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The free fall of an apple: conceptual subtleties and implications for physics teaching

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Abstract

The study of free fall is thoroughly present in physics teaching at all levels. From the point of view of Newtonian dynamics it appears to be extremely simple, as it consists of a two-body problem with a constant force generating a constant acceleration. However, there are several important conceptual subtleties and hidden assumptions involved in this problem, which are rarely discussed in educational settings. In this work we present some of these subtleties and argue that explicitly addressing them has significant pedagogical benefits.

Keywords: free fall, universal law of gravitation, inertial frames, gravitational force exerted by a spherical shell

(Some figures may appear in colour only in the online journal)

1. The free fall of an apple in classical mechanics

Classical mechanics has been taught in the last 300 hundred years based on the work of Isaac Newton as presented in his book *Mathematical Principles of Natural Philosophy*, first published in 1687, usually known by its first Latin name, *Principia* (Newton 1934). Aside from some excerpts of Newton's original reasoning, it will be presented here in modern vector notation and in the International System of Units. We will consider, in particular, the simplest problem of Newtonian dynamics, namely, the free fall of an apple. It is fair to say that a rather

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representative way of how the topic is presented in introductory mechanics courses goes as follows.

When a force $\vec{F}$ acts on a body of mass m, it moves with an acceleration $\vec{a}$. This acceleration is related to the force by Newton's second law of motion, namely, $\vec{F} = m\vec{a}$. The simplest problem of dynamics is when there is a constant force acting on the test body in such a way that it will move with a constant acceleration. The most important situation is that of free fall in which the constant force is simply the weight $\vec{W}$ of the body given by Newton's law of gravitation, namely: $\vec{W} = GMm\hat{r}/R^2 = m\vec{g}$. Here $G = 6.67 \times 10^{-11}$ m^3 kg^{-1} s^{-2} is the so-called constant of universal gravitation, M is the mass of the Earth, $\hat{r}$ is the unit vector pointing from the apple to the centre of the Earth, R is the Earth's radius and the constant $\vec{g}$ is called the gravitational field of the Earth, which points towards its centre. By inserting the values of G, M and R we obtain the magnitude of $\vec{g}$ at the Earth's surface as given by 9.8 m s^{-2}. Combining these two equations and cancelling the masses, we conclude that all bodies in the same location will fall freely towards the centre of the Earth with the same acceleration, namely, $\vec{a} = \vec{g}$. This acceleration has a magnitude of 9.8 m s^{-2}, independent of the weights or chemical compositions of the falling bodies. This fact, which is far from being intuitive, was first grasped by Galileo Galilei, who presented it in his book of 1638, *Two New Sciences*. The first to publish a precise value of the free fall acceleration, based on pendulum experiments, was Christian Huygens in his book of 1673, *The Pendulum Clock*.

Although the reasoning appears very simple, there are many subtle aspects in this problem that are normally not considered in traditional textbooks (Lehavi and Galili 2009). Some of these aspects are discussed in the physics education literature, including the mass dependence of g for an observer on the ground (Lehavi and Galili 2009), the distinction between gravitational and inertial masses (Coelho 2007, 2012, Lehavi and Galili 2009), the fact that g is not constant for long distances (Stewart 1998, Gallant and Carlson 1999), the influence of the Earth's rotation in the value of g (French 1983), among others. In this work we stress other subtle aspects of this problem. Although we do not claim that the subtleties discussed here are new, we do argue that no paper in the physics education literature presents them in such a concise way and discusses which ones were already found in Newton's original work.

The following sections are guided by plausible questions that may be asked to this classical example and have a twofold goal. First, we raise some arguments from a more modern perspective to highlight the pertinence of the question and then we consider whether or not the issue was relevant to Newton. When sketching Newton's answers, we will try to be as faithful as possible to his original reasoning and notation, but without compromising the understanding for a modern reader. Overall, we argue that exploring such conceptual subtleties with students should contribute for the development of critical reasoning, a major goal of science education.

2. Is free fall a simple two-body problem?

This problem is normally considered as a simple two-body problem, that is, the interaction between the Earth and the apple. But this supposition is not true. As a matter of fact, there are innumerable bodies around the Earth. These other bodies include the Sun, the Moon and the planets of the solar system. However, as Newton's law of gravitation is inversely proportional to the square of the distance between the interacting bodies, the force exerted by these faraway bodies on the apple is usually neglected in comparison with the force exerted by the Earth.

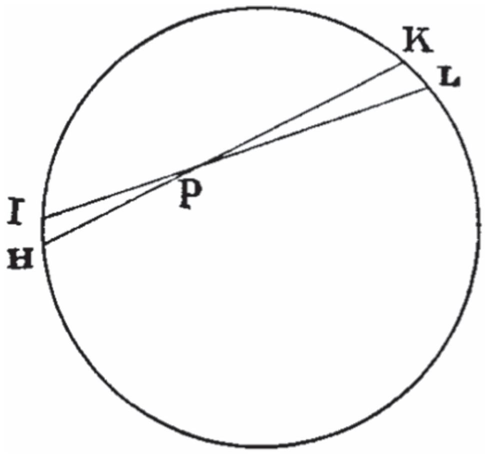

Figure 1. Newton's original figure to prove theorem 30.

But what about the stars and galaxies? We might think that we can also neglect their gravitational influence due to the fact that they are also very far away from the apple, at distances much greater than the Earth's radius. However, the argument does not apply in this situation because we do not know how many stars and galaxies exist in the Universe. The force of a single galaxy acting on the apple may be much smaller than the gravitational force exerted by the Earth. But the force exerted by all the galaxies might have a magnitude comparable to the gravitational force exerted by the Earth. The force exerted by all galaxies on the apple might even be infinitely large, if there is an infinite number of galaxies in the Universe. In principle we must include the gravitational influence exerted by all these bodies in any dynamical problem.

Although the influence of the distant bodies in dynamical problems is usually neglected, Newton himself did take them into account. At his time the galaxies were not yet known, but the argument he utilised for stars can also be applied for galaxies and for other bodies which may be discovered in the future. Following Newton, we can consider the stars and galaxies as composing a series of spherical shells around the solar system. The radii of these shells can go on to infinity. In theorem 30 of the *Principia*, Newton considered a test body anywhere inside a spherical shell. He then considered the gravitational force exerted by each portion of the shell acting on the internal test body.

His argument goes roughly as follows. Consider a point P inside a homogenous spherical shell (figure 1). Through P draw two lines HK and IL intercepting very small arcs HI and KL. Because the angle with vertex on P is very small, one can treat the arcs HI and KL as line segments and the triangles PIH and PKL as similar, thus writing

$$\frac{KL}{IH} = \frac{PK}{PH}. \tag{1}$$

In other words, the lengths of the arcs are proportional to their distances to P. But instead of a circle, the situation involves a spherical shell with constant mass density. So if we want to know how much mass is on each side, we need to relate the *areas* associated with those arcs and not their lengths. A possible way to conceive these areas is to think of two *cones* with vertices at P and circular bases with diameters HI and KL. The areas are, of course, proportional to the squares of the base diameters. Since the superficial mass density is constant, one can write the ratio of the masses at the bases of these two cones as given by

$$\frac{m_{KL}}{m_{IH}} = \left(\frac{KL}{IH}\right)^2 \tag{2}$$

which from (1) we obtain

$$\frac{m_{KL}}{m_{IH}} = \left(\frac{PK}{PH}\right)^2. \tag{3}$$

Now, according to Newton, the gravitational force is directly proportional to the product of the masses and inversely proportional to the square of the distance. When comparing the forces exerted by the masses at *KL* and *IH* acting on a test body in *P*, we need to consider both the masses (m_{KL}, m_{IH}) and distances (*PK*, *PH*). Although *KL* is more distant from *P*, which would lead to a smaller force, its area is greater. Therefore, m_{KL} is more massive than m_{IH}. Due to (3), one effect cancels precisely the other. That is, the force exerted by m_{KL} on a test body in *P* is equal and opposite the force exerted by m_{IH}. There is no net force acting on the test body in *P*. This geometrical reasoning is very close to Newton's original argument. Nowadays the theorem is usually demonstrated using Gauss' law for the gravitational field.

It is important to stress that this result is only valid for central forces that vary as the inverse square of the distance. The theorem is not valid, for instance, for a force law that varies inversely with the distance between the interacting bodies or for a force that falls as $1/r^3$. Suppose that Newton's law of gravitation represents only a portion of a more general force law containing other terms. If these other terms depend on the distance differently from $1/r^2$, then this theorem will not be valid for this more general force law.

In sum, theorem 30 shows that the total or net force exerted by the spherical shell is zero, no matter the position of the internal test particle. This is one of the most important cosmological results of the *Principia*. According to Newtonian dynamics, even if there is an infinite number of stars and galaxies in the Universe, it is possible to neglect their joint gravitational influence on the apple due to this theorem, by considering the stars and galaxies scattered at random in all directions of the sky. This theorem is what allows us to treat the free fall of an apple as a two-body problem, although the whole discussion is often neglected in physics teaching. Nevertheless, it is important and instructive to see how it played an essential role in Newton's original work.

Newton himself was completely aware of the cosmological significance of his theorem 30. In book III of the *Principia,* he mentioned that the aphelions and nodes of the orbits of the planets are fixed relative to the set of fixed stars. He mentioned that one of the reasons for this fact was that the fixed stars, being everywhere promiscuously dispersed in the heavens, destroy their mutual actions on the planets by their contrary attractions, quoting specifically his theorem 30.

3. Why can the Earth's mass be considered as concentrated in its centre?

This problem is normally presented as a simple two-body problem, with the Earth and the apple treated as material points concentrated at their centres of gravity. As regards the apple, this supposition seems reasonable, due to the fact that it is small in comparison with the Earth's radius. But the Earth itself is huge and different portions of the Earth are at different distances from the apple. It is not obvious that we can consider the Earth with all its mass concentrated at its centre. The force exerted on the apple by the real Earth might be different from the force exerted on the apple by a single hypothetical particle with the Earth's mass

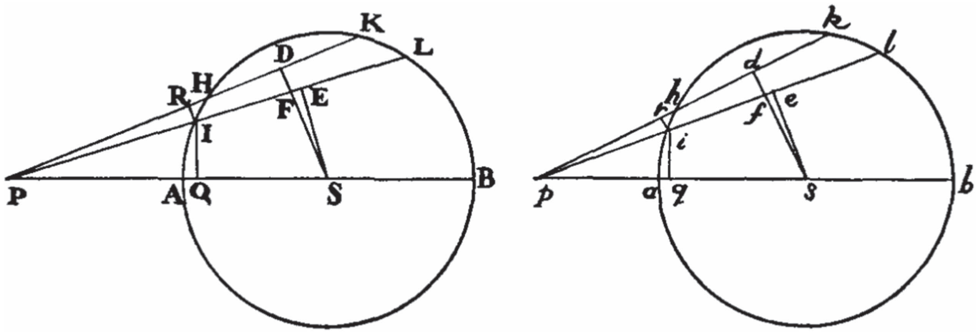

Figure 2. Newton's original figure to prove theorem 31.

concentrated at the centre of the Earth. Newton was very aware of this issue and was the first to prove that one can indeed assume that the Earth's mass is concentrated in its centre.

However, Newton's original proof is much harder to follow in this case. We will sketch its steps to give an overview of its content. The interested reader is referred to the *Principia* and to Chandrasekhar (1995, pp 269–273).

It is already quite demanding to understand the reasons for the peculiar geometric construction of theorem 31 (see figure 2). It depicts two *equal* spherical surfaces with centres at *S* and *s*. Newton considers corpuscles placed at *P* and *p* along the diameters *BSA* and *bsa*. He draws the lines *KHP*, *LIP*, *khp* and *lip*. The goal of the proof is to relate the gravitational forces exerted by the spherical zones generated by the revolution of arcs *HI* and *hi* acting at corpuscles located at different points (*P* and *p*). The two spheres have the same diameters, namely, $AB = ab$. The distance *PS* is different from the distance *ps*. It is assumed that the arcs *HK* and *hk*, as well as *IL* and *il*, are *equal*. As *PS* is different from *ps*, the angle *KPS* is different from the angle *kps*. He draws *SFD*, *IR*, *sfd* and *ir* orthogonal to the lines *KP* and *kp*, while *SE* and *se* are orthogonal to *LP* and *lp*. Lines *IQ* and *iq* are drawn orthogonal to the diameters *AB* and *ab*. From the very beginning Newton states that everything is to be considered for *vanishing* angles *DPE* and *dpe*, which means that we are dealing with his characteristic 'last ratios' kind of reasoning. This assumption allows him to conclude that the lines *DF* and *df*, as well as *SE* and *se,* are *equal*.

The proof starts with the common procedure of finding relations between segments from similar triangles. Since $\Delta PRI \sim \Delta PFD$, $\Delta pri \sim \Delta pfd$ and $DF = df$ one has

$$\frac{PI}{PF} = \frac{RI}{DF}, \tag{4}$$

$$\frac{pf}{pi} = \frac{df \text{ or } DF}{ri} \tag{5}$$

multiplying (4) and (5) one has

$$\frac{PI.pf}{PF.pi} = \frac{RI}{ri} = \frac{\text{arc } IH}{\text{arc } ih} = \frac{IH}{ih}. \tag{6}$$

The last terms of the equality are justified by the vanishing angles assumption. Considering now another set of similar triangles, namely $\Delta PIQ \sim \Delta PES$, $\Delta piq \sim \Delta pes$, and $SE = se$. In this case one has

$$\frac{PI}{PS} = \frac{IQ}{SE}, \tag{7}$$

$$\frac{ps}{pi} = \frac{se \text{ or } SE}{iq}. \tag{8}$$

Multiplying (7) and (8) one has

$$\frac{PI.ps}{PS.pi} = \frac{IQ}{iq}. \tag{9}$$

Multiplying (6) and (9) one has

$$\frac{PI^2.pf.ps}{pi^2.PF.PS} = \frac{HI.IQ}{hi.iq}. \tag{10}$$

The ratio on the right-hand side of (10) has an important meaning. It is the ratio between the areas described by the arcs HI and hi when each semicircle revolves about the diameter. And just like in the previous theorem, the area will be directly related with the quantity of mass because the surface mass density is taken to be constant.

Since the gravitational force exerted by the spherical zone on a test mass at P (or p) is directly proportional to its area (mass) and inversely proportional to the square of the distance PI (or pi), the ratio between the forces acting at the corpuscles located at P and p is

$$\frac{dF_{PI}}{dF_{pi}} = \frac{\frac{HI\,.\,IQ}{PI^2}}{\frac{hi\,.\,iq}{pi^2}} \tag{11}$$

which from (10) is equal to

$$\frac{dF_{PI}}{dF_{pi}} = \frac{pf.ps}{PF.PS}. \tag{12}$$

Each elementary gravitational force is directed in the line connecting the point P (or p) and the infinitesimal arc HI (or hi). One useful strategy is to decompose the (inclined) force into two perpendicular components. Due to the spherical symmetry, the vertical components will cancel out for the whole spherical zone and the resultant force will be horizontal, i.e. towards the centre of the sphere. This is geometrically expressed as

$$\frac{dF_{PS}}{dF_{ps}} = \frac{dF_{PI}}{dF_{pi}} \frac{PI/PQ}{pi/pq}. \tag{13}$$

Once again similar triangles ($\Delta PIQ \sim \Delta PSF$ and $\Delta piq \sim \Delta psf$) are used to obtain the following relationship:

$$\frac{dF_{PS}}{dF_{ps}} = \frac{dF_{PI}}{dF_{pi}} \frac{PS/PF}{ps/pf}. \tag{14}$$

Substituting (12) in (14) one finally obtains

$$\frac{dF_{PS}}{dF_{ps}} = \frac{ps^2}{PS^2}. \tag{15}$$

Newton argues that by the same process the same ratio would be obtained when considering the revolution of the arcs KL and kl, as well as with the other spherical zones that constitute the spheres. Thus, the resultant forces exerted by the whole spheres acting at corpuscles located at points P and p are inversely proportional to the squares of the distances of these corpuscles to the centre of the spheres. This means that the situation is equivalent to having the mass of the spherical shell concentrated at its centre, as it was to be proven. As the result is valid for any spherical shell of arbitrary radius, it will also be valid for a (solid) sphere composed of many spherical shells.

Even if Newton's geometric reasoning may seem too complicated for the present-day reader, modern derivations using calculus are not necessarily easier to follow[4]. In sum, this suffices to show that assuming that the Earth's mass is concentrated in its centre is not at all trivial. Once again it is valid only for force laws that are central and vary with the distance as $1/r^2$.

4. Why can we cancel the gravitational mass of the apple with its inertial mass?

The force $\vec{F}_{21}$ exerted by particle 2 of *gravitational* mass m_{g2} and acting on particle 1 of *gravitational* mass m_{g1}, when they are separated by a distance r_{12} which is much larger than the sizes or diameters of these particles, can be expressed as

$$\vec{F}_{21} = -\frac{Gm_{g1}m_{g2}}{r_{12}^2}\hat{r}_{12} = -\vec{F}_{12}. \tag{16}$$

Here $\hat{r}_{12}$ is the unit vector pointing from 2 to 1 and $\vec{F}_{12}$ represents the force exerted by 1 on 2. After integrating equation (16) over the whole Earth of radius R (theorem 31), the total force $\vec{F}_{E1}$ exerted by the Earth and acting on a particle 1 at the surface of the Earth can be expressed as

$$\vec{F}_{E1} = \vec{W}_1 = -\frac{Gm_{g1}m_{gE}}{R^2}\hat{r} = m_{g1}\vec{g}, \tag{17}$$

where m_{gE} is the gravitational mass of the Earth and $\vec{g} = -Gm_{gE}\hat{r}/R^2$ is the gravitational field of the Earth at the location of the particle. By replacing the known values $G = 6.67 \times 10^{-11}\ \mathrm{m^3\,kg^{-1}\,s^{-2}}$, $m_{gE} = 5.98 \times 10^{24}\ \mathrm{kg}$ and $R = 6.37 \times 10^6\ \mathrm{m}$ we obtain the previous value of $g = |\vec{g}| = 9.8\ \mathrm{m\,s^{-2}}$.

However, the mass that appears in the right-hand side of Newton's second law of motion, $\vec{F} = m\vec{a}$, is the *inertial* mass m_i of the test particle. Combining equation (17) with Newton's second law of motion for particle 1 yields

$$\vec{W}_1 = m_{g1}\vec{g} = m_{i1}\vec{a}_1. \tag{18}$$

Which leads to

$$\vec{a}_1 = \frac{m_{g1}}{m_{i1}}\vec{g}. \tag{19}$$

In principle, we *cannot* cancel the masses m_{g1} and m_{i1} as they have no conceptual relation with one another. The mass m_{g1} is a gravitational property of body 1 which is determined experimentally with an equal-arm balance. It is related to the gravitational interaction between the apple and the Earth. For instance, if we measure with this balance the weights W_1 and W_2 of bodies 1 and 2 at the same location of the surface of the Earth, the ratio of their gravitational masses is defined as: $m_{g1}/m_{g2} = W_1/W_2$.

The mass m_{i1}, on the other hand, is an inertial property of body 1. It is determined experimentally by its acceleration relative to an inertial frame of reference due to any kind of

[4] Derivations of the shell theorems (30 and 31) are standard exercises in calculus courses at university (see, for instance, French 1971, pp 261–265). These theorems can be more easily proved using Gauss's law.

force acting on it. For instance, suppose that a spring of elastic constant k and relaxed length ℓ_o is held horizontally over a frictionless table, with one of its extremities connected to a fixed support attached to the Earth. Suppose that body 1 is connected to its other extremity. Suppose that body 1 suffers an initial horizontal acceleration a_1 when the spring is stretched to a length ℓ and released from rest. By Hooke's law combined with Newton's second law of motion we obtain $k(\ell - \ell_o) = m_{i1}a_1$. We then remove body 1 and connect body 2 to the same spring. Suppose that body 2 suffers an initial horizontal acceleration a_2 when the spring is stretched to a length ℓ and released from rest. The ratio of their inertial masses is defined as $m_{i1}/m_{i2} = a_2/a_1$.

In principle, there is no relation between W_1/W_2 and a_1/a_2. These ratios might have no relation with one another. From classical mechanics and utilising the fact that the gravitational field $\vec{g} = -Gm_{gE}\hat{r}/R^2$ depends only on the Earth, we obtain from equation (18) the following relation:

$$\frac{W_1}{W_2} = \frac{m_{g1}}{m_{g2}} = \frac{m_{i1}}{m_{i2}}\frac{a_1}{a_2}. \tag{20}$$

This relation is all that can be obtained from classical mechanics.

However, Galileo found out experimentally that all bodies fall freely to the ground at the same location of the Earth with the same acceleration, no matter their weights, shapes or chemical compositions:

$$a_1 = a_2. \tag{21}$$

This result is very counter-intuitive. Consider a coin and a feather falling freely in vacuum. They have different weights, different chemical compositions, different shapes, different textures, etc. Since any specific property of the coin is different from the corresponding specific property of the feather, it would be natural to predict that they would fall freely towards the ground with different accelerations. However, both fall with the same acceleration in the same gravitational field of the Earth. This should be not taken for granted in physics teaching.

Utilising this experimental fact that $a_1 = a_2$ in equation (20) yields $m_{g1}/m_{g2} = m_{i1}/m_{i2}$ for all bodies. It is this discovery of Galileo that $a_1 = a_2$ that allows the cancelation of the gravitational mass m_{g1} appearing in the weight $m_{g1}\vec{g}$ of the body with the inertial mass m_{i1} appearing in the right-hand side of Newton's second law of motion $m_{i1}\vec{a}_1$. Newton's laws of motion together with his law of universal gravitation do *not* allow by themselves this cancelation of the two masses of body 1. We need to supplement Newton's laws with the result of Galileo's free fall experiment in order to cancel the gravitational mass of a body with its inertial mass.

To emphasise this fact, we can compare the accelerations in vacuum of an alpha particle and a proton in the same electric field. These two particles might be moving, for instance, inside the same ideal capacitor. The alpha particle and the proton do not move with the same acceleration. The alpha particle has twice the charge of the proton and four times its mass, as it is composed of two protons and two neutrons. The observed acceleration of the proton in an electric field is twice the acceleration of the alpha particle in the same electric field. However, if both particles were falling in the gravitational field of the Earth, they would fall with the same acceleration.

The inertial mass of a test body is only proportional to its gravitational mass. It has no relation with the electric charge of the test body, nor with any of its magnetic properties, nor with the electric current which may be flowing through it, nor with any of its elastic or nuclear properties etc. This empirical fact suggests that the inertia of a body may be due to its gravitational interaction with other bodies in the Universe. This idea is known as Mach's principle and has been implemented mathematically with Weber's force for gravitation (Assis 2014). An analysis of the influence of Mach's ideas in the teaching of classical mechanics in representative university textbooks is found in Assis and Zylbersztajn (2001).

It is important to stress that the distinction between gravitational and inertial masses is not found in Newton's original work. In the *Principia* Newton utilised a single mass concept. This mass was also called the quantity of matter of the body. It corresponds to our inertial mass, namely, the mass appearing in the linear momentum of a body, $\vec{p} = m_i\vec{v}$, and also in Newton's second law of motion, $\vec{F} = m_i\vec{a}$. Newton put two pendulums of equal shape and length to oscillate near the Earth's surface. They were filled with the same weight of different substances. One of them was filled with wood. The second pendulum was filled with an equal weight of gold. He observed that these two pendulums oscillated with the same frequencies. The same happened when the second pendulum was filled with an equal weight of silver, lead, glass, sand, common salt, water and wheat. Newton then concluded that the mass of a body was proportional to its weight. His pendulum experiment is analogous to Galileo's free fall experiment discussed in this paper. Both experiments lead to the same conclusion, namely, that the inertial mass of a body is proportional to its weight (as expressed by Newton), or that the gravitational mass of a body is proportional to its inertial mass (as discussed in modern textbooks).

5. The acceleration of the apple is relative to what?

Now that we clarified these aspects, we can conclude that all bodies fall freely towards the ground with the same acceleration given by

$$\vec{a} = \vec{g} = -\frac{Gm_{gE}}{R^2}\hat{r} = -9.8 \text{ m s}^{-2}\hat{r}. \tag{22}$$

That is, all bodies fall freely near the surface of the Earth with the same acceleration[5] of 9.8 m s^{-2}. But this acceleration of 9.8 m s^{-2} is the acceleration of the apple *relative to what*? We might think that it is the acceleration of the apple relative to the ground. Another assumption could be that it is the acceleration relative to the centre of mass of the apple–Earth system. However, in principle, neither of these assumptions are completely true in Newton's original reasoning.

In order to arrive at equation (22) we utilised Newton's second law of motion, $\vec{F} = m_i\vec{a}$. Newton applied his formulation of mechanics not only for test bodies moving relative to the ground, but also to the planets of the solar system. He considered, for instance, the annual orbit of the Earth around the Sun. In this last situation he would not consider the ground as the frame of reference, because obviously the Earth is not moving relative to itself. Another frame of reference was necessary in order to study the orbit of the Earth around the Sun.

According to Newton, the correct frame of reference to utilise in his laws of motion is what he called 'absolute space'. In his own words, 'absolute space, in its own nature, without relation to anything external, remains always similar and immovable'. Thus, this Newtonian absolute space is not connected with the Earth or the ground, has no relation with the Sun, is

[5] For greater heights the constant acceleration approximation is no longer valid (Stewart 1998, Gallant and Carlson 1999). Moreover, if one considers the common centre of mass of the apple–Earth system, it is possible to show that the free fall acceleration of the apple measured by an observer on the Earth's ground depends on the apple's mass (Lehavi and Galili 2009).

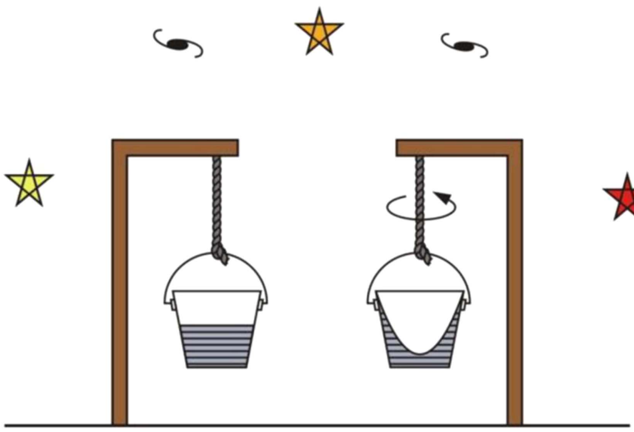

Figure 3. Newton's bucket experiment. Reproduced with permission from Assis (1999).

not related to the frame of fixed stars nor is it related with any other material body in the Universe. It is essentially the empty free space or the vacuum.

Therefore, the free fall acceleration of the test bodies in Galileo's experiment, according to Newton, takes place *relative to empty free space* and *not* relative to the ground. Usually, didactical presentations of the topic either (i) simply ignore the question of the frame of reference with respect to which g is measured, (ii) mention that it is relative to the ground (which, according to Newton's original argument, is wrong), or (iii) refer to an inertial frame of reference attached to the fixed stars. However, as we can see from Newton himself, the fixed stars cannot be taken to be absolute space, since the latter is immaterial. In the next section, we show how Newton claims to prove the existence of absolute space with his famous bucket experiment.

6. How to measure the acceleration of a body relative to absolute empty space?

Obviously this Newtonian concept of the acceleration of the test body relative to absolute space raises many questions. We do not see the vacuum nor empty space. How to detect the motion relative to nothing? How to measure it? How can we know the motion of the Earth relative to empty space? etc.

Newton was aware of this problem. He performed a famous bucket experiment in order to deal with it. He considered a bucket partially filled with water and hanging by a rope. In the beginning, the bucket B and the water W remained at rest, without any angular velocity relative to the ground, namely, $\omega_B^I = \omega_W^I = 0$. In this initial configuration I, the water surface was horizontal. He twisted the rope and released the bucket. In the beginning of the rotation, only the bucket was spinning relative to the ground and the water remained at rest and plain. The vessel then gradually communicated its motion to the water, which began to revolve around the axis of the bucket, receding from the middle and ascending to the sides of the bucket. After a while, the bucket and the water were rotating together relative to the ground with a constant angular velocity ω_o. In this final configuration F we have $\omega_B^F = \omega_W^F = \omega_o \neq 0$ and the water surface was concave, high at the sides of the bucket and low at the axis of rotation (see figure 3).

Newton discussed what caused this parabolic shape. The simple answer is that the parabolic shape was due to the rotation of the water. But rotation relative to what? There are three material suspects, namely: (a) rotation of the water relative to the bucket, (b) rotation of the water relative to the ground, and (c) rotation of the water relative to the distant bodies of the cosmos like the fixed stars. Let us follow Newton's reasoning and consider each suspect separately.

(a) Newton argued that the parabolic shape was not due to the rotation of the water relative to the bucket. After all, although $\omega_B^I = \omega_W^I = 0$ and $\omega_B^F = \omega_W^F = \omega_o \neq 0$, in both cases there is no relative motion between the water and the bucket, namely, $\omega_B^I - \omega_W^I = \omega_B^F - \omega_W^F = 0$. Therefore, the ascent of the water towards the sides of the bucket was not due to its relative motion relative to the bucket.

(b) We might think that this ascent was due to the rotation of the water relative to the ground because when the water was at rest it was flat, while when it was spinning relative to the Earth it became parabolic. However, according to Newton's law of gravitation and his theorem 31 mentioned above, the Earth attracts any molecule of the water downwards, no matter if the molecule is at rest or moving relative to the ground. That is, the weight of the molecule points towards the centre of the Earth and has always the same value, no matter if the molecule is at rest or spinning around the axis of the bucket. Even when the water is spinning, the Earth exerts no centrifugal force pressing the water against the walls of the bucket. In conclusion, the parabolic shape of the water when it was spinning was not caused by the Earth.

(c) Finally we might think that this ascent of the water was due to its rotation relative to the frame of fixed stars (or due to its rotation relative to the set of distant galaxies). After all, when the water was not spinning relative to the stars its surface was flat, while when the water was spinning relative to the frame of fixed stars its surface acquired a parabolic shape. However, according to Newton's law of gravitation and his theorem 30 discussed above, the set of fixed stars exerts no net force on any molecule of the water because the stars are scattered all over the sky, like a series of spherical shells around the Earth. The same reasoning applies to the set of distant galaxies. That is, they exert no resultant force on any molecule of water, no matter if the water is at rest or spinning relative to this set of distant galaxies.

Newton then argued that this concave shape of the water surface was due to its rotation relative to *absolute space*, which had no relation with anything material. That is, his absolute space might be considered equivalent to empty free space. It was not related to the ground, nor to the set of fixed stars, nor to any set of distant bodies around the Earth. The concavity of the water surface was a measure of its absolute rotation relative to empty space.

We are fully aware that the whole discussion is rather strange for a modern reader. But in order to appreciate the importance of this experiment and the notion of absolute space for the theoretical edifice of the *Principia,* it is instructive to see that this discussion appears in the very beginning of this work, even before Newton's famous laws of motion. Nevertheless, it is interesting to notice that the bucket experiment is practically absent from modern physics textbooks, which leads to a reflection about processes involving a didactical transposition (Chevallard 1991). One cannot help wondering whether or not this exclusion has been a conscious choice made by textbook authors and what are their motivations.

Ernst Mach criticised Newton's arguments in his book *The Science of Mechanics*, originally published in 1883 (Mach 1960). According to Mach the concavity of the water was due to its rotation relative to the distant bodies in the cosmos, it was not due to its rotation relative to Newton's absolute and empty free space. Although Mach was not able to implement this suggestion, his idea became known as Mach's principle. It is possible to implement mathematically Mach's principle utilising Weber's law applied to gravitation (Assis 2014).

7. Concluding remarks

The free fall of an apple is often presented as an extremely simple application of Newton's laws where a constant force leads to a constant acceleration. In this work, we have shown that this is far from being the case. We have argued that it is not at all trivial (i) to consider it a two-body problem, (ii) to assume that the Earth's mass is concentrated in its centre, (iii) to cancel the gravitational mass of the apple with its inertial mass, and (iv) to assume that the acceleration of the apple is relative to the ground. We have also stressed how important these issues were for Newton's original work. The absence of an explicit discussion of some of these conceptual subtleties in traditional textbooks illustrates features of the process called didactical transposition (Chevallard 1991) or reconstruction (Duit *et al* 2012) by the educational literature.

If one agrees that physics should contribute for the education of critical citizens and the development of logical reasoning, then raising some of those issues in educational settings can be very beneficial. Although the notion of 'critical citizens' has been commonly associated with societal issues (e.g. energy consumption and global warming), we are confident that it is also possible to develop critical reasoning from an 'internalist' approach focused solely on physics concepts (see, for instance, Viennot 2014). The fact that free fall is such a widely taught topic makes it a powerful gateway to deep conceptual discussions; there are simply many rich opportunities missed when some of those issues are ignored.

But of course one cannot deny that transforming the discussion presented here into an adequate didactical discourse intended for students being introduced to the topic is not an easy task. Any plausible learning theory will argue that we learn from the simple to the complex. Therefore, exposing the students to all the subtleties of the problem already in their first contact with it may not be the best strategy. Nevertheless, from the perspective of the teacher there is an important difference between not knowing the subtleties (and even implying that the situation is indeed very simple) and being aware of them and making the conscious choice of omitting and/or briefly mentioning some of these subtleties when appropriate. Thus, the discussions presented in this article may be even more relevant for teacher education, especially in order to develop an ability that Chevallard (1991) called *epistemological surveillance* and defined as follows:

> […] a tool that allows revised, take away, to question evidences, doubt about the simple ideas, abandon familiarity, hence misleading its object of study. In a word, is what enables exercising its epistemological surveillance (Chevallard 1991, p 16).

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