Wick calculus

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In quantum field theory the "normal ordering" of operators is routinely used, which amounts to shuffling all creation operators to the left. Potentially confusing is the occurrence in the literature of normal-ordered functions, sometimes called "Wick transforms." In this paper, we introduce the reader to some basic results, ideas, and mathematical subtleties about normal ordering of operators and functions. The intended audience are instructors of quantum field theory and researchers who are interested but not specialists in mathematical physics.
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I. INTRODUCTION

Normal ordering was introduced in quantum field theory by Wick in 1950 to avoid some infinities in the vacuum expectation values of field operators expressed in terms of creation and annihilation operators.[1] The simplest example of such infinities can be discussed starting from only nonrelativistic quantum mechanics and the simple harmonic oscillator; an infinite number of harmonic oscillators make up a free quantum field. (The reader who wishes a quick reminder of some basic quantum field-theoretical concepts may find comfort in Appendix A).

In addition to the usual concept of quantizing by promoting fields to operators, modern quantum field theory uses functional integrals as basic objects. In the functional integral formalism we calculate physical quantities such as scattering cross sections and decay constants by integrating over some polynomials in the fields and their derivatives (see, for example, Refs. 2 and 3). In the examples considered here, the fields we want to integrate are equivalent to functions on spacetime. As mentioned, a first step toward removing infinities and giving mathematical meaning to calculations in the operator formalism is normal ordering. The analog of this ordering for functional integrals is called the "Wick transform".

In this paper we introduce the reader to an interesting connection, which is usually not discussed in introductory treatments of quantum field theory, among normal ordering, Wick transforms, and Hermite polynomials. We will also be concerned with some basic questions that are usually glossed over, i.e., what does the functional integral really mean? Although a complete answer is not known and beyond the scope of this paper, we intend to give some flavor of the first steps toward addressing this question and how the Wick transform has been put to work in this regard.

The paper is organized as follows: We will use the harmonic oscillator to exhibit the connection between Wick-ordered polynomials and the familiar Hermite polynomials. Then we turn to Wick transforms in the functional integral formalism of field theory, where we show that there is again a connection with Hermite polynomials. Several approaches to Wick transforms are compared and shown to be equivalent. In passing, we observe how the standard quantum field theory result known as "Wick's theorem" follows directly in this framework from well-known properties of the Hermite polynomials. Finally, we provide a brief example of how the Wick transform can be utilized in a physical application.

II. WICK OPERATOR ORDERING

A. Simple harmonic oscillator

The Hamiltonian operator for the simple harmonic oscillator in nonrelativistic quantum mechanics has the form:

$$H = \tfrac{1}{2}(P^2 + Q^2), \tag{1}$$

where we have, as usual, hidden Planck's constant $\hbar$, the mass m, and the angular frequency ω in the definitions of the dimensionless momentum and position operators,

$$P \equiv \frac{1}{\sqrt{\hbar m\omega}}\hat{p}, \quad Q \equiv \sqrt{\frac{m\omega}{\hbar}}\hat{q}, \tag{2}$$

so that

$$[P,Q] = -i. \tag{3}$$

If we define the creation and annihilation operators $a^\dagger$ and a by

$$a^\dagger = \frac{1}{\sqrt{2}}(Q - iP), \tag{4}$$

$$a = \frac{1}{\sqrt{2}}(Q + iP), \tag{5}$$

so that

$$[a,a^\dagger] = 1, \tag{6}$$

we find that

$$H = \tfrac{1}{2}(P^2 + Q^2) = \tfrac{1}{2}(a^\dagger a + aa^\dagger) = a^\dagger a + \tfrac{1}{2}. \tag{7}$$

The eigenvalues of this Hamiltonian operator (see for example, Ref. 4) satisfy the sequence:

$$E_n = n + \tfrac{1}{2}, \quad n = 0,1,2, \ldots . \tag{8}$$

In particular, the ground state energy (or zero-point energy), which is the lowest eigenvalue of the Hamiltonian, is nonzero:

$$H|0\rangle = \tfrac{1}{2}|0\rangle. \tag{9}$$

This zero-point energy has observable physical consequences. As an illustration, it is possible to measure the zero-point motion of the atoms in a crystal by studying the dispersion of light in the crystal. Classical theory predicts that the oscillations of the atoms in the crystal, and therefore also dispersion effects, cease to exist when the temperature is near absolute zero. However, experiments demonstrate that the dispersion of light reaches a finite nonzero value at very low temperatures.

In quantum field theory a free scalar field can be viewed as an infinite collection of harmonic oscillators as described in Appendix A. If we proceed as before, each oscillator will give a contribution to the zero-point energy, resulting in an infinite energy, which seems like it could be a problem.

One way to remedy the situation is to define the ground state as a state of zero energy, which can be achieved by redefining the Hamiltonian. We subtract the contribution of the ground state and define a Wick-ordered (or normal-ordered) Hamiltonian, denoted by putting a colon on each side, by

$$:H: = :\tfrac{1}{2}(a^\dagger a + aa^\dagger): \equiv \tfrac{1}{2}(a^\dagger a + aa^\dagger) - \tfrac{1}{2}\langle 0|a^\dagger a + aa^\dagger|0\rangle \equiv a^\dagger a. \tag{10}$$

In this example the definition of Wick ordering can be thought of as a redefinition of the ground state of the harmonic oscillator.

In the last equality in Eq. (10) we see that all creation operators end up on the left. A general prescription for Wick ordering in quantum field theory in a creation/annihilation operator formalism is "Permute all the $a^\dagger$ and a, treating them as if they commute, so that in the end all $a^\dagger$ are to the left of all a." The resulting expression is the same:

$$:H: = a^\dagger a. \tag{11}$$

B. Wick ordering and Hermite polynomials

The first connection between Wick ordering and Hermite polynomials arises when we study powers of the (dimensionless) position operator Q. For the harmonic oscillator the eigenvalue of Q^2 gives the variance of the oscillator from rest. We have $Q=(a^\dagger+a)/\sqrt{2}$, but to avoid cluttering the equations with factors of $\sqrt{2}$, we will study powers of just $(a^\dagger+a)$:

$$(a^\dagger + a)^2 = a^{\dagger 2} + a^\dagger a + aa^\dagger + a^2, \tag{12a}$$

$$= a^{\dagger 2} + 2a^\dagger a + a^2 + [a,a^\dagger], \tag{12b}$$

$$= :(a^\dagger + a)^2: + [a,a^\dagger], \tag{12c}$$

$$= :(a^\dagger + a)^2: + 1 \quad [\text{by Eq. (10)}]. \tag{12d}$$

If we arrange terms in a similar way for higher powers of $(a^\dagger+a)$, we find:

$$(a^\dagger + a)^3 = :(a^\dagger + a)^3: + 3(a^\dagger + a), \tag{13a}$$

$$(a^\dagger + a)^4 = :(a^\dagger + a)^4: + 6:(a^\dagger + a)^2: + 3. \tag{13b}$$

We can summarize the results using the notation $a^\dagger+a=q$,

$$q^2 = :q^2: + 1, \tag{14a}$$

$$q^3 = :q^3: + 3:q:, \tag{14b}$$

$$q^4 = :q^4: + 6:q^2: + 3. \tag{14c}$$

Because we can recursively replace normal-ordered terms on the right by expressions on the left that are not normal-ordered (for example $:q^2:$ can be replaced by q^2-1), we can also invert these relations:

$$:q^2: = q^2 - 1 = \mathrm{He}_2(q), \tag{15a}$$

$$:q^3: = q^3 - 3q = \mathrm{He}_3(q), \tag{15b}$$

$$:q^4: = q^4 - 6q^2 + 3 = \mathrm{He}_4(q), \tag{15c}$$

where the polynomials $\mathrm{He}_n(q)$ are a scaled version of the more familiar form of the Hermite polynomials H_n:

$$\mathrm{He}_n(x) = 2^{-n/2}\mathrm{H}_n(x/\sqrt{2}). \tag{16}$$

In some of the mathematical physics literature, the He_n are often just called H_n. Some of the many useful properties are collected in Appendix B for easy reference (a more complete collection is given in Ref. 5, for example).

Because of the relation between operator Wick ordering and Hermite polynomials, the literature sometimes defines "Wick ordering" in terms of Hermite polynomials:

$$:q^n: \equiv \mathrm{He}_n(q). \tag{17}$$

Although q is an operator composed of noncommuting operators a and $a^\dagger$, this alternative definition naturally generalizes to Wick ordering of functions. We will explore this idea in the next section.

One reason that the connection to Hermite polynomials is not mentioned in the standard quantum field theory literature is the fact (which was also Wick's motivation) that the normal-ordered part is precisely the part that will vanish when we take the vacuum expectation value. The traditional way to define normal-ordering, the one given at the end of Sec. II A ("put $a^\dagger$ to the left of a"), yields for powers of q:

$$:q^n: = \sum_{i=1}^{n} \binom{n}{i} (a^\dagger)^{n-i} a^i, \tag{18}$$

which vanishes for any nonzero power n when applied to the vacuum state $|0\rangle$.

In other words, because we know that normal-ordered terms vanish upon taking the vacuum expectation value, we may not be interested in their precise form. However, if the expectation value is not taken in vacuum (for example, in a particle-scattering experiment), this part does not vanish in general, and there are many instances where the actual normal-ordered expression itself is the one of interest.

III. FUNCTIONAL INTEGRALS AND THE WICK TRANSFORM

Most contemporary courses on quantum field theory discuss functional integrals (sometimes called path integrals; however, only in nonrelativistic quantum mechanics do we really integrate over paths). In a functional-integral setting, the counterpart of the Wick ordering in the operator formalism is the Wick transform. This transform applies to functions and functionals. It can, like its quantum-mechanics counterpart [Eq. (17)], be defined by means of Hermite polynomials. But first, we briefly skip ahead and explain why such a transform will prove to be useful.

A. Integration over products of fields

In the functional integral formalism, physical quantities such as scattering cross sections and decay constants are calculated by integrating over some polynomial in the fields and their derivatives. The algorithmic craft of such calculations is described in textbooks such as those of Peskin and Schroeder[2] and Ryder.[3] Although there are examples of physical effects that can be studied with functional integral methods but not with ordinary canonical quantization,[6] in the scope of this paper we can only give examples of some results that can be derived more quickly or transparently using functional integrals.

The polynomials we will consider in this section are those of Euclidean fields (fields defined on four-dimensional Euclidean space $\mathbb{R}^4$). Similar formalisms exist for Minkowski fields (fields defined on spacetime) with minor changes in the equations (see for example, Ref. 2). Because functional integrals over Minkowski fields are less mathematically developed than integrals over Euclidean fields, we shall restrict our attention to Wick transforms of functions and functionals of Euclidean fields—primarily polynomials and exponentials. The Wick transform, like the Hermite polynomials, has orthogonality properties that turn out to be useful in quantum field theory. First, we have to introduce a few mathematical concepts.

1. Gaussian measures

Here, our aim is to specify the notation and to briefly remind the reader how to integrate over Euclidean fields, without going into too much detail. The standard mathematical framework to perform such integration is the theory of Gaussian measures in Euclidean field theory.[7,8]

As a first try, we might define a field in the functional integral as a function ϕ on $\mathbb{R}^4$. Fields in the functional integral might seem like functions at first glance, but can produce divergences that cannot, for instance, be multiplied in the way that functions can. A more useful way to regard a quantum field in the functional integral formalism is as a distribution Φ acting on a space of test functions f:

$$\Phi(f) \equiv \langle \Phi, f \rangle, \tag{19}$$

where the bracket $\langle , \rangle$ denotes duality, that is, Φ is such that it yields a number when applied to a smooth test function f. In many situations the distribution Φ is equivalent to a function ϕ, which means this number is the ordinary integral:

$$\langle \Phi, f \rangle = \int_{\mathbb{R}^4} d^4x \, \phi(x) f(x). \tag{20}$$

A familiar example of a distribution is the Dirac distribution $\Phi = \delta$, for which we have

$$\Phi(f) = \langle \delta, f \rangle = f(0). \tag{21}$$

Just like a function, Φ in general belongs to an infinite-dimensional space. To be able to integrate over this space [not to be confused with the integral in Eq. (20), which is an ordinary integral over spacetime] we need a measure, some generalization of the familiar d^4x in the ordinary integral in Eq. (20). To this end, we need to introduce the covariance. Given the action of the theory, we readily calculate the classical equations of motion, extract the differential operator D, and invert it to find the covariance C. In general, the covariance is a positive continuous nondegenerate bilinear form on the space of test functions. A simple example is the covariance for a free Klein–Gordon field,

$$C(f,g) = \int d^4x d^4y f(x) D_{\mathrm{F}}(x-y) g(y), \tag{22}$$

where D_{F} is the textbook Feynman propagator (see Ref. 2 for example). In the following we will often encounter the covariance at coincident test functions, here denoted as $C(f,f)$.

To get to the point, a Gaussian measure $d\mu_C$ is defined by its covariance C as

$$\int_{\mathbf{Y}} d\mu_C(\Phi) \exp(-i\langle \Phi, f \rangle) = \exp[-\tfrac{1}{2} C(f,f)], \tag{23}$$

over a space $\mathbf{Y}$ of distributions Φ.

For comparison, the usual Gaussian measure on $\mathbb{R}^d$ is defined by,[9]

$$\frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \frac{d^dx}{(\det D)^{1/2}} e^{-(1/2)Q(x)} e^{-i\langle x', x\rangle} = e^{-(1/2)W(x')}, \tag{24}$$

where $\langle x', x\rangle_{\mathbb{R}^d} = x'_\mu x^\mu$ is the duality in $\mathbb{R}^d$ with $x \in \mathbb{R}^d$ and $x' \in \mathbb{R}_d$, $Q(x)$ is a quadratic form on $\mathbb{R}^d$,

$$Q(x) = D_{\mu\nu} x^\mu x^\nu = \langle Dx, x \rangle_{\mathbb{R}^d}, \tag{25}$$

and $W(x')$ is a quadratic form on $\mathbb{R}_d$

$$W(x') = x'_\mu C^{\mu\nu} x'_\nu = \langle x', Cx' \rangle_{\mathbb{R}^d}, \tag{26}$$

such that

$$DC = CD = 1. \tag{27}$$

In the more familiar $\mathbb{R}^d$ case [Eq. (24)], the combination of the standard measure and kinetic term corresponds to the measure $d\mu_C$ we introduced earlier. That is, $d^dx/(\det D)^{1/2} e^{-1/2Q(x)}$ is analogous to $d\mu_C(\Phi)$. The explicit separation of $d\mu_C$ into $d^dx/(\det D)^{1/2}$ and $e^{-1/2Q(x)}$ will turn out to be unnecessary for our discussion. By defining the measure $d\mu_C$ through Eq. (23), we have not even specified what such a separation would mean.

The covariance at incident points can be expressed as the following integral, obtained by expanding Eq. (23):

$$C(f,f) = \int d\mu_C(\Phi) \langle \Phi, f \rangle^2. \tag{28}$$

The integral on the left-hand side of Eq. (23) is the generating function of the Gaussian measure; let us denote this integral by $Z(f)$. By successively expanding Eq. (23), the nth moment of the Gaussian measure can be compactly written as

$$\int d\mu_C(\Phi) \langle \Phi, f \rangle^n = \left(-i \frac{d}{d\lambda}\right)^n Z(\lambda f) \bigg|_{\lambda=0} = \begin{cases} (n-1)!!\, C(f,f)^{n/2} & n \text{ even} \\ 0 & n \text{ odd}, \end{cases} \tag{29}$$

where $n!! = n(n-2)(n-4)\cdots$ is the semifactorial. For convenience, we introduce the following notation for the average with respect to the Gaussian measure μ_C:

$$\langle F[\Phi(f)]\rangle_{\mu_C} \equiv \int d\mu_C(\Phi) F[\Phi(f)]. \tag{30}$$

Note the difference between the brackets $\langle\ \rangle_{\mu_C}$ used for the average and the brackets $\langle,\rangle$ used for duality.

2. Wick transforms: Definitions

We can use this set of definitions to define a Wick transform of functionals of fields. Our goal is to provide some idea of how to address the difficult problem of making sense out of products of distributions and integrals of such products. These products are ubiquitous in quantum field theory, although their exact meaning is not usually discussed in introductory textbooks. To simplify the calculations, we define the Wick transform of a power $\Phi(f)^n \equiv \langle \Phi^n, f\rangle$ so as to satisfy an orthogonality property with respect to Gaussian integration. We recall the orthogonality properties of Hermite polynomials (Appendix B) and the definition [Eq. (17)] and define the Wick transform in terms of Hermite polynomials:

$$:\Phi(f)^n:_C = C(f,f)^{n/2}\mathrm{He}_n\left(\frac{\Phi(f)}{\sqrt{C(f,f)}}\right). \tag{31}$$

Note that this definition depends on the covariance C, and that there is no analogous dependence in the analogous harmonic-oscillator definition [Eq. (17)].

The orthogonality of two Wick-transformed polynomials is expressed by

$$\int d\mu_C(\Phi) :\Phi(f)^n:_C :\Phi(g)^m:_C = \delta_{m,n} n!\,(\langle \Phi(f)\Phi(g)\rangle_{\mu_C})^n. \tag{32}$$

An entertaining problem is to prove Eq. (32), which we will do in Sec. III A 3 (paragraph 2).

The Wick transform can also be defined recursively by the following equations:[10]

$$:\Phi(f)^0:_C = 1, \tag{33a}$$

$$\frac{\delta}{\delta\Phi} :\Phi(f)^n:_C = n :\Phi(f)^{n-1}:_C, \quad n = 1,2,\ldots, \tag{33b}$$

$$\int d\mu_C(\Phi) :\Phi(f)^n:_C = 0, \quad n = 1,2,\ldots, \tag{33c}$$

where the functional derivative with respect to a distribution is

$$\frac{\delta}{\delta\Phi}\Phi(f) = f. \tag{34}$$

Let us check that the Wick transform $:\Phi(f)^n:_C$ defined by Eq. (33) is the same as the Wick transform given in terms of Hermite polynomials in Eq. (31). This equivalence is to be expected, because the Hermite polynomials satisfy similar recursion relations, but it is a useful problem to check that it works. To begin, we establish a property of Wick exponentials.

Let $:\exp(\alpha\Phi(f)):_C$ be the formal series:

$$:\exp(\alpha\Phi(f)):_C \equiv 1 + \alpha :\Phi(f):_C + \tfrac{1}{2}\alpha^2 :\Phi(f)^2:_C + \ldots, \tag{35}$$

where normal ordering is defined by Eq. (33).

Problem 1. Show that

$$:\exp(\alpha\Phi(f)):_C = \frac{\exp(\alpha\Phi(f))}{\langle \exp(\alpha\Phi(f))\rangle_{\mu_C}}. \tag{36}$$

Solution. We can evaluate the right-hand side of Eq. (36) by expanding the numerator and denominator in a power series and dividing one power series by the other.

$$\frac{\sum_{k=0}^{\infty} b_k x^k}{\sum_{k=0}^{\infty} a_k x^k} = \frac{1}{a_0}\sum_{k=0}^{\infty} c_k x^k, \tag{37}$$

where $c_n + \frac{1}{a_0}\sum_{k=1}^{n} c_{n-k}a_k - b_n = 0$. A comparison of the resulting series, term by term, to the power series expansion of the left-hand side proves Eq. (36).

Problem 2. Show the equivalence of Eqs. (33) and (31).

Solution. We can explicitly calculate the denominator in Eq. (36):

$$\langle \exp[\alpha\Phi(f)]\rangle_{\mu_C} = \int d\mu_C(\Phi)\exp(\alpha\Phi(f))$$

$$= \int d\mu_C(\Phi)\sum_n \frac{\alpha^n}{n!}\Phi(f)^n \tag{38a}$$

$$= \sum_n \frac{\alpha^{2n}}{n!}\int d\mu_C(\Phi)\Phi(f)^{2n}, \tag{38b}$$

[by Eq. (29)]

$$= \exp\left[\frac{1}{2}\alpha^2 C(f,f)\right]. \quad \text{[by Eq. (28)]} \tag{38c}$$

Thus, from Eq. (36) we find

$$:\exp[\alpha\Phi(f)]:_C = \exp\left(\alpha\Phi(f) - \tfrac{1}{2}\alpha^2 C(f,f)\right). \tag{39}$$

If we multiply the power series expansions[11] of $\exp[\alpha\Phi(f)]$ and $\exp[1/2\alpha^2 C(f,f)]$ and compare the result term by term to the series expansion of the left-hand side of Eq. (39), we obtain

$$:\Phi(f)^n:_C = \sum_{m=0}^{\left[\frac{n}{2}\right]} \frac{n!}{m!\,(n-2m)!}\Phi(f)^{n-2m}\left(-\frac{1}{2}C(f,f)\right)^m. \tag{40}$$

We rewrite Eq. (40) as

$$:\Phi(f)^n:_C = C(f,f)^{n/2}\sum_{m=0}^{\left[\frac{n}{2}\right]} (-1)^m \frac{n!}{2^m m!\,(n-2m)!} \times \left(\frac{\Phi(f)}{\sqrt{C(f,f)}}\right)^{n-2m}, \tag{41}$$

use the expression (B1) for the defining series of the Hermite polynomials given in Appendix B and recover Eq. (31).

3. Wick transforms: Properties

Many properties of Wick-ordered polynomials can be conveniently derived using the formal exponential series. For simplicity, we assume that all physical quantities that we wish to calculate (for example, scattering cross sections) are written with normalization factors of $1/C(f,g)$, which in effect lets us set the coincident-point covariance to unity: $C(f,f)=1$. The covariance can be restored by comparison with Eq. (41). The properties in which we are interested are useful problems:

Problem 3. Show that $:\exp(\Phi(f)+\Phi(g)):_C =\exp(-\langle\Phi(f)\Phi(g)\rangle_{\mu_C}):\exp(\Phi(f)):_C:\exp(\Phi(g)):_C$.

Solution.

$$:\exp(\Phi(f)):_C:\exp(\Phi(g)):_C, \tag{42a}$$

$$=\exp(\Phi(f)+\Phi(g))\exp(-\tfrac{1}{2}[\langle\Phi(f)^2\rangle_{\mu_C}+\langle\Phi(g)^2\rangle_{\mu_C}]), \tag{42b}$$

$$=:\exp(\Phi(f)+\Phi(g)):_C\exp(\langle\Phi(f)\Phi(g)\rangle_{\mu_C}), \tag{42c}$$

where we have used Eq. (39) in the first equality, and, after completing the square in the second factor of Eq. (42b), again in the second equality. Dividing both sides of Eq. (42) by the second factor in Eq. (42c) completes the proof.

Problem 4. Show that $\langle:\Phi(f)^n:_C:\Phi(g)^m:_C:\rangle_{\mu_C} =\delta_{nm}n!\langle\Phi(f)\Phi(g)\rangle^n_{\mu_C}$

Solution. If we take the expectation value of both sides of Eq. (42), we find

$$\langle:\exp(\Phi(f)):_C:\exp(\Phi(g)):_C\rangle_{\mu_C}=\exp[\langle\Phi(f)\Phi(g)\rangle_{\mu_C}] \tag{43}$$

using Eq. (33). Expanding the exponentials on both sides and comparing term by term completes the proof.

Problem 5. Show that $:\Phi(f)^{n+1}:_C=n:\Phi(f)^{n-1}:_C -\Phi(f):\Phi(f)^n:_C$.

Solution. This recursive relationship is a consequence of the equivalence of Wick-ordered functions and Hermite polynomials. The expression follows from the recursion relation for Hermite polynomials given in Appendix B.

Problem 6. The definition of Wick transforms given in Eq. (33) can be generalized to several fields in a straightforward manner. We quote here some results without proof (details can be found in Ref. 10). The reader may find it interesting to check that it works:

$$\begin{aligned}:\Phi(f_1)\ldots\Phi(f_{n+1}):=\,&:\Phi(f_1)\ldots\Phi(f_n):\Phi(f_{n+1})\\&-\sum_{k=1}^{n}C(f_k,f_{n+1}):\Phi(f_1)\ldots\Phi(f_{k-1})\\&\times\Phi(f_{k+1})\ldots\Phi(f_n):\end{aligned} \tag{44}$$

$$\int d\mu_C(\Phi):\Phi(f_1)\ldots\Phi(f_n):=0, \tag{45}$$

$$\int d\mu_C(\Phi):\Phi(f_1)\ldots\Phi(f_n)::\Phi(g_1)\ldots\Phi(g_m):=0 \quad \text{for } n\neq m. \tag{46}$$

These latter multifield expressions reproduce, within this functional framework, what is usually referred to as Wick's theorem in the creation/annihilation-operator formalism. In this formalism it takes some effort to show this theorem; here we find it somewhat easier by relying on familiar properties of the Hermite polynomials.

B. Wick transforms and functional Laplacians

We can also define Wick transforms of functions by the following exponential operator expression, which is convenient in many cases (for example in two-dimensional quantum field theory settings, such as in Refs. 12 and 13):

$$:\phi^n(x):_C\equiv e^{-\frac{1}{2}\Delta_C}\phi^n(x), \tag{47}$$

where the functional Laplacian is defined by

$$\Delta_C=\int d^4xd^4x'\,C(x,x')\frac{\delta}{\delta\phi(x)}\frac{\delta}{\delta\phi(x')}, \tag{48}$$

which, again, depends on the covariance C.

Instead of proving Eq. (47), which is straightforward, we just illustrate the equivalence of definition (47) and definition (31) for the common example of a ϕ^4 power for which the definition becomes

$$:\phi^4(y):_C=\exp\left(-\frac{1}{2}\int d^4xd^4x'\,C(x,x')\frac{\delta}{\delta\phi(x)}\frac{\delta}{\delta\phi(x')}\right)\phi^4(y). \tag{49}$$

We expand the exponential and find

$$\begin{aligned}:\phi^4(y):_C=\,&\phi^4+\left(-\frac{1}{2}\right)\left(\int d^4xd^4x'\,C(x,x')\right.\\&\left.\times\frac{\delta}{\delta\phi(x)}\frac{\delta}{\delta\phi(x')}\right)\phi^4(y)\\&+\frac{1}{2!}\left(-\frac{1}{2}\right)^2\left(\int d^4xd^4x'\,C(x,x')\right.\\&\left.\times\frac{\delta}{\delta\phi(x)}\frac{\delta}{\delta\phi(x')}\right)^2\phi^4(y).\end{aligned} \tag{50}$$

All higher terms in the expansion are zero. We can now evaluate each term separately. The second term is the integral

$$\int d^4xd^4x'\,C(x,x')\frac{\delta}{\delta\phi(x)}\frac{\delta}{\delta\phi(x')}\phi^4(y)$$

$$=\int d^4xd^4x'\,C(x,x')\frac{\delta}{\delta\phi(x)}4\phi^3(y)\delta(x'-y), \tag{51a}$$

$$=\int d^4x,C(x,y)12\phi^2(y)\delta(x-y)=12\phi^2(y), \tag{51b}$$

if the covariance is normalized to unity. We use this result in the term:

$$\int d^4xd^4x',C(x,x')\frac{\delta}{\delta\phi(x)}\frac{\delta}{\delta\phi(x')}12\phi^2(y)$$

$$=\int d^4xd^4x'\,C(x,x')\frac{\delta}{\delta\phi(x)}24\phi(y)\delta(x'-y), \tag{52a}$$

$$=24\int d^4x C(x,y)\delta(x-y)=24, \tag{52b}$$

with the same normalization of the covariance. We collect these results with the appropriate coefficients from the expansion:

$$:\phi^4(y):_C=\phi^4(y)-\frac{1}{2}\cdot 12\phi^2(y)+\frac{1}{2!}\frac{1}{2^2}\cdot 24, \tag{53a}$$

$$=\phi^4(y)-6\phi^2(y)+3=\mathrm{He}_4[\phi(y)], \tag{53b}$$

which completes the example.

C. Further reading

Although we hope to have given some flavor of some of the techniques and ideas of quantum field theory mathematical-physics style, we have given only a few examples and demonstrated some simple identities. For more on the mathematical connection between Wick transforms on function spaces and Wick-ordering of annihilation and creation operators, we recommend Refs. 7 and 8. For readers more interested in fermions than the scalar fields we have discussed, we recommend Ref. 14 as an introduction to Wick ordering.

D. An application: Specific heat

In this section we discuss an example of a physical application of some of the results we have discussed. By using the connection Eq. (31) between the Wick transform and Hermite polynomials, we show how standard properties of those polynomials can be exploited to simplify certain calculations.

Consider the familiar generating function of Hermite polynomials [but for the scaled polynomials Eq. (16)]:

$$e^{x\alpha-\alpha^2/2}=\sum \mathrm{He}_n(x)\frac{\alpha^n}{n!}. \tag{54}$$

This generating function gives a shortcut to calculations in two-dimensional quantum field theory, where normal ordering is often the only form of renormalization necessary. An important issue is the scaling dimension of the normal-ordered exponential $:e^{ip\phi}:$, where p is a momentum. (The real part of this operator can represent the energy of a system where ϕ is the quantum field.) In other words, the question is if we rescale the momentum $p\to\Lambda p$, equivalent to a rescaling $x\to\Lambda^{-1}x$ in coordinate space, how does the operator $:e^{ip\phi}:$ scale? Because an exponential is dimensionless, we might guess that the answer is that it does not scale at all, that is, the scaling dimension is zero. This answer is not so due to quantum effects induced by the normal ordering. In terms of the functional integral we can easily calculate the effect of the normal-ordering (here, the Wick transform):

$$:e^{ip\phi}:_C=\sum_{n=0}^{\infty}\frac{1}{n!}(ip)^n:\phi^n:_C, \tag{55a}$$

$$=\sum_{n=0}^{\infty}\frac{1}{n!}(ip)^n C^{n/2}\mathrm{He}_n(\phi/\sqrt{C})=e^{\frac{1}{2}p^2C}e^{ip\phi}, \tag{55b}$$

where we used the definition Eq. (31) and the generating function Eq. (54). The covariance C is a logarithm in two dimensions; that is, the solution of the two-dimensional Laplace equation is a logarithm. To regulate divergences when $p\to\infty$, we introduce a cutoff Λ for the momentum, which makes $C=\ln\Lambda$. If we substitute this result into Eq. (55b), we obtain

$$:e^{ip\phi}:=\Lambda^{p^2/2}e^{ip\phi}. \tag{56}$$

Thus the anomalous scaling dimension, usually denoted by γ, is $\gamma=p^2/2$ for the exponential operator. This result is important in conformal field theory (see for example Ref. 15). Here the $p^2/2$ comes from the $\alpha^2/2$ in the generating function Eq. (54).

How could such a quantum effect be measured? Consider the two-dimensional Ising model with random bonds (Ref. 13, p. 719). This model is the familiar Ising model, but the coupling between the spins is allowed to fluctuate, that is, the coupling becomes a space-dependent Euclidean field. The energy of the system is described by (the real part of) the exponential operator $:e^{ip\phi}:$. The anomalous dimension in Eq. (56) leads to an expression for the specific heat [see Ref. 13, Eq. (356)], where the derivation is given using renormalization group methods. The specific heat is in principle directly measurable as a function of the temperature, or more conveniently, as a function of $\theta=(T-T_c)/T_c$, a dimensionless deviation from the critical temperature. The renormalization group description predicts a double logarithm dependence on θ that could not have been found by simple perturbation theory; it uses as input the result from Eq. (56).

Admittedly, the telegraphic description in the previous paragraph does not do justice to the full calculation of the specific heat in the two-dimensional Ising model with random bonds. Our purpose here is only to show how the Wick transform reproduces the quantum effect in Eq. (56), and then to give some flavor of how this effect is measurable.

IV. CONCLUSION

We have shown that the scope of normal ordering has expanded to settings beyond the original one of ordering operators and we have discussed an interesting connection to Hermite polynomials, which is usually not mentioned in courses on quantum field theory. Several different definitions of Wick ordering of functions have been discussed and their equivalence established.

From a physics point of view, Wick ordering is part of the process of renormalization, which systematically removes infinities from a theory (that is, removes infinite terms from the perturbation expansion of the functional integral). In general, Wick ordering needs to be supplemented by additional rules for renormalization (see for example, Ref. 2) From a mathematics point of view, Wick ordering is helpful to make sense of the polynomials of fields that are integrated over in the functional integral.

For a deeper understanding and further applications of these ideas, the interested reader is invited to consult the cited literature, which is a selection of texts we have found particularly useful. In particular, for the physics of functional integrals we recommend Ref. 7. For a more mathematically oriented treatment we have found Ref. 8 to be quite useful.

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APPENDIX A: WICK-ORDERING OF OPERATORS IN QUANTUM FIELD THEORY

We briefly remind the reader how the need for Wick ordering arises in the operator formulation of quantum field theory. All of this material is standard and can be found in introductory books on quantum field theory (for example, Ref. 2), albeit in lengthier and more thorough form. We set $\hbar = 1$ throughout.

Consider a real scalar field $\phi(t,\mathbf{x})$ of mass m defined at all points of four-dimensional Minkowski spacetime and satisfying the Klein-Gordon equation:

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2\right)\phi(t,\mathbf{x}) = 0. \tag{A1}$$

The differential operator in parenthesis is one instance of what we call D in the text. The classical Hamiltonian of this scalar field is:

$$H = \frac{1}{2}\sum_{\mathbf{x}} [(\pi(t,\mathbf{x}))^2 + (\nabla\phi(t,\mathbf{x}))^2 + m^2\phi^2(t,\mathbf{x})], \tag{A2}$$

where π is the variable canonically conjugate to ϕ, $\pi = \partial\phi/\partial t$. We can think of the first term as the kinetic energy and the second as the shear energy. This classical system is quantized in the canonical quantization scheme by treating the field ϕ as an operator, and imposing equal-time commutation relations:

$$[\phi(t,\mathbf{x}), \phi(t,\mathbf{x}')] = 0, \tag{A3a}$$

$$[\pi(t,\mathbf{x}), \pi(t,\mathbf{x}')] = 0, \tag{A3b}$$

$$[\phi(t,\mathbf{x}), \pi(t,\mathbf{x}')] = i\delta^3(\mathbf{x}-\mathbf{x}'). \tag{A3c}$$

The plane-wave solutions of the Klein–Gordon equation are known as the field modes, $u_{\mathbf{k}}(t,\mathbf{x})$. Together with their respective complex conjugates $u^*_{\mathbf{k}}(t,\mathbf{x})$ they form a complete orthonormal basis. Hence, the field ϕ can be expanded as

$$\phi(t,\mathbf{x}) = \sum_{\mathbf{k}} [a_{\mathbf{k}}u_{\mathbf{k}}(t,\mathbf{x}) + a^\dagger_{\mathbf{k}}u^*_{\mathbf{k}}(t,\mathbf{x})]. \tag{A4}$$

The equal time commutation relations for ϕ and π are then equivalent to

$$[a_{\mathbf{k}}, a_{\mathbf{k}'}] = 0, \tag{A5a}$$

$$[a^\dagger_{\mathbf{k}}, a^\dagger_{\mathbf{k}'}] = 0, \tag{A5b}$$

$$[a_{\mathbf{k}}, a^\dagger_{\mathbf{k}'}] = \delta_{\mathbf{k}\mathbf{k}'}. \tag{A5c}$$

These operators are defined on a Fock space, which is a Hilbert space made of n-particle states ($n=0,1,\ldots$). The normalized basis ket vectors, denoted by $|\rangle$, can be constructed starting from the vector $|0\rangle$. The vacuum state $|0\rangle$ has the property that it is annihilated by all the $a_{\mathbf{k}}$ operators:

$$a_{\mathbf{k}}|0\rangle = 0, \quad \forall\,\mathbf{k}. \tag{A6}$$

In terms of the frequency $\omega_{\mathbf{k}} = c\sqrt{|\mathbf{k}|^2 + m^2}$, the Hamiltonian operator obtained from Eq. (A2) is

$$\hat{H} = \frac{1}{2}\sum_{\mathbf{k}} (a^\dagger_{\mathbf{k}}a_{\mathbf{k}} + a_{\mathbf{k}}a^\dagger_{\mathbf{k}})\omega_{\mathbf{k}} = \sum_{\mathbf{k}}\left(a^\dagger_{\mathbf{k}}a_{\mathbf{k}} + \frac{1}{2}\right)\omega_{\mathbf{k}}, \tag{A7}$$

where in the last step we used the commutation relations from Eq. (A5c). A calculation of the vacuum energy reveals a potential problem:

$$\langle 0|\hat{H}|0\rangle = \langle 0|0\rangle\sum_{\mathbf{k}}\frac{1}{2}\omega_{\mathbf{k}} = \sum_{\mathbf{k}}\frac{1}{2}\omega_{\mathbf{k}} \to \infty\,, \tag{A8}$$

where we have used the normalization condition $\langle 0|0\rangle = 1$. This infinite constant can be removed as described in the text.

Propagation amplitudes in quantum field theory (and hence scattering cross sections and decay constants) are given in terms of expectation values of time-ordered products of field operators. These time-ordered products arise in the interaction Hamiltonian of an interacting quantum field theory. The goal is to calculate propagation amplitudes for these interactions using time-dependent perturbation theory familiar from quantum mechanics. To leading order in the coupling constant, these products can be simplified, and the zero-point constant energy removed by using Wick's theorem.

The way Wick ordering is applied in practice to calculations in quantum field theory is through Wick's theorem, which gives a decomposition of time-ordered products of field operators into sums of normal-ordered products of field operators (again, we refer to Ref. 2). Wick's theorem appears in the functional-integral formulation of the theory in Sec. III A 3.

APPENDIX B: PROPERTIES OF $\mathrm{He}_n(x)$

Here we list a few useful properties of the scaled Hermite polynomials. More properties can be found in Ref. 5.

Defining series:

$$\mathrm{He}_n(x) = \sum_{m=0}^{\left[\frac{n}{2}\right]} (-1)^m \frac{n!}{m!\,2^m(n-2m)!}x^{n-2m}, \tag{B1}$$

where $[n/2]$ is the integer part of $n/2$.

Orthogonality:

$$\int_{-\infty}^{\infty} dx e^{-x^2/2}\mathrm{He}_n(x)\mathrm{He}_m(x) = \delta_{mn}\sqrt{2\pi}n!\,. \tag{B2}$$

Generating function:

$$\exp(x\alpha - \tfrac{1}{2}\alpha^2) = \sum_{n=0}^{\infty}\mathrm{He}_n(x)\frac{\alpha^n}{n!}. \tag{B3}$$

Recursion relation:

$$\mathrm{He}_{n+1}(x) = x\mathrm{He}_n(x) - n\mathrm{He}_{n-1}(x). \tag{B4}$$

The first five polynomials are

$$\mathrm{He}_0(x) = 1, \tag{B5a}$$

$$\mathrm{He}_1(x) = x, \tag{B5b}$$

$$\mathrm{He}_2(x) = x^2 - 1, \tag{B5c}$$

$$\mathrm{He}_3(x) = x^3 - 3x, \tag{B5d}$$

$$\mathrm{He}_4(x) = x^4 - 6x^2 + 3. \tag{B5e}$$

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[1]G. C. Wick, "The evaluation of the collision matrix," Phys. Rev. **80**, 268–272 (1950).

[2]Michael Peskin and Daniel V. Schroeder, *An Introduction to Quantum Field Theory* (Addison-Wesley, Reading, MA, 1995). See in particular the chapter on functional integrals in quantum field theory.

[3]Lewis H. Ryder, *Quantum Field Theory* (Cambridge University Press, Cambridge, 1985).

[4]Jun J. Sakurai, *Modern Quantum Mechanics*, rev. ed. (Addison-Wesley, Reading, MA, 1994).

[5]Milton Abramovitz and Irene A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1965).

[6]In particular, contributions to the S-matrix that have an essential singularity at zero coupling constant cannot be found by standard perturbative expansion around zero coupling. Yet these "nonperturbative" contributions can be studied using functional integrals. Rates for decays that would be strictly forbidden without these effects can be calculated; see for example, Ref. 3, Sec. 10.5.

[7]James Glimme and Arthur Jaffe, *Quantum Physics*, 2nd ed. (Springer-Verlag, New York, 1981).

[8]Svante Janson, *Gaussian Hilbert Spaces* (Cambridge University Press, Cambridge, 1997).

[9]To avoid dimension-dependent numerical terms (powers of π, powers of 2) in the definition (24) of the Gaussian measure we can, alternatively, define it by $\int_{\mathbb{R}^d} d^d x / (\det D)^{1/2} e^{-\pi Q(x)} e^{-2\pi i \langle x', x \rangle} = e^{-\pi W(x')}$. This definition can be convenient in the simplest Fourier transforms for those who forget where the 2π goes: $\hat{f}(p) = \int dx e^{-2\pi i p x} f(x)$ yields an inverse $f(x) = \int dp e^{2\pi i p x} \hat{f}(p)$, without the prefactor.

[10]Barry Simon, *The $P(\phi)_2$ Euclidean (Quantum) Field Theory* (Princeton University Press, Princeton, NJ, 1974).

[11]Similar to the division of power series, the multiplication of a power series is $(\Sigma_{k=0}^{\infty} a_k x^k)(\Sigma_{k=0}^{\infty} b_k x^k) = \Sigma_{n=0}^{\infty} d_n x^n$, where $d_n = \Sigma_{m=0}^{n} a_m b_{n-m}$.

[12]Joseph Polchinski, *String Theory: An Introduction to the Bosonic String*, Vol. 1 (Cambridge University Press, Cambridge, 1998).

[13]Claude Itzykson and Jean-Michel Drouffe, *Statistical Field Theory*, Vols. 1 and 2 (Cambridge University Press, Cambridge, 1989).

[14]Manfred Salmhofer, *Renormalization: An Introduction* (Springer-Verlag, Berlin, 1999).

[15]Edward Witten, "Perturbative quantum field theory," in *Quantum Fields and Strings: A Course for Mathematicians*, edited by P. Deligne *et al.* (American Mathematical Society, Providence, RI, 1999), Vol. 1, p. 451.

Inertia Demonstration. We still do this demonstration today. A stiff card is placed on a pillar and a marble is balanced atop the card. A quick blow with the hand or a snap with the fingertip on the edge of the card knocks it out of the way, and the marble drops into a small depression in the top of the tower. For reliable demonstrations, the digit is replaced by a strip of spring steel. This example is at Grinnell College in Iowa. Suggestion: make a half dozen of these (they can be made with blocks of wood and a section of an old hacksaw blade) and put them in the introductory lab for students to play with before lab starts. (Photograph and Notes by Thomas B. Greenslade, Jr., Kenyon College)