The wedge rests on a smooth horizontal table. Find (**a**) the equation of motion of m and M and (**b**) the reaction of the wedge on m.

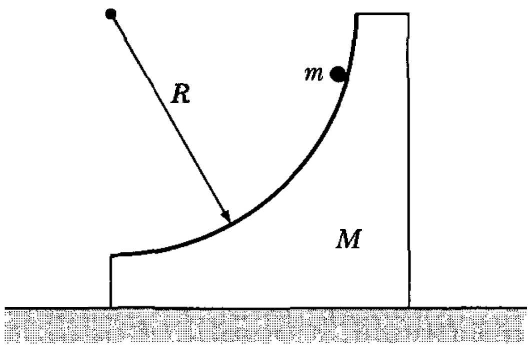

FIGURE 7-C Problem 7-34.

7-35. Four particles are directed upward in a uniform gravitational field with the following initial conditions:

$$\begin{aligned}
&(1)\quad z(0) = z_0; && p_z(0) = p_0\\
&(2)\quad z(0) = z_0 + \Delta z_0; && p_z(0) = p_0\\
&(3)\quad z(0) = z_0; && p_z(0) = p_0 + \Delta p_0\\
&(4)\quad z(0) = z_0 + \Delta z_0; && p_z(0) = p_0 + \Delta p_0
\end{aligned}$$

Show by direct calculation that the representative points corresponding to these particles always define an area in phase space equal to $\Delta z_0\, \Delta p_0$. Sketch the phase paths, and show for several times $t > 0$ the shape of the region whose area remains constant.

7-36. Discuss the implications of Liouville's theorem on the focusing of beams of charged particles by considering the following simple case. An electron beam of circular cross section (radius R_0) is directed along the z-axis. The density of electrons across the beam is constant, but the momentum components transverse to the beam (p_x and p_y) are distributed uniformly over a circle of radius p_0 in momentum space. If some focusing system reduces the beam radius from R_0 to R_1, find the resulting distribution of the transverse momentum components. What is the physical meaning of this result? (Consider the angular divergence of the beam.)

7-37. Use the method of Lagrange undetermined multipliers to find the tensions in both strings of the double Atwood machine of Example 7.8.

7-38. The potential for an anharmonic oscillator is $U = kx^2/2 + bx^4/4$ where k and b are constants. Find Hamilton's equations of motion.

7-39. An extremely limber rope of uniform mass density, mass m and total length b lies on a table with a length z hanging over the edge of the table. Only gravity acts on the rope. Find Lagrange's equation of motion.

7-40. A double pendulum is attached to a cart of mass $2m$ that moves without friction on a horizontal surface. See Figure 7-D. Each pendulum has length b and mass bob m. Find the equations of motion.

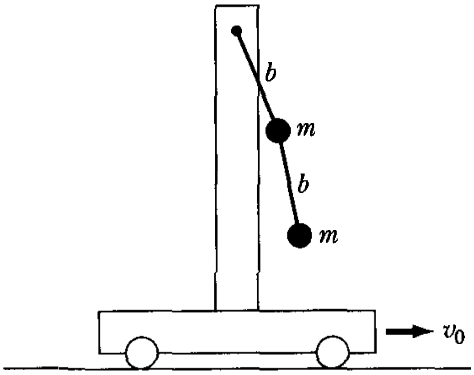

FIGURE 7-D Problem 7-40.

7-41. A pendulum of length b and mass bob m is oscillating at small angles when the length of the pendulum string is shortened at a velocity of α $(db/dt = -\alpha)$. Find Lagrange's equations of motion.

We integrate the last term using the definite integral, $\int \ln x\, dx = x \ln x - x$, to obtain after collecting terms,

$$H_{bo} = -\frac{g(m_0 - m_f)^2}{2\alpha^2} + \frac{u}{\alpha}\left[m_f \ln\left(\frac{m_f}{m_0}\right) + m_0 - m_f\right] \tag{9.167}$$

If we insert the numbers from the last example, we find the same answer for the burnout height.

The speed at burnout can be determined directly from Equation 9.165.

$$v_{bo} = -gT + u \ln\left(\frac{m_0}{m_f}\right)$$

$$= -\frac{g(m_0 - m_f)}{\alpha} + u \ln\left(\frac{m_0}{m_f}\right) \tag{9.168}$$

PROBLEMS

9-1. Find the center of mass of a hemispherical shell of constant density and inner radius r_1 and outer radius r_2.

9-2. Find the center of mass of a uniformly solid cone of base diameter $2a$ and height h.

9-3. Find the center of mass of a uniformly solid cone of base diameter $2a$ and height h and a solid hemisphere of radius a where the two bases are touching.

9-4. Find the center of mass of a uniform wire that subtends an arc θ if the radius of the circular arc is a, as shown in Figure 9-A.

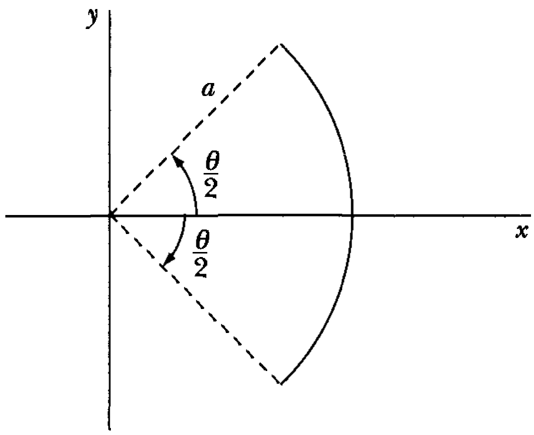

FIGURE 9-A Problem 9-4.

9-5. The center of gravity of a system of particles is the point about which external gravitational forces exert no net torque. For a uniform gravitational force, show that the center of gravity is identical to the center of mass for the system of particles.

9-6. Consider two particles of equal mass m. The forces on the particles are $\mathbf{F}_1 = 0$ and $\mathbf{F}_2 = F_0\mathbf{i}$. If the particles are initially at rest at the origin, what is the position, velocity, and acceleration of the center of mass?

9-7. A model of the water molecule H_2O is shown in Figure 9-B. Where is the center of mass?

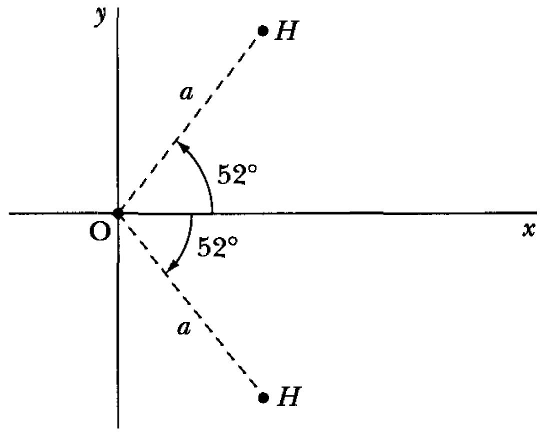

FIGURE 9-B Problem 9-7.

9-8. Where is the center of mass of the isosceles right triangle of uniform areal density shown in Figure 9-C?

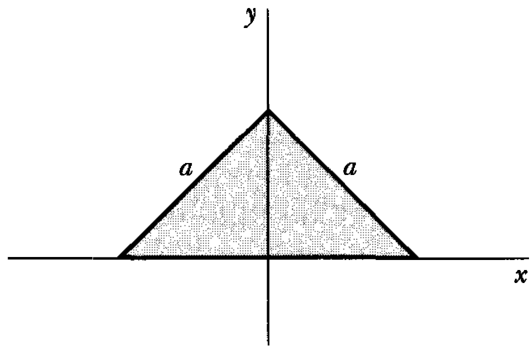

FIGURE 9-C Problem 9-8.

9-9. A projectile is fired at an angle of 45° with initial kinetic energy E_0. At the top of its trajectory, the projectile explodes with additional energy E_0 into two fragments. One fragment of mass m_1 travels straight down. What is the velocity (magnitude and direction) of the second fragment of mass m_2 and the velocity of the first? What is the ratio of m_1/m_2 when m_1 is a maximum?

9-10. A cannon in a fort overlooking the ocean fires a shell of mass M at an elevation angle θ and muzzle velocity v_0. At the highest point, the shell explodes into two fragments (masses $m_1 + m_2 = M$), with an additional energy E, traveling in the original horizontal direction. Find the distance separating the two fragments when they land in the ocean. For simplicity, assume the cannon is at sea level.

9-11. Verify that the second term on the right-hand side of Equation 9.9 indeed vanishes for the case $n = 3$.

9-12. Astronaut Stumblebum wanders too far away from the space shuttle orbiter while repairing a broken communications satellite. Stumblebum realizes that the orbiter is moving away from him at 3 m/s. Stumblebum and his maneuvering unit have a mass of 100 kg, including a pressurized tank of mass 10 kg. The tank includes only 2 kg of gas that is used to propel him in space. The gas escapes with a constant velocity of 100 m/s.

(a) Will Stumblebum run out of gas before he reaches the orbiter?

(b) With what velocity will Stumblebum have to throw the empty tank away to reach the orbiter?

9-13. Even though the total force on a system of particles (Equation 9.9) is zero, the net torque may not be zero. Show that the net torque has the same value in any coordinate system.

9-14. Consider a system of particles interacting by magnetic forces. Are Equations 9.11 and 9.31 valid? Explain.

9-15. A smooth rope is placed above a hole in a table (Figure 9-D). One end of the rope falls through the hole at $t = 0$, pulling steadily on the remainder of the rope. Find the velocity and acceleration of the rope as a function of the distance to the end of the rope x. Ignore all friction. The total length of the rope is L.

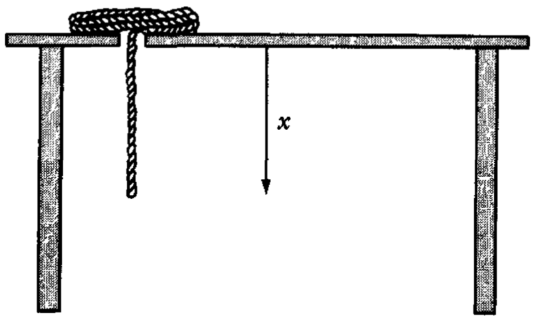

FIGURE 9-D Problem 9-15.

9-16. For the energy-conserving case of the falling chain in Example 9.2, show that the tension on either side of the bottom bend is equal and has the value $\rho\dot{x}^2/4$.

9-17. Integrate Equation 9.17 in Example 9.2 numerically and make a plot of the speed versus the time using dimensionless parameters, $\dot{x}/\sqrt{2gb}$ vs. $t/\sqrt{2b/g}$ where $\sqrt{2b/g}$ is the free fall time, $t_{\text{free fall}}$. Find the time it takes for the free end to reach the bottom. Define natural units by $\tau \equiv t\sqrt{g/2b}$, $\alpha \equiv x/2b$ and integrate $d\tau/d\alpha$ from $\alpha = \varepsilon$ (some small number greater than 0) to $\alpha = 1/2$. One can't integrate numerically from $\alpha = 0$ because of a singularity in $d\tau/d\alpha$. The expression $d\tau/d\alpha$ is

$$\frac{d\tau}{d\alpha} = \sqrt{\frac{1 - 2\alpha}{2\alpha(1 - \alpha)}}$$

9-18. Use a computer to make a plot of the tension versus time for the falling chain in Example 9.2. Use dimensionless parameters (T/Mg) versus $t/t_{\text{free fall}}$, where $t_{\text{free fall}} = \sqrt{2b/g}$. Stop the plot before T/Mg becomes greater than 50.

9-19. A chain such as the one in Example 9.2 (with the same parameters) of length b and mass ρb is suspended from one end at a point that is a height b above a table so that the free end barely touches the tabletop. At time $t = 0$, the fixed end of the chain is released. Find the force that the tabletop exerts on the chain after the original fixed end has fallen a distance x.

9-20. A uniform rope of total length $2a$ hangs in equilibrium over a smooth nail. A very small impulse causes the rope to slowly roll off the nail. Find the velocity of the rope as it just clears the nail. Assume the rope is prevented from lifting off the nail and is in free fall.

9-21. A flexible rope of length 1.0 m slides from a frictionless table top as shown in Figure 9-E. The rope is initially released from rest with 30 cm hanging over the edge of the table. Find the time at which the left end of the rope reaches the edge of the table.

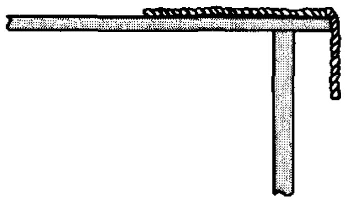

FIGURE 9-E Problem 9-21.

9-22. A deuteron (nucleus of deuterium atom consisting of a proton and a neutron) with speed 14.9 km/s collides elastically with a neutron at rest. Use the approximation that the deuteron is twice the mass of the neutron. (**a**) If the deuteron is scattered through a LAB angle $\psi = 10°$, what are the final speeds of the deuteron and neutron? (**b**) What is the LAB scattering angle of the neutron? (**c**) What is the maximum possible scattering angle of the deuteron?

9-23. A particle of mass m_1 and velocity u_1 collides with a particle of mass m_2 at rest. The two particles stick together. What fraction of the original kinetic energy is lost in the collision?

9-24. A particle of mass m at the end of a light string wraps itself about a fixed vertical cylinder of radius a (Figure 9-F). All the motion is in the horizontal plane (disregard gravity). The angular velocity of the cord is ω_0 when the distance from the particle to the point of contact of the string and cylinder is b. Find the angular velocity and tension in the string after the cord has turned through an additional angle θ.

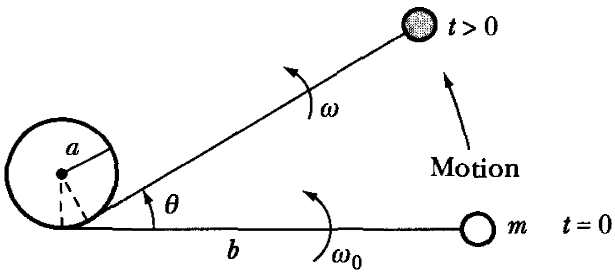

FIGURE 9-F Problem 9-24.

9-25. Slow-moving neutrons have a much larger absorption rate in ^{235}U than fast neutrons produced by $^{235}U^*$ fission in a nuclear reactor. For that reason, reactors consist of moderators to slow down neutrons by elastic collisions. What elements are best to be used as moderators? Explain.

9-26. The force of attraction between two particles is given by

$$\mathbf{f}_{12} = k\left[(\mathbf{r}_2 - \mathbf{r}_1) - \frac{r}{v_0}(\dot{\mathbf{r}}_2 - \dot{\mathbf{r}}_1)\right]$$

where k is a constant, v_0 is a constant velocity, and $r \equiv |\mathbf{r}_2 - \mathbf{r}_1|$. Calculate the internal torque for the system; why does this quantity not vanish? Is the system conservative?

9-27. Derive Equation 9.90.

9-28. A particle of mass m_1 elastically collides with a particle of mass m_2 at rest. What is the maximum fraction of kinetic energy loss for m_1? Describe the reaction.

9-29. Derive Equation 9.91.

9-30. A tennis player strikes an incoming tennis ball of mass 60 g as shown in Figure 9-G. The incoming tennis ball velocity is $v_i = 8$ m/s, and the outgoing velocity is $v_f = 16$ m/s.
(**a**) What impulse was given to the tennis ball?
(**b**) If the collision time was 0.01 s, what was the average force exerted by the tennis racket?

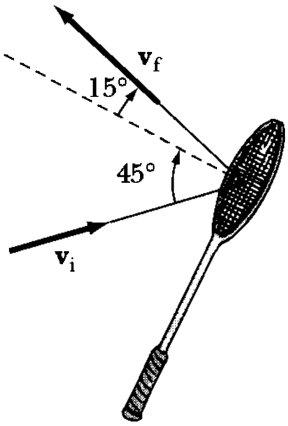

FIGURE 9-G Problem 9-30.

9-31. Derive Equation 9.92.

9-32. A particle of mass m and velocity u_1 makes a head-on collision with another particle of mass $2m$ at rest. If the coefficient of restitution is such to make the loss of total kinetic energy a maximum, what are the velocities v_1 and v_2 after the collision?

9-33. Show that T_1/T_0 can be expressed in terms of $m_2/m_1 \equiv \alpha$ and $\cos\psi \equiv y$ as

$$\frac{T_1}{T_0} = (1+\alpha)^{-2}\left(2y^2 + \alpha^2 - 1 + 2y\sqrt{\alpha^2 + y^2 - 1}\right)$$

Plot T_1/T_0 as a function of ψ for $\alpha = 1$, 2, 4, and 12. These plots correspond to the energies of protons or neutrons after scattering from hydrogen ($\alpha = 1$), deuterium ($\alpha = 2$), helium ($\alpha = 4$), and carbon ($\alpha = 12$), or of alpha particles scattered from helium ($\alpha = 1$), oxygen ($\alpha = 4$), and so forth.

9-34. A billiard ball of initial velocity u_1 collides with another billiard ball (same mass) initially at rest. The first ball moves off at $\psi = 45°$. For an elastic collision, what are the velocities of both balls after the collision? At what LAB angle does the second ball emerge?

9-35. A particle of mass m_1 with initial laboratory velocity u_1 collides with a particle of mass m_2 at rest in the LAB system. The particle m_1 is scattered through a LAB angle ψ and has a final velocity v_1, where $v_1 = v_1(\psi)$. Find the surface such that the time of travel of the scattered particle from the point of collision to the surface is independent of the scattering angle. Consider the cases (**a**) $m_2 = m_1$, (**b**) $m_2 = 2m_1$, and (**c**) $m_2 = \infty$. Suggest an application of this result in terms of a detector for nuclear particles.

9-36. In an elastic collision of two particles with masses m_1 and m_2, the initial velocities are $\mathbf{u}_1$ and $\mathbf{u}_2 = \alpha\mathbf{u}_1$. If the initial kinetic energies of the two particles are equal, find the conditions on u_1/u_2 and m_1/m_2 such that m_1 is at rest after the collision. Examine both cases for the sign of α.

9-37. When a bullet fires in a gun, the explosion subsides quickly. Suppose the force on the bullet is $F = (360 - 10^7 t^2 \text{s}^{-2})$ N until the force becomes zero (and remains zero). The mass of the bullet is 3 g.
(**a**) What impulse acts on the bullet?
(**b**) What is the muzzle velocity of the gun?

9-38. Show that

$$\frac{T_1}{T_0} = \frac{m_1^2}{(m_1 + m_2)^2} \cdot S^2$$

where

$$S \equiv \cos\psi + \frac{\cos(\theta - \psi)}{\left(\dfrac{m_1}{m_2}\right)}$$

9-39. A particle of mass m strikes a smooth wall at an angle θ from the normal. The coefficient of restitution is ε. Find the velocity and the rebound angle of the particle after leaving the wall.

9-40. A particle of mass m_1 and velocity u_1 strikes head-on a particle of mass m_2 at rest. The coefficient of restitution is ε. Particle m_2 is tied to a point a distance a away as shown in Figure 9-H. Find the velocity (magnitude and direction) of m_1 and m_2 after the collision.

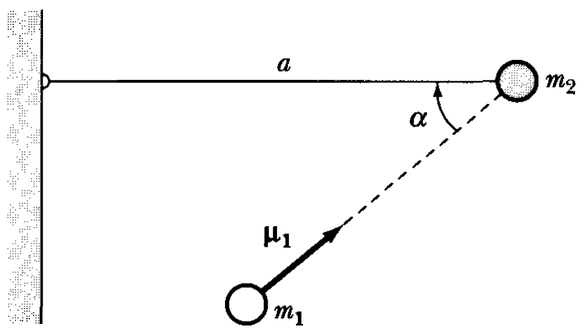

FIGURE 9-H Problem 9-40.

9-41. A rubber ball is dropped from rest onto a linoleum floor a distance h_1 away. The rubber ball bounces up to a height h_2. What is the coefficient of restitution? What fraction of the original kinetic energy is lost in terms of ε?

9-42. A steel ball of velocity 5 m/s strikes a smooth, heavy steel plate at an angle of 30° from the normal. If the coefficient of restitution is 0.8, at what angle and velocity does the steel ball bounce off the plate?

9-43. A proton (mass m) of kinetic energy T_0 collides with a helium nucleus (mass $4m$) at rest. Find the recoil angle of the helium if $\psi = 45°$ and the inelastic collision has $Q = -T_0/6$.

9-44. A uniformly dense rope of length b and mass density μ is coiled on a smooth table. One end is lifted by hand with a constant velocity v_0. Find the force of the rope held by the hand when the rope is a distance a above the table ($b > a$).

9-45. Show that the equivalent of Equation 9.129 expressed in terms of θ rather than ψ is

$$\sigma(\theta) = \sigma(\psi) \cdot \frac{1 + x\cos\theta}{(1 + 2x\cos\theta + x^2)^{3/2}}$$

9-46. Calculate the differential cross section $\sigma(\theta)$ and the total cross section σ_t for the elastic scattering of a particle from an impenetrable sphere; the potential is given by

$$U(r) = \begin{cases} 0, & r > a \\ \infty, & r < a \end{cases}$$

9-47. Show that the Rutherford scattering cross section (for the case $m_1 = m_2$) can be expressed in terms of the recoil angle as

$$\sigma_{\text{LAB}}(\zeta) = \frac{k^2}{T_0^2} \cdot \frac{1}{\cos^3\zeta}$$

9-48. Consider the case of Rutherford scattering in the event that $m_1 \gg m_2$. Obtain an approximate expression for the differential cross section in the LAB coordinate system.

9-49. Consider the case of Rutherford scattering in the event that $m_2 \gg m_1$. Obtain an expression of the differential cross section in the CM system that is correct to first order in the quantity m_1/m_2. Compare this result with Equation 9.140.

9-50. A fixed force center scatters a particle of mass m according to the force law $F(r) = k/r^3$. If the initial velocity of the particle is u_0, show that the differential scattering cross section is

$$\sigma(\theta) = \frac{k\pi^2(\pi - \theta)}{mu_0^2\theta^2(2\pi - \theta)^2 \sin\theta}$$

9-51. It is found experimentally that in the elastic scattering of neutrons by protons ($m_n \cong m_p$) at relatively low energies, the energy distribution of the recoiling protons in the LAB system is constant up to a maximum energy, which is the energy of the incident neutrons. What is the angular distribution of the scattering in the CM system?

9-52. Show that the energy distribution of particles recoiling from an elastic collision is always directly proportional to the differential scattering cross section in the CM system.

9-53. The most energetic α-particles available to Ernest Rutherford and his colleagues for the famous Rutherford scattering experiment were 7.7 MeV. For the scattering of 7.7 MeV α-particles from ^{238}U (initially at rest) at a scattering angle in the lab of 90° (all calculations are in the LAB system unless otherwise noted), find the following:
(a) the recoil scattering angle of ^{238}U.
(b) the scattering angles of the α-particle and ^{238}U in the CM system.
(c) the kinetic energies of the scattered α-particle and ^{238}U.
(d) the impact parameter b.
(e) the distance of closest approach r_{min}.
(f) the differential cross section at 90°.
(g) the ratio of the probabilities of scattering at 90° to that of 5°.

9-54. A rocket starts from rest in free space by emitting mass. At what fraction of the initial mass is the momentum a maximum?

9-55. An extremely well-constructed rocket has a mass ratio (m_0/m) of 10. A new fuel is developed that has an exhaust velocity as high as 4500 m/s. The fuel burns at a constant rate for 300 s. Calculate the maximum velocity of this single-stage rocket, assuming constant acceleration of gravity. If the escape velocity of a particle from the earth is 11.3 km/s, can a similar single-stage rocket with the same mass ratio and exhaust velocity be constructed that can reach the moon?

9-56. A water droplet falling in the atmosphere is spherical. Assume that as the droplet passes through a cloud, it acquires mass at a rate equal to kA where k is a constant(>0) and A its cross-sectional area. Consider a droplet of initial radius r_0 that enters a cloud with a velocity v_0. Assume no resistive force and show **(a)** that the radius increases linearly with the time, and **(b)** that if r_0 is negligibly small then the speed increases linearly with the time within the cloud.

9-57. A rocket in outer space in a negligible gravitational field starts from rest and accelerates uniformly at a until its final speed is v. The initial mass of the rocket is m_0. How much work does the rocket's engine do?

9-58. Consider a single-stage rocket taking off from Earth. Show that the height of the rocket at burnout is given by Equation 9.166. How much farther in height will the rocket go after burnout?

9-59. A rocket has an initial mass of m and a fuel burn rate of α (Equation 9.161). What is the minimum exhaust velocity that will allow the rocket to lift off immediately after firing?

9-60. A rocket has an initial mass of 7×10^4 kg and on firing burns its fuel at a rate of 250 kg/s. The exhaust velocity is 2500 m/s. If the rocket has a vertical ascent from resting on the earth, how long after the rocket engines fire will the rocket lift off? What is wrong with the design of this rocket?

9-61. Consider a multistage rocket of n stages, each with exhaust speed u. Each stage of the rocket has the same mass ratio at burnout ($k = m_i/m_f$). Show that the final speed of the nth stage is $nu \ln k$.

9-62. To perform a rescue, a lunar landing craft needs to hover just above the surface of the moon, which has a gravitational acceleration of $g/6$. The exhaust velocity is 2000 m/s, but fuel amounting to only 20 percent of the total mass may be used. How long can the landing craft hover?

9-63. A new projectile launcher is developed in the year 2023 that can launch a 10^4 kg spherical probe with an initial speed of 6000 m/s. For testing purposes, objects are launched vertically.

(a) Neglect air resistance and assume that the acceleration of gravity is constant. Determine how high the launched object can reach above the surface of Earth.

(b) If the object has a radius of 20 cm and the air resistance is proportional to the square of the object's speed with $c_w = 0.2$, determine the maximum height reached. Assume the density of air is constant.

(c) Now also include the fact that the acceleration of gravity decreases as the object soars above Earth. Find the height reached.

(d) Now add the effects of the decrease in air density with altitude to the calculation. We can very roughly represent the air density by $\log_{10}(\rho) = -0.05h + 0.11$ where ρ is the air density in kg/m^3 and h is the altitude above Earth in km. Determine how high the object now goes.

9-64. A new single-stage rocket is developed in the year 2023, having a gas exhaust velocity of 4000 m/s. The total mass of the rocket is 10^5 kg, with 90% of its mass being fuel. The fuel burns quickly in 100 s at a constant rate. For testing purposes, the rocket is launched vertically at rest from Earth's surface. Answer parts **(a)** through **(d)** of the previous problem.

9-65. In a typical model rocket (Estes Alpha III) the Estes C6 solid rocket engine provides a total impulse of 8.5 N-s. Assume the total rocket mass at launch is 54 g and that it has a rocket engine of mass 20 g that burns evenly for 1.5 s. The rocket diameter is 24 mm. Assume a constant burn rate of the propellent mass (11 g), a rocket exhaust speed 800 m/s, vertical ascent, and drag coefficient $c_w = 0.75$. Determine

(a) The speed and altitude at engine burnout,

(b) Maximum height and time it occurs,

(c) Maximum acceleration,

(d) Total flight time, and

(e) Speed at ground impact.

Produce a plot of altitude and speed versus time. For simplicity, because the propellent mass is only 20% of the total mass, assume a constant mass during rocket burning.

9-66. For the previous problem, take into account the change of rocket mass with time and omit the effect of gravity. **(a)** Find the rocket's speed at burn out. **(b)** How far has the rocket traveled at that moment?

9-67. Complete the derivation for the burnout height H_{bo} in Example 9.13. Use the numbers for the *Saturn V* rocket in Example 9.12 and use Equations 9.167 and 9.168 to determine the height and speed at burnout.

PROBLEMS

11-1. Calculate the moments of inertia I_1, I_2, and I_3 for a homogeneous sphere of radius R and mass M. (Choose the origin at the center of the sphere.)

11-2. Calculate the moments of inertia I_1, I_2, and I_3 for a homogeneous cone of mass M whose height is h and whose base has a radius R. Choose the x_3-axis along the axis of symmetry of the cone. Choose the origin at the apex of the cone, and calculate the elements of the inertia tensor. Then make a transformation such that the center of mass of the cone becomes the origin, and find the principal moments of inertia.

11-3. Calculate the moments of inertia I_1, I_2, and I_3 for a homogeneous ellipsoid of mass M with axes' lengths $2a > 2b > 2c$.

11-4. Consider a thin rod of length l and mass m pivoted about one end. Calculate the moment of inertia. Find the point at which, if all the mass were concentrated, the moment of inertia about the pivot axis would be the same as the real moment of inertia. The distance from this point to the pivot is called the **radius of gyration.**

11-5. **(a)** Find the height at which a billiard ball should be struck so that it will roll with no initial slipping. **(b)** Calculate the optimum height of the rail of a billiard table. On what basis is the calculation predicated?

11-6. Two spheres are of the same diameter and same mass, but one is solid and the other is a hollow shell. Describe in detail a nondestructive experiment to determine which is solid and which is hollow.

11-7. A homogeneous disk of radius R and mass M rolls without slipping on a horizontal surface and is attracted to a point a distance d below the plane. If the force of attraction is proportional to the distance from the disk's center of mass to the force center, find the frequency of oscillations around the position of equilibrium.

11-8. A door is constructed of a thin homogeneous slab of material: it has a width of 1 m. If the door is opened through 90°, it is found that on release it closes itself in 2 s. Assume that the hinges are frictionless, and show that the line of hinges must make an angle of approximately 3° with the vertical.

11-9. A homogeneous slab of thickness a is placed atop a fixed cylinder of radius R whose axis is horizontal. Show that the condition for stable equilibrium of the slab, assuming no slipping, is $R > a/2$. What is the frequency of small oscillations? Sketch the potential energy U as a function of the angular displacement θ. Show that there is a minimum at $\theta = 0$ for $R > a/2$ but not for $R < a/2$.

11-10. A solid sphere of mass M and radius R rotates freely in space with an angular velocity ω about a fixed diameter. A particle of mass m, initially at one pole, moves with a constant velocity v along a great circle of the sphere. Show that, when the particle has reached the other pole, the rotation of the sphere will have been retarded by an angle

$$\alpha = \omega T\left(1 - \sqrt{\frac{2M}{2M + 5m}}\right)$$

where T is the total time required for the particle to move from one pole to the other.

11-11. A homogeneous cube, each edge of which has a length l, is initially in a position of unstable equilibrium with one edge in contact with a horizontal plane. The cube is then given a small displacement and allowed to fall. Show that the angular velocity of the cube when one face strikes the plane is given by

$$\omega^2 = A\frac{g}{l}\left(\sqrt{2} - 1\right)$$

where $A = 3/2$ if the edge cannot slide on the plane and where $A = 12/5$ if sliding can occur without friction.

11-12. Show that none of the principal moments of inertia can exceed the sum of the other two.

11-13. A three-particle system consists of masses m_i and coordinates (x_1, x_2, x_3) as follows:

$$m_1 = 3m, \quad (b, 0, b)$$

$$m_2 = 4m, \quad (b, b, -b)$$

$$m_3 = 2m, \quad (-b, b, 0)$$

Find the inertia tensor, principal axes, and principal moments of inertia.

11-14. Determine the principal axes and principal moments of inertia of a uniformly solid hemisphere of radius b and mass m about its center of mass.

11-15. If a physical pendulum has the same period of oscillation when pivoted about either of two points of unequal distances from the center of mass, show that the length of the simple pendulum with the same period is equal to the sum of separations of the pivot points from the center of mass. Such a physical pendulum, called **Kater's reversible pendulum,** at one time provided the most accurate way (to about 1 part in 10^5) to measure the acceleration of gravity.* Discuss the advantages of Kater's pendulum over a simple pendulum for such a purpose.

11-16. Consider the following inertia tensor:

$$\{\mathbf{I}\} = \begin{Bmatrix} \frac{1}{2}(A+B) & \frac{1}{2}(A-B) & 0 \\ \frac{1}{2}(A-B) & \frac{1}{2}(A+B) & 0 \\ 0 & 0 & C \end{Bmatrix}$$

*First used in 1818 by Captain Henry Kater (1777–1835), but the method was apparently suggested somewhat earlier by Bohnenberger. The theory of Kater's pendulum was treated in detail by Friedrich Wilhelm Bessel (1784–1846) in 1826.

Perform a rotation of the coordinate system by an angle θ about the x_3-axis. Evaluate the transformed tensor elements, and show that the choice $\theta = \pi/4$ renders the inertia tensor diagonal with elements A, B, and C.

11-17. Consider a thin homogeneous plate that lies in the x_1-x_2 plane. Show that the inertia tensor takes the form

$$\{\mathbf{I}\} = \begin{Bmatrix} A & -C & 0 \\ -C & B & 0 \\ 0 & 0 & A + B \end{Bmatrix}$$

11-18. If, in the previous problem, the coordinate axes are rotated through an angle θ about the x_3-axis, show that the new inertia tensor is

$$\{\mathbf{I}\} = \begin{Bmatrix} A' & -C' & 0 \\ -C' & B' & 0 \\ 0 & 0 & A' + B' \end{Bmatrix}$$

where

$$A' = A\cos^2\theta - C\sin 2\theta + B\sin^2\theta$$

$$B' = A\sin^2\theta + C\sin 2\theta + B\cos^2\theta$$

$$C' = C\cos 2\theta - \frac{1}{2}(B - A)\sin 2\theta$$

and hence show that the x_1- and x_2-axes become principal axes if the angle of rotation is

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{2C}{B - A}\right)$$

11-19. Consider a plane homogeneous plate of density ρ bounded by the logarithmic spiral $r = ke^{\alpha\theta}$ and the radii $\theta = 0$ and $\theta = \pi$. Obtain the inertia tensor for the origin at $r = 0$ if the plate lies in the x_1-x_2 plane. Perform a rotation of the coordinate axes to obtain the principal moments of inertia, and use the results of the previous problem to show that they are

$$I_1' = \rho k^4 P(Q - R), \quad I_2' = \rho k^4 P(Q + R), \quad I_3' = I_1' + I_2'$$

where

$$P = \frac{e^{4\pi\alpha} - 1}{16(1 + 4\alpha^2)}, \quad Q = \frac{1 + 4\alpha^2}{2\alpha}, \quad R = \sqrt{1 + 4\alpha^2}$$

11-20. A uniform rod of length b stands vertically upright on a rough floor and then tips over. What is the rod's angular velocity when it hits the floor?

11-21. The proof represented by Equations 11.54–11.61 is expressed entirely in the summation convention. Rewrite this proof in matrix notation.

11-22. The **trace** of a tensor is defined as the sum of the diagonal elements:

$$\mathrm{tr}\{\mathbf{I}\} \equiv \sum_k I_{kk}$$

Show, by performing a similarity transformation, that the trace is an invariant quantity. In other words, show that

$$\mathrm{tr}\{\mathbf{I}\} = \mathrm{tr}\{\mathbf{I}'\}$$

where $\{\mathbf{I}\}$ is the tensor in one coordinate system and $\{\mathbf{I}\}'$ is the tensor in a coordinate system rotated with respect to the first system. Verify this result for the different forms of the inertia tensor for a cube given in several examples in the text.

11-23. Show by the method used in the previous problem that the *determinant* of the elements of a tensor is an invariant quantity under a similarity transformation. Verify this result also for the case of the cube.

11-24. Find the frequency of small oscillations for a thin homogeneous plate if the motion takes place in the plane of the plate and if the plate has the shape of an equilateral triangle and is suspended **(a)** from the midpoint of one side and **(b)** from one apex.

11-25. Consider a thin disk composed of two homogeneous halves connected along a diameter of the disk. If one half has density ρ and the other has density 2ρ, find the expression for the Lagrangian when the disk rolls without slipping along a horizontal surface. (The rotation takes place in the plane of the disk.)

11-26. Obtain the components of the angular velocity vector $\boldsymbol{\omega}$ (see Equation 11.102) directly from the transformation matrix $\boldsymbol{\lambda}$ (Equation 11.99).

11-27. A symmetric body moves without the influence of forces or torques. Let x_3 be the symmetry axis of the body and $\mathbf{L}$ be along x_3'. The angle between $\boldsymbol{\omega}$ and x_3 is α. Let $\boldsymbol{\omega}$ and $\mathbf{L}$ initially be in the x_2-x_3 plane. What is the angular velocity of the symmetry axis about $\mathbf{L}$ in terms of I_1, I_3, ω, and α?

11-28. Show from Figure 11-9c that the components of $\boldsymbol{\omega}$ along the fixed (x_i') axes are

$$\omega_1' = \dot{\theta}\cos\phi + \dot{\psi}\sin\theta\sin\phi$$

$$\omega_2' = \dot{\theta}\sin\phi - \dot{\psi}\sin\theta\cos\phi$$

$$\omega_3' = \dot{\psi}\cos\theta + \dot{\phi}$$

11-29. Investigate the motion of the symmetric top discussed in Section 11.11 for the case in which the axis of rotation is vertical (i.e., the x_3'- and x_3-axes coincide). Show that the motion is either stable or unstable depending on whether the quantity $4I_1Mhg/I_3^2\omega_3^2$ is less than or greater than unity. Sketch the effective potential $V(\theta)$ for the two cases, and point out the features of these curves that determine whether the motion is stable. If the top is set spinning in the stable configuration, what is the effect as friction gradually reduces the value of ω_3? (This is the case of the "sleeping top.")

11-30. Refer to the discussion of the symmetric top in Section 11.11. Investigate the equation for the turning points of the nutational motion by setting $\dot{\theta} = 0$ in Equation 11.162. Show that the resulting equation is a cubic in $\cos\theta$ and has two real roots and one imaginary root for θ.

11-31. Consider a thin homogeneous plate with principal momenta of inertia

$$I_1 \text{ along the principal axis } x_1$$

$$I_2 > I_1 \text{ along the principal axis } x_2$$

$$I_3 = I_1 + I_2 \text{ along the principal axis } x_3$$

Let the origins of the x_i and x_i' systems coincide and be located at the center of mass O of the plate. At time $t = 0$, the plate is set rotating in a force-free manner with an angular velocity Ω about an axis inclined at an angle α from the plane of the plate and perpendicular to the x_2-axis. If $I_1/I_2 \equiv \cos 2\alpha$, show that at time t the angular velocity about the x_2-axis is

$$\omega_2(t) = \Omega \cos\alpha \tanh(\Omega\, t \sin\alpha)$$

11-32. Solve Example 11.2 for the case when the physical pendulum does not undergo small oscillations. The pendulum is released from rest at 67° at time $t = 0$. Find the angular velocity when the pendulum angle is at 1°. The mass of the pendulum is 340 g, the distance L is 13 cm, and the radius of gyration k is 17 cm.

11-33. Do a literature search and explain how a cat can always land on its feet when dropped from a position at rest with its feet pointing upward. Estimate the minimum height a cat needs to fall in order to execute such a maneuver.

11-34. Consider a symmetrical rigid body rotating freely about its center of mass. A frictional torque ($N_f = -b\omega$) acts to slow down the rotation. Find the component of the angular velocity along the symmetry axis as a function of time.