

$$\begin{aligned}
& -i \frac{V_0 t}{4 \pi^2} \int \frac{dk'}{(\mathbf{K}_2 + ik')(k - k' - i\epsilon)} \\
& = \frac{V_0 t}{4 \pi^2} \left( \int \pi \delta(k - k') \frac{1}{\mathbf{K}_2 + ik'} dk' \right. \\
& \quad \left. - i \int \frac{\mathcal{P}}{k - k'} \frac{1}{\mathbf{K}_2 + ik'} dk' \right) = \frac{V_0 t}{4 \pi (\mathbf{K}_2 + ik)} \\
& \quad - i \frac{V_0 t}{4 \pi^2} \int \frac{dk'}{(k - k')(\mathbf{K}_2 + ik')} \\
& = \frac{V_0 t}{2 \pi (\mathbf{K}_2 + ik)}. \tag{A1}
\end{aligned}$$

(For brevity we have set  $V_1=0$  and  $V_0=V_2-V_1$ .) Using the same logic as in Sec. II the well-known results

$$r = \frac{\kappa_1 - i\mathbf{K}_2}{\kappa_1 + i\mathbf{K}_2}, \quad t = \frac{2\kappa_1}{\kappa_1 + i\mathbf{K}_2} \tag{A2}$$

are obtained.

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<sup>2</sup>S. Flügge, *Rechenmethoden der Quantentheorie* (Springer, Berlin, 1965), p. 35.  
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<sup>8</sup>V. S. Vladimirov, *Equations of Mathematical Physics* (Mir, Moscow, 1984), p. 90.  
<sup>9</sup>H. A. Bethe and E. E. Salpeter, *Quantum Mechanics of One- and Two-Electron Atoms* (Springer, New York, 1957), p. 41.  
<sup>10</sup>Note that sometimes the limit  $\epsilon \rightarrow 0$  (not  $0_+$ ) is taken. There is no difference with regard to the contents. The index “+” accentuates the plus and minus signs in the formula.  
<sup>11</sup>A. Messiah, in Ref. 6, pp. 79 and 80.  
<sup>12</sup>A. Messiah, in Ref. 6, p. 81.  
<sup>13</sup>K. T. R. Davies, R. W. Davies, and G. D. White, “Dispersion relations for causal Green’s functions: Derivations using the Poincaré–Bertrand theorem and its generalizations,” *J. Math. Phys.* **31** (6), 1356–1373 (1990) (see especially the Appendix).  
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<sup>15</sup>V. S. Vladimirov, in Ref. 8, Chap. 5.  
<sup>16</sup>J. A. Stovneeng and E. H. Hauge, “Time-dependent resonant tunneling of wave packets in the tight-binding model,” *Phys. Rev. B* **44**, 13 582–13 594 (1991).  
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<sup>19</sup>H. Triebel, *Höhere Analysis* (Deutscher Verlag der Wissenschaft, Berlin, 1972), p. 110.

## On the thermodynamic efficiencies of reversible cycles with sloping, straight-line processes

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(Received 17 September 1993; accepted 14 December 1993)

Heat and efficiency calculations for negatively sloping, straight-line paths in  $PV$  diagrams seem to be consistently misleading or incorrect in many introductory physics texts. The source of the error is identified and thoroughly explored with several examples.

### I. INTRODUCTION

Almost everyone teaching freshman/sophomore introductory thermodynamics encounters  $PV$ -diagram cycles similar to Fig. 1. The constant-volume and constant-pressure processes are straightforward; it is the negatively sloping, straight-line path 1→2 that requires some thought. The inventors of these problems typically pick simple, whole-number factors for the volume and pressure changes. Figure 1 uses a factor of 2 for  $P$  and  $V$  so that the end points of the sloping path are at equal temperatures,  $T_1$ .

Given the cycle in Fig. 1, and assuming ideal gases, students are generally asked questions about the work done and the heat involved (we prefer the single word “heat” to refer to energy transferred due to temperature differences, rather

than the somewhat redundant but widely used phrase “heat flow”). Along path 1→2 the work is simply the area under the path,  $W_{12} = (3/4)P_1V_1$ . The heat is found from the First Law:  $Q_{12} = \Delta U_{12} + W_{12} = W_{12}$ , since the end points of the path are at the same temperature. Although physically unrealistic, this type of problem is invented for the pedagogic value of having students apply the First Law. However, it is the often misleading implications about the heat involved in such a problem that concerns us. In addition, we believe textbooks have missed an opportunity to discuss some interesting physics.

Students can also be asked to compute the thermal efficiency of a cycle similar to Fig. 1, although this problem is much less frequently encountered. We ask readers of this

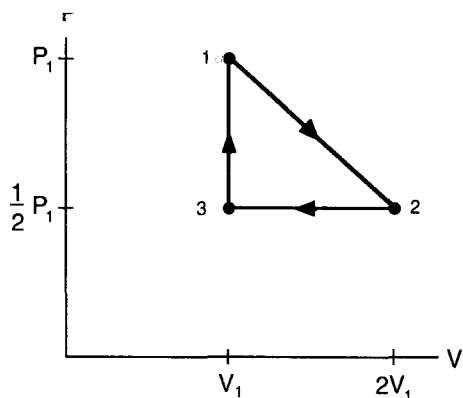


Fig. 1. A  $PV$  diagram of a reversible cycle.

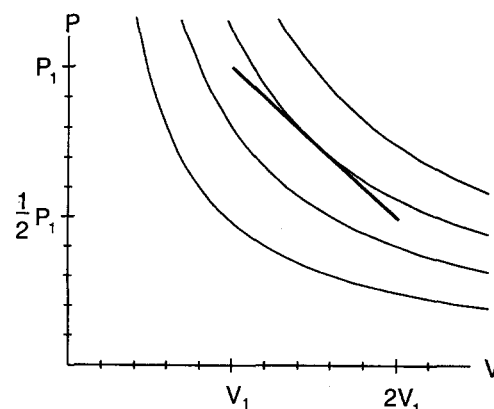


Fig. 3. Comparison of path 1→2 of Fig. 1 and various isotherms. Only one isotherm is tangent to the path and it gives the maximum temperature attained.

article who teach thermodynamics to stop for a moment and compute the efficiency of an ideal, monatomic gas undergoing the reversible cycle shown in Fig. 1.

We have asked some of our colleagues to solve the same problem. All who responded to us got the answer wrong. We have examined the introductory texts on our bookshelves for problems similar to Fig. 1. The few texts that we found give wrong answers to their own problems, although one text<sup>1</sup> seems to have gotten the essence of everything correct, but nevertheless provides the wrong answer for the efficiency. From our unscientific survey we would guess that many readers of this article attempting the problem of Fig. 1 also computed an incorrect value of  $\epsilon=1/6$  for the efficiency. The correct value is  $\epsilon=16/97$ .

## II. ANALYSIS

To begin the investigation into the source of the commonly made error in this type of problem we first consider a different reversible cycle, as shown in Fig. 2. Here the straight-line path ends at  $P_2=0.315P_1$  and  $V_2=2V_1$ ; these values have been chosen so that the return curve can be an

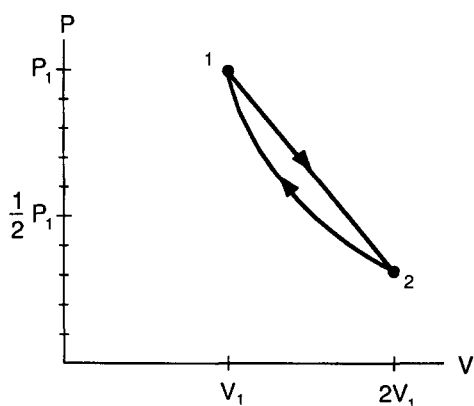


Fig. 2. A reversible cycle using a straight path and an adiabat.

adiabat. If the new cycle is analyzed in the same manner as is typically done for the first case, then the heat along the straight-line path is  $Q_{12}=0.102P_1V_1$ . The dilemma occurs when computing the efficiency,  $\epsilon=1-Q_{out}/Q_{in}\equiv 1$ , since  $Q_{out}$  must apparently be zero. Either we have here an engine of 100% efficiency and an astounding discovery has been made, or else there is something wrong with the analysis; of course, the above simple-minded analysis is wrong.

The difficulty must occur for the sloping straight-line path, and in particular because the slope is negative. Textbook problems<sup>2,3</sup> also exist for triangular cycles with positively sloping straight-line paths, but in these cases naive assumptions yield the correct result. This forms the basis for our earlier statement about the misleading nature of a negatively sloping, straight-line process:  $Q_{12}$  is positive and therefore the assumption is almost always made that  $Q$  is positive along the entire path 1→2. Returning to Fig. 2, it must be obvious that path 1→2 of Fig. 2 involves both  $Q_{in}$  and  $Q_{out}$ , otherwise the Second Law is violated. If straight paths of negative slope are to be used for student assignments then we encourage instructors and authors to explore with their students the correct thermodynamic processes occurring, and in particular how to determine whether there is  $Q_{in}$  or  $Q_{out}$  along the path. This is explored below.

For the discussion that follows we will return to the cycle of Fig. 1. Even though the end points of path 1→2 are at equal temperatures, the path is obviously not isothermal. Figure 3 is a plot of path 1→2 along with a number of isotherms. Only one isotherm is tangent to the path; the point of tangency gives the maximum temperature encountered along the path. The actual gas temperature can be determined from the ideal gas law,  $T=(T_1/P_1V_1)PV$ . The pressure for a straight-line path can be written as  $P=mV+b$ , with slope  $m$  and intercept  $b$ . For the case of Fig. 1 the values are  $m=-(1/2)P_1/V_1$  and  $b=(3/2)P_1$ . Finally,  $T=(T_1/P_1V_1)(bV+mV^2)=(T_1/2V_1)(3V_1V-V^2)$ ; Figure 4 shows the variation in temperature with volume. Figures 3 and 4 both show that the temperature is a maximum at  $V_T=(3/2)V_1$ . Since the temperature decreases beyond  $V_T$ , one might naively conclude that  $dQ$  reverses sign at this point along the path. But this is not the case. We will rewrite

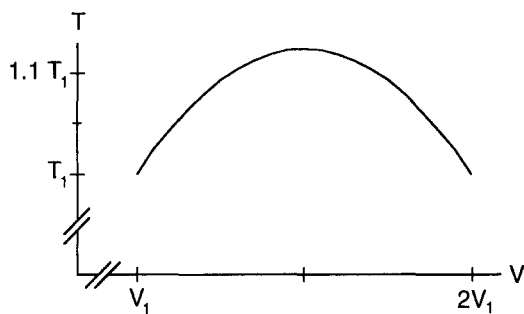


Fig. 4. Temperature along path 1→2 of Fig. 1. The maximum temperature is  $(9/8)T_1$  at volume  $(3/2)V_1$ .

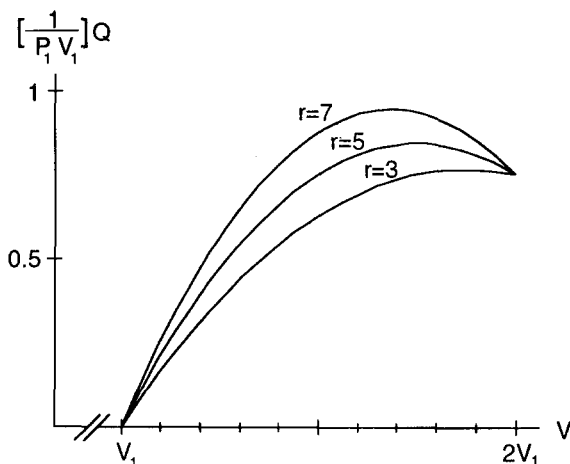


Fig. 6. Heat along the path 1→2 of Fig. 1.

the First Law,  $dQ = dU + dW$ , in terms of  $dV$ . By the definition of an ideal gas,  $dU = nC_V dT = n(r/2)R dT$ , where  $r$  is the number of degrees of freedom. Differentiating the ideal gas law we have:  $nR dT = P dV + V dP = (mV + b)dV + V(m dV)$ . Thus  $dU = (r/2)[2mV + b]dV$ . Similarly,  $dW = P dV = (mV + b)dV$ . Finally, the First Law becomes

$$dQ = \left[ \frac{(r+2)}{2} b + m(r+1)V \right] dV, \quad (1a)$$

$$= P_1 \left[ \frac{3(r+2)}{4} - \frac{(r+1)}{2V_1} V \right] dV. \quad (1b)$$

Figure 5 is a plot of Eq. (1b), while Fig. 6 is a plot of  $Q$ , as  $V$  varies from  $V_1$  to  $2V_1$ . For the specific case of Fig. 1 the change in internal energy is  $\Delta U_{12} = 0$  and thus  $Q_{12}$  must be independent of  $r$ , as shown in Fig. 6.

We now define  $V_A$  as the volume at which  $Q$  is a maximum.  $V_A$  can be found from Eq. (1) by setting  $dQ = 0$ . The result agrees with Figs. 5 and 6

$$V_A = - \frac{(r+2)b}{2m(r+1)}, \quad (2a)$$

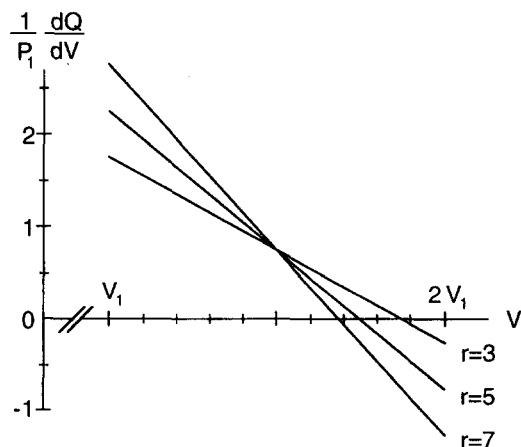


Fig. 5. Change in heat as a function of volume for the path 1→2 of Fig. 1;  $r$  = number of degrees of freedom.

$$= \frac{3(r+2)}{2(r+1)} V_1. \quad (2b)$$

The significance of  $V_A$  is found by considering adiabatics. Every point in a  $PV$  diagram has only one adiabatic that passes through it. Figure 7 shows a random point, its associated adiabatic, and various paths leading away from the selected point. The two paths tangent to the adiabatic have  $dQ = 0$ . Paths above the tangent have heat entering the system, while the reverse is true for paths below the tangent; Refs. 4, 5, and 6 are examples of texts which carefully recognize this fact.

Figure 8 plots path 1→2 of Fig. 1 along with a number of  $r=3$  adiabatics. Only one adiabatic is tangent to the path 1→2. We now show that the location of this tangent point is given by  $V_A$ . For this assertion to be true then at  $V_A$  the slope of path 1→2 must equal the slope of the adiabatic. The slope of an adiabatic is  $(dP/dV)_Q = -\gamma P/V$ , where the ratio of specific heats is  $\gamma = (r+2)/r$ . Substituting  $P = P_A$

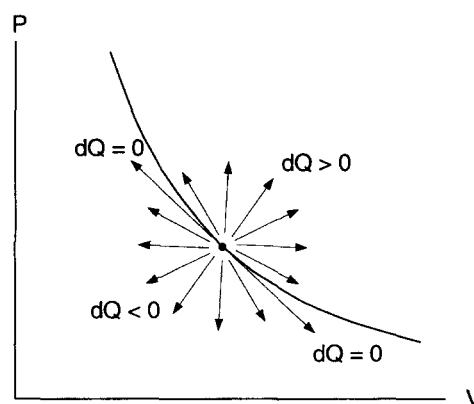


Fig. 7. Whether heat enters or leaves a system depends on the path taken relative to the unique adiabatic through each point of a  $PV$  diagram.

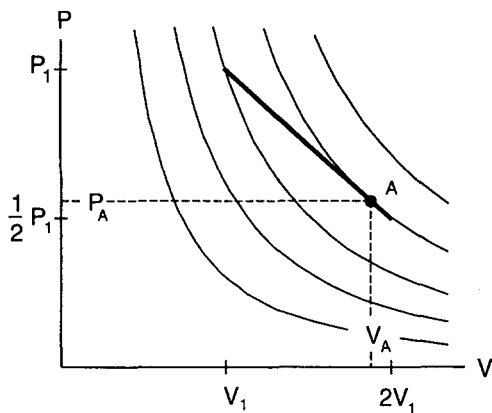


Fig. 8. The sloping path of Fig. 1 and various  $r=3$  adiabatics. The point  $(P_A, V_A)$  is also shown.

$=mV_A + b$  and using Eq. (2a) for  $V=V_A$   $=-(r+2)b/[2m(r+1)]$  gives  $(dP/dV)_Q = m\gamma r/(r+2) = m$ , which is the desired result.

Adiabatic curves play a critical role in the analysis of many thermodynamic processes. For example, wherever an adiabatic is tangent to a path in a  $PV$  diagram there will be a sign change in the heat. This fact seems to be ignored, or at least seriously underemphasized, in introductory texts.

### III. DISCUSSION

Only when the significance of the point  $(P_A, V_A)$  is understood can the efficiency of the cycle in Fig. 1 be correctly determined. The efficiency is  $\epsilon = W_{\text{tot}}/Q_{\text{in}}$ , but now we realize that  $Q_{\text{in}} = Q_{1A} + Q_{31}$ , and  $Q_{1A}$  is gotten by integrating Eq. (1) from  $V_1$  to  $V_A$ . From Eq. (2b),  $V_A = 1.875V_1$ . The efficiency could also have been computed using  $\epsilon = 1 - Q_{\text{out}}/Q_{\text{in}}$ , where we must use  $Q_{\text{out}} = Q_{A2} + Q_{23}$ . Again,  $Q_{A2}$  is obtained by integrating Eq. (1), but now the integration is from  $V_A$  to  $2V_1$ . In either case, the result is  $\epsilon = 1/6.0625 = 0.165$  for an ideal monatomic gas; compare this to the erroneous value given earlier,  $\epsilon = 1/6$ .

Alternatively, it is possible to derive the above results without recourse to Eqs. (1) and (2); this approach may be more accessible to introductory classes and we would recommend the following as an example to present to students. Determining the efficiency of the cycle in Fig. 1 starts with finding the net work for the cycle, which is simply the area enclosed by the right-triangle:  $W_{\text{tot}} = (1/2) \text{base} \times \text{height} = (1/4)P_1V_1$ . Next,  $Q_{\text{in}}$  is required. Clearly, the process  $3 \rightarrow 1$  is a purely heat in process, and  $2 \rightarrow 3$  is a purely heat out process. For a monatomic gas  $Q_{31} = nC_V\Delta T_{31} = (3/2)V_1\Delta P_{31} = (3/4)P_1V_1$ . This paper has made it clear that care must be taken to determine  $Q_{\text{in}}$  and  $Q_{\text{out}}$  during a negatively sloping straight-line process; this is the unusual, but crucial and necessary step that requires determining  $V_A$ . At  $V_A$  the slope of path  $1 \rightarrow 2$  [ $m = -(1/2)P_1/V_1$ ] equals the slope of the adiabat [ $-\gamma P/V = -\gamma(mV_A + b)/V_A$ ]. Equating the slopes gives  $V_A = -(5/8)b/m = (15/8)V_1$ , using  $\gamma = 5/3$  for a monatomic gas; the same result can be gotten from Eq. (2b), found by setting  $dQ = 0$ . From the equation of the line,  $P_A = (9/16)P_1$ .  $Q_{1A}$  is calculated using the First Law, simple geometry, the ideal gas equation of state,

and the results just above for  $V_A$  and  $P_A$ :  $Q_{1A} = (U_A - U_1) + W_{1A}$ ; here  $U_1 = (3/2)P_1V_1$ ,  $U_A = (3/2)P_AV_A = (405/256)P_1V_1$ , and  $W_{1A} = (1/2)(P_1 - P_A)(V_A - V_1) + (V_A - V_1)P_A = (175/256)P_1V_1$ . Substituting into  $Q_{1A}$  gives  $Q_{1A} = (49/64)P_1V_1$ . The result for  $Q_{\text{in}}$  is  $Q_{\text{in}} = Q_{31} + Q_{1A} = (3/4)P_1V_1 + (49/64)P_1V_1 = (97/64)P_1V_1$ . Finally, the efficiency of this cycle is  $\epsilon = W_{\text{tot}}/Q_{\text{in}} = (1/4)/(97/64) = 16/97 = 0.165$ , as before.

The computed efficiency also depends on  $r$ , as can be seen in Figs. 5 and 6. Consider the cycle of Fig. 1, but modify path  $1 \rightarrow 2$  so that  $P_2 = P_1/2$  and  $V_2 = 3V_1$ . From Eq. (2a) it will be found that  $V_A$  is not part of path  $1 \rightarrow 2$ , indicating that an adiabatic is nowhere tangent to the path. Thus  $Q_{12}$  is entirely  $Q_{\text{in}}$  and naive calculations for  $\epsilon$  will be correct and give  $\epsilon = 1/6$  for a monatomic gas. However, if an  $r=5$  diatomic gas had been used then  $V_A$  becomes  $(35/12)V_1$  and the correct efficiency is  $1/8.01$  rather than the incorrect<sup>7</sup> value of  $1/8$ .

Consider another example where the correct and incorrect efficiencies differ only slightly. Modify the basic cycle so that  $P_2 = P_1$ ,  $V_2 = 3V_1$ , and  $P_3 = P_1/3$ ,  $V_3 = 3V_1$ . Equation (2a) gives  $V_A = (5/2)V_1$  for  $r=3$  along the straight line path  $3 \rightarrow 1$ . For this problem  $Q_{\text{in}} = Q_{12} + Q_{3A}$  and the correct efficiency is  $4/31$  rather than the incorrect<sup>8,9</sup> value of  $2/15$ .

Simple paths were used in these examples; paths are usually designed with whole-number factors to make plotting and calculations easy. An accidental consequence of the paths chosen in the above textbook examples was that the point  $A$  was near an end of a path, and thus the correct and incorrect efficiencies were almost the same. But dramatic changes can occur with what at first appear to be innocuous variations in the selected path. As an example, modify the basic cycle of the Fig. 1 so that  $P_2 = P_1/3$ ,  $V_2 = 2V_1$ . From Eq. (2a),  $V_A = 1.562V_1$  for  $r=3$  and the correct efficiency is  $\epsilon = 1/4.26$ . If the assumption had been made that  $Q_{12}$  is entirely  $Q_{\text{in}}$  then the resulting efficiency would be  $\epsilon = 1/3.5$ , for a roughly 20% discrepancy from the correct value. If the same problem is repeated using  $r=6$  then the correct and incorrect efficiencies become  $\epsilon = 1/7.29$  and  $1/5$ , respectively, a roughly 45% discrepancy.

Finally, for the cycle shown in Fig. 2 there is a point  $A$  on the straight line given by  $V_A = 1.537V_1$ , leading to a correctly calculated efficiency of 0.259 for an ideal monatomic gas.

### IV. CONCLUSIONS

For all of the problems that we have found in texts of the type discussed in this article, the incorrect efficiencies differ only slightly from the correct values. But the differences are nevertheless important from a pedagogical point of view. The common, naive answer that texts seem prone to give is wrong and is based on wrong assumptions. It is the unquestioned and wrong assumptions that disturb us, not the "mere" computing of correct or incorrect answers. These assumptions show up frequently and have become so ingrained that many instructors and authors apparently have forgotten some basic physics. Consider the symmetric cycle of Fig. 9. Homework problems exist in which students are told to assume  $Q_{\text{in}}$  occurs along the upper half of the cycle (above the dotted line in the figure), while  $Q_{\text{out}}$  occurs for the lower half. In fact, it is the adiabatics tangent at points 1 and 2 that separate  $Q_{\text{in}}$  and  $Q_{\text{out}}$ , creating a much different problem. Such cycles are intended as learning exercises, but there

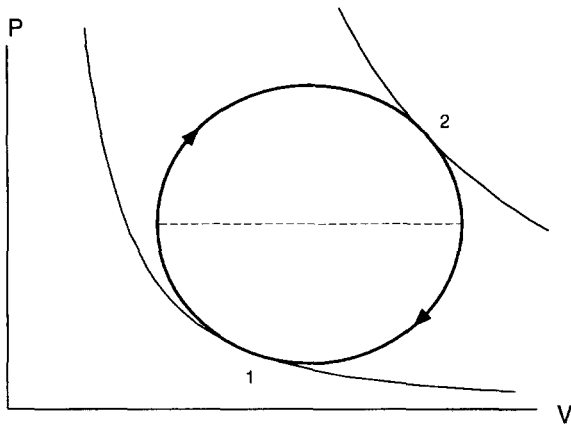


Fig. 9. A symmetrical cycle. The adiabatic curves tangent at points 1 and 2 indicate where the heat changes sign.

are very few students sophisticated enough to realize that the assumptions they are asked to use are wrong.

We are also concerned when students are presented with the simpler but more frequently encountered question, "What is the heat  $Q_{12}$  for path 1→2 of Fig. 1?" Again, very few students have the sophistication to realize that  $Q_{12}$  may have (and in this case does have) both positive and negative components. We encourage instructors and authors to correct

the misleading impressions inherent in this type of process, and instead try to describe correctly the relationship between adiabatics and the  $Q_{in}$  and  $Q_{out}$  taking place along such paths. That relationship is simple to summarize: if there exists a point of tangency (called A in this paper) between an adiabatic and a reversible process on a PV diagram, then  $dQ=0$  at A and there is a change in sign for the heat. This change from  $Q_{in}$  to  $Q_{out}$  (or vice versa) for a process is crucial in correctly determining the thermodynamic efficiency of a cycle.

## ACKNOWLEDGMENTS

We wish to thank the authors of Ref. 1. A problem in their text gave the initial motivation for this paper.

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## THE HYPOTHETICO-DEDUCTIVE SCHEME

Now let us turn to certain shortcomings, real or fancied, of the hypothetico-deductive scheme. If it is a formal objection to classical inductivism that it sets no upper limit to the amount of factual information we should assemble, so also is it a defect of the hypothetico-deductive scheme that it sets no upper limit to the number of hypotheses we might propound to account for our observations. To substitute Whewell's system for Mill's is, on the face of it, to trade in an infinitude of irrelevant facts for an infinitude of inane hypotheses. Mill meant it as a criticism, not as a comment, when he said:

An hypothesis being a mere supposition, there are no other limits to hypotheses than those of the human imagination; we may, if we please, imagine, by way of accounting for an effect, some cause of a kind utterly unknown, and acting according to a law altogether fictitious.

In real life, of course, just as the crudest inductive observations will always be limited by some unspoken criterion of relevance, so also the hypotheses that enter our minds will as a rule be plausible and not, as in theory they could be, idiotic. But this implies the existence of some internal censorship which restricts hypotheses to those that are not absurd, and the internal circuitry of this process is quite unknown. The critical process in scientific reasoning is not therefore wholly logical in character, though it can be made to appear so when we look back upon a completed episode of thought.

Peter Medawar, *Pluto's Republic* (Oxford, New York, 1984), pp. 106–107.